

Calculus 1 Series and power series: Solution

Note: The parts between pairs of $\langle\langle \rangle\rangle$ are explanatory notes. These are not “official” parts of the solution, they merely illustrate how we think about the problem. When writing an “official” solution of a problem, such parts should be left out.

1 a). This looks like a geometric series:

$$\begin{aligned} \sum_{k=3}^{\infty} \frac{3^{k+1}}{2^{2k+1}} &= \sum_{k=3}^{\infty} \frac{3 \cdot 3^k}{2^5 (2^2)^k} = \frac{3}{2^5} \sum_{k=3}^{\infty} \left(\frac{3}{4}\right)^k = \frac{3}{2^5} \left(\frac{3}{4}\right)^3 \sum_{k=0}^{\infty} \left(\frac{3}{4}\right)^k = \langle\langle \left|\frac{3}{4}\right| < 1 \rangle\rangle = \frac{3}{2^5} \left(\frac{3}{4}\right)^3 \frac{1}{1-\frac{3}{4}} \\ &= \frac{3}{2^5} \frac{3^3}{4^3} \cdot 4 = \frac{3^4}{2^9}. \end{aligned}$$

1 b). $\sum_{k=2}^{\infty} \frac{1}{k(k+1)} = \sum_{k=2}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right)$. This looks like a telescopic series, we try it:

$$s_N = \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \dots + \left(\frac{1}{N-1} - \frac{1}{N}\right) + \left(\frac{1}{N} - \frac{1}{N+1}\right) = \frac{1}{2} - \frac{1}{N+1} \rightarrow \frac{1}{2},$$

$$\text{so } \sum_{k=2}^{\infty} \frac{1}{k(k+1)} = \frac{1}{2}.$$

2 a). This is a typical problem for the limit ratio test:

$$\lambda = \lim_{k \rightarrow \infty} \left(\frac{a_{k+1}}{a_k}\right) = \lim_{k \rightarrow \infty} \left(\frac{(2(k+1)-1)/e^{k+1}}{(2k-1)/e^k}\right) = \lim_{k \rightarrow \infty} \left(\frac{2k+1}{2k-1} \frac{e^k}{e^{k+1}}\right) = \lim_{k \rightarrow \infty} \left(\frac{2+1/k}{2-1/k} \frac{1}{e}\right) = \frac{1}{e}.$$

Since $\lambda = \frac{1}{e} < 1$, the series $\sum \frac{2k-1}{e^k}$ converges.

Also the limit root test would work:

$$\rho = \lim_{k \rightarrow \infty} \left(\sqrt[k]{a_k}\right) = \lim_{k \rightarrow \infty} \left(\sqrt[k]{\frac{2k-1}{e^k}}\right) = \lim_{k \rightarrow \infty} \left(\frac{\sqrt[k]{2k-1}}{e}\right) = \frac{1}{e}.$$

We cheated a bit, we should check that

$$\begin{aligned} \lim_{k \rightarrow \infty} \left(\sqrt[k]{2k-1}\right) &= \lim_{k \rightarrow \infty} \left((2k-1)^{1/k}\right) = \langle\langle \infty^0 \rangle\rangle = e^{\lim_{k \rightarrow \infty} \left(\frac{\ln(2k-1)}{k}\right)} \\ &= \langle\langle \frac{\infty}{\infty} \implies \text{l'H} \rangle\rangle = e^{\lim_{k \rightarrow \infty} \left(\frac{2}{2k-1}\right)} = e^{1/\infty} = e^0 = 1. \end{aligned}$$

Anyway, we have $\rho = \frac{1}{e} < 1$ and we get convergence of the series.

Integral test is also possible, since the function $f(x) = \frac{2x-1}{e^x}$ is non-increasing (check its derivative) on $[2, \infty)$:

$$\begin{aligned} \int_2^{\infty} \frac{2x-1}{e^x} dx &= \int_2^{\infty} (2x-1)e^{-x} dx = \left| \begin{array}{l} u = 2x-1 \\ u' = 2 \end{array} \quad \begin{array}{l} v' = e^{-x} \\ v = -e^{-x} \end{array} \right| = \left[-(2x-1)e^{-x} \right]_2^{\infty} + 2 \int_2^{\infty} e^{-x} dx \\ &= 3e^{-2} - \lim_{x \rightarrow \infty} \left(\frac{(2x-1)}{e^x}\right) + 2 \left[-e^{-x} \right]_2^{\infty} = \langle\langle \text{l'H} \rangle\rangle = \frac{3}{e^2} - 0 + 2[e^{-2} - 0] = 5e^{-2}. \end{aligned}$$

Integral came up finite, hence also the series $\sum \frac{2k-1}{e^k}$ converges.

Since $a_k = \frac{2k-1}{e^k} \geq 0$, we have $|a_k| = a_k$, so convergence and absolute convergence coincide. Conclusion: The series converges absolutely.

2 b). For alternating series we have just one test, the AST. For that we need that $b_k = \frac{3^k}{(k+2)!}$ be non-negative and non-increasing. This is fortunately true, for $k \geq 1$ we even have a decreasing sequence:

$$b_{k+1} = \frac{3^{k+1}}{(k+3)!} = \frac{3}{k+3} \frac{3^k}{(k+2)!} = \frac{3}{k+3} b_k < b_k.$$

So we can use AST, and since $b_k \rightarrow 0$, the series converges.

What is the type of convergence, is it also absolute? Does $\sum \frac{3^k}{(k+2)!}$ converge? Because of the factorial we cannot use the integral test, nor the root test, factorials can be handled only by the limit ratio test:

$$\lambda = \lim_{k \rightarrow \infty} \left(\frac{a_{k+1}}{a_k} \right) = \lim_{k \rightarrow \infty} \left(\frac{3^{k+1}/(k+3)!}{3^k/(k+2)!} \right) = \lim_{k \rightarrow \infty} \left(\frac{(k+2)!}{(k+3)!} \frac{3^{k+1}}{3^k} \right) = \lim_{k \rightarrow \infty} \left(\frac{3}{k+3} \right) = 0.$$

We have $\lambda = 0 < 1$, so the series $\sum \frac{3^k}{(k+2)!}$ converges. Therefore the series $\sum (-1)^k \frac{3^k}{(k+2)!}$ converges absolutely, not conditionally.

By the way, an experienced series solver would guess that the factorial in the denominator would be decisive, so he would go directly to testing absolute convergence via the ratio test, then by a theorem the series itself automatically converges and we don't have to use AST.

2 c). For alternating series we have only one test, namely AST, but for that we need $b_k = \frac{2^k}{k^2}$ to be non-increasing. However, $f(x) = \frac{2^x}{x^2}$ is increasing for $x \geq 3$, so AST is out. We can try for absolute convergence, does $\sum \frac{2^k}{k^2}$ converge? Here we can use both the root and ratio test, we start with the former:

$$\sqrt[k]{a_k} = \sqrt[k]{\frac{2^k}{k^2}} = \frac{2}{[\sqrt[k]{k}]^2} \rightarrow \frac{2}{1^2} = 2 \text{ as } k \rightarrow \infty.$$

We have $\rho = 2 > 1$, so the series $\sum \frac{2^k}{k^2}$ diverges. Thus the series $\sum (-1)^k \frac{2^k}{k^2}$ does not converge absolutely. The limit ratio test would yield the same answer:

$$\frac{a_{k+1}}{a_k} = \frac{2^{k+1}/(k+1)^2}{2^k/k^2} = \frac{2^{k+1}}{2^k} \left(\frac{k}{k+1} \right)^2 = 2 \left(\frac{1+1/k}{1} \right)^2 \rightarrow 2 \cdot 1^2 = 2 \text{ pro } k \rightarrow \infty.$$

We still do not know whether the series converges (but not absolutely) or diverges. Since we have no other tests for alternating series, it will probably diverge, which is something we can tell from the limit of a_k . And indeed, $\frac{2^k}{k^2} \rightarrow \infty$ [for instance using two l'Hospital rules], so it is definitely not true that $a_k = (-1)^k \frac{2^k}{k^2} = \pm \frac{2^k}{k^2} \rightarrow 0$. The given series therefore diverges.

2 d). We have a series with non-negative numbers again, so convergence and absolute convergence coincide and we can use our favorite tests. There are only powers in expressions for a_k , so both root and ratio tests might work. However, since even and odd terms are of different types and the ratio test would mix them up, it will be probably easier to try the root test:

$$\sqrt[k]{a_k} = \begin{cases} \frac{1}{2}, & k \text{ odd,} \\ \frac{1}{3}, & k \text{ even.} \end{cases}$$

The limit of $\sqrt[k]{a_k}$ does not exist, so the limit root test failed. However, we can try the ordinary root test. If we put $q = \frac{1}{2}$, then $q < 1$ and we also see that $\sqrt[k]{a_k} \leq q$ for all $k \geq 1$. By the root test the given series converges (also absolutely due to $a_k \geq 0$).

By the way, how would the ratio test fare? A bit of careful reasoning shows that

$$\frac{a_{k+1}}{a_k} = \begin{cases} \frac{2^k}{3^{k+1}} = \frac{1}{3} \left(\frac{2}{3} \right)^k, & k \text{ odd,} \\ \frac{3^k}{2^{k+1}} = \frac{1}{2} \left(\frac{3}{2} \right)^k, & k \text{ even.} \end{cases}$$

Since $\frac{1}{3} \left(\frac{2}{3} \right)^k \rightarrow 0$, while $\frac{1}{2} \left(\frac{3}{2} \right)^k \rightarrow \infty$, there is no limit of the expression $\frac{a_{k+1}}{a_k}$ and the limit ratio test fails. Here we cannot even use the ordinary ratio test, since the terms $\frac{a_{k+1}}{a_k}$ alternate being greater than 1 and less than 1, so none of the two variants is true. Ratio failed in any form.

3 a). First we rearrange the series in order to see its center and have it in a nice form:

$$\sum \frac{1}{k^p} (2x - 4)^k = \sum \frac{1}{k^p} [2(x - 2)]^k = \sum \frac{2^k}{k^p} (x - 2)^k.$$

The center is $x_0 = 2$. Now it's time for absolute convergence, so we need to investigate the series $\sum \left| \frac{2^k}{k^p} (x - 2)^k \right| = \sum \frac{2^k}{k^p} |x - 2|^k$. Since we have only powers there, the limit root test is probably best.

$$\sqrt[k]{a_k} = \frac{2}{[\sqrt[k]{k}]^p} |x - 2| \rightarrow \frac{2}{1^p} |x - 2| = 2|x - 2| = \rho.$$

As we can see, so far the value of the parameter p did not play any role. In order for the series with absolute value to converge we need $\rho = 2|x - 2| < 1$, which happens exactly if $|x - 2| < \frac{1}{2}$. Thus we get the radius of convergence $r = \frac{1}{2}$.

How about endpoints? They are $2 \pm \frac{1}{2}$. What if we substitute $x = \frac{5}{2}$? We get $\sum \frac{1}{k^p}$, so we see that p begins to matter. It is best to do each case separately.

Case $p = 0$, the series $\sum (2x - 4)^k$. We have $r = \frac{1}{2}$, we try endpoints.

$x = \frac{5}{2}$: $\sum 1 = \infty$, the series diverges.

$x = \frac{3}{2}$: $\sum (-1)^k$, the series diverges.

Conclusion: The series converges absolutely and converges on $(\frac{3}{2}, \frac{5}{2})$.

Case $p = 1$, the series $\sum \frac{1}{k} (2x - 4)^k$. We have $r = \frac{1}{2}$, we try endpoints.

$x = \frac{5}{2}$: $\sum \frac{1}{k} = \infty$, the series diverges (harmonic series).

$x = \frac{3}{2}$: $\sum (-1)^k \frac{1}{k}$, the series converges (this one we know, it is handled via AST).

Conclusion: The series converges absolutely on $(\frac{3}{2}, \frac{5}{2})$, converges on $[\frac{3}{2}, \frac{5}{2})$.

Case $p = 2$, the series $\sum \frac{1}{k^2} (2x - 4)^k$. We have $r = \frac{1}{2}$, we try endpoints.

$x = \frac{5}{2}$: $\sum \frac{1}{k^2}$, the series converges (we know this one, too, and if we forget, we use integral test).

$x = \frac{3}{2}$: $\sum (-1)^k \frac{1}{k^2}$, the series converges (via AST).

So the series converges for both endpoints, hence it converges absolutely at $2 \pm \frac{1}{2}$.

Conclusion: The series absolutely converges and converges on $[\frac{3}{2}, \frac{5}{2}]$.

3 b). First we rearrange the series in order to see its center and have it in a nice form:

$$\sum \frac{3^{k+1}}{k!} (2x + 1)^k = \sum \frac{3 \cdot 3^k}{k!} [2(x + \frac{1}{2})]^k = \sum \frac{3 \cdot 3^k 2^k}{k!} (x + \frac{1}{2})^k = \sum \frac{3 \cdot 6^k}{k!} (x - (-\frac{1}{2}))^k.$$

The center is $x_0 = -\frac{1}{2}$. Now it's time for absolute convergence, so we need to investigate the series $\sum \left| \frac{3 \cdot 6^k}{k!} (x - (-\frac{1}{2}))^k \right| = \sum \frac{3 \cdot 6^k}{k!} |x + \frac{1}{2}|^k$. Factorial clearly indicates that we should use the ratio test, we try the limit version:

$$\frac{a_{k+1}}{a_k} = \frac{3 \cdot 6^{k+1} / (k+1)! \cdot |x + \frac{1}{2}|^{k+1}}{3 \cdot 6^k / k! \cdot |x + \frac{1}{2}|^k} = \frac{3 \cdot 6^{k+1}}{3 \cdot 6^k} \frac{|x + \frac{1}{2}|^{k+1}}{|x + \frac{1}{2}|^k} \frac{k!}{(k+1)!} = 6 |x + \frac{1}{2}| \frac{1}{k+1} \rightarrow 0 = \lambda.$$

In order for the series with absolute value to converge we need $\lambda = 0 < 1$, which is always true. Thus we get the radius of convergence $r = \infty$ and the given series converges absolutely (and hence also converges) on \mathbb{R} .

3 c). First we rearrange the series in order to see its center and have it in a nice form:

$$\sum \frac{(kx)^k}{3^{k+1}} = \sum \frac{k^k}{3^{k+1}} x^k = \sum \frac{k^k}{3^{k+1}} (x - 0)^k.$$

The center is $x_0 = 0$. Now it's time for absolute convergence, so we need to investigate the series $\sum \left| \frac{k^k}{3^{k+1}} x^k \right| = \sum \frac{k^k}{3 \cdot 3^k} |x|^k$. Since we have only powers, probably the limit root test will be best.

$$\sqrt[k]{a_k} = \frac{k}{\sqrt[3]{3} \cdot 3} |x| = \left\langle \left\langle \frac{\infty}{1 \cdot 3} |x| \right\rangle \right\rangle \rightarrow \begin{cases} \infty, & |x| \neq 0, \\ 0, & |x| = 0. \end{cases}$$

In order for the series with absolute value to converge we need $\rho = \lim_{k \rightarrow \infty} (\sqrt[k]{a_k}) < 1$, which is true only for $|x| = 0$, that is, $x = 0$. Thus we get the radius of convergence $r = 0$.

Conclusion: The series converges absolutely and converges on the set $\{0\}$.

4. First we rearrange the series in order to see its center and have it in a nice form:

$$\begin{aligned} \sum \frac{\sqrt{k^2 + 1}}{4^{k-1}} \left(1 - \frac{x}{2}\right)^k &= \sum \frac{\sqrt{k^2 + 1}}{4^{k-1}} \left(\frac{2-x}{2}\right)^k = \sum \frac{4\sqrt{k^2 + 1}}{4^k 2^k} [-(x - 2)]^k \\ &= \sum (-1)^k \frac{4\sqrt{k^2 + 1}}{8^k} (x - 2)^k. \end{aligned}$$

The center is $x_0 = 2$. Now it's time for absolute convergence, so we need to investigate the series $\sum |(-1)^k \frac{4\sqrt{k^2 + 1}}{8^k} (x - 2)^k| = \sum \frac{4\sqrt{k^2 + 1}}{8^k} |x - 2|^k$. Here both ratio and root seem possible, but the complicated root will probably be better in a ratio:

$$\begin{aligned} \frac{a_{k+1}}{a_k} &= \frac{4\sqrt{(k+1)^2 + 1}/8^{k+1} \cdot |x - 2|^{k+1}}{4\sqrt{k^2 + 1}/8^k \cdot |x - 2|^k} = \sqrt{\frac{k^2 + 2k + 2}{k^2 + 1} \frac{8^k}{8^{k+1}} \frac{|x - 2|^{k+1}}{|x - 2|^k}} \\ &= \sqrt{\frac{1 + 2/k + 2/k^2}{1 + 1/k^2} \frac{1}{8}} |x - 2| \rightarrow \sqrt{1} \frac{1}{8} |x - 2| = \frac{1}{8} |x - 2| = \lambda. \end{aligned}$$

In order for the series with absolute value to converge we need $\lambda = \frac{1}{8} |x - 2| < 1$, which is true for $|x - 2| < 8$. Thus we get the radius of convergence $r = 8$.

Endpoints are 2 ± 8 .

$x = 10$: $\sum 4(-1)^k \sqrt{k^2 + 1}$, the series diverges, since $\sqrt{k^2 + 1} \rightarrow \infty$ and so $c_k = 4(-1)^k \sqrt{k^2 + 1} \rightarrow 0$ is not true.

$x = -6$: $\sum 4\sqrt{k^2 + 1} \geq \sum 4 = \infty$, the series diverges.

Conclusion: The series converges absolutely and converges on $(-6, 10)$.

5 a). We need to get $(x - 1)$:

$$\begin{aligned} (2x + 1)e^{4x-1} - 2 &= (2(x - 1) + 3)e^{4(x-1)+3} - 2 = 2e^3(x - 1)e^{4(x-1)} + 3e^3e^{4(x-1)} - 2 \\ &= \left\langle \text{for } y = 4(x - 1) \in \mathbb{R} \implies x \in \mathbb{R}, r = \infty \right\rangle \\ &= 2e^3(x - 1) \sum_{k=0}^{\infty} \frac{[4(x - 1)]^k}{k!} + 3e^3 \sum_{k=0}^{\infty} \frac{[4(x - 1)]^k}{k!} - 2 \\ &= \sum_{k=0}^{\infty} \frac{2e^3 4^k}{k!} (x - 1)^{k+1} + \sum_{k=0}^{\infty} \frac{3e^3 4^k}{k!} (x - 1)^k - 2 = \left\langle \begin{matrix} k + 1 \mapsto k^* \\ k \mapsto k^* - 1 \end{matrix} \right\rangle \\ &= \sum_{k=1}^{\infty} \frac{2e^3 4^{k-1}}{(k - 1)!} (x - 1)^k + \sum_{k=0}^{\infty} \frac{3e^3 4^k}{k!} (x - 1)^k - 2 \end{aligned}$$

$$\begin{aligned}
&= \sum_{k=1}^{\infty} \frac{k 2e^3 4^{k-1}}{k!} (x-1)^k + \sum_{k=0}^{\infty} \frac{3e^3 4^k}{k!} (x-1)^k - 2 \\
&= \sum_{k=1}^{\infty} \frac{k 2e^3 4^{k-1}}{k!} (x-1)^k + \left[\frac{3e^3 4^0}{0!} (x-1)^0 + \sum_{k=1}^{\infty} \frac{3e^3 4^k}{k!} (x-1)^k \right] - 2 \\
&= [3e^3 - 2] + \sum_{k=1}^{\infty} \frac{e^3 4^{k-1} (2k + 12)}{k!} (x-1)^k; \quad x \in \mathbb{R}.
\end{aligned}$$

5 b). We need to get $(x - (-1)) = (x + 1)$:

$$\begin{aligned}
(x+1) \sin\left(\frac{\pi}{2}x\right) + x &= (x+1) \sin\left(\frac{\pi}{2}(x+1-1)\right) + (x+1) - 1 \\
&= (x+1) \sin\left(\frac{\pi}{2}(x+1) - \frac{\pi}{2}\right) + (x+1) - 1 = \left\langle\left\langle \sin\left(\alpha - \frac{\pi}{2}\right) = -\cos(\alpha), \alpha = \frac{\pi}{2}(x+1) \right\rangle\right\rangle \\
&= -(x+1) \cos\left(\frac{\pi}{2}(x+1)\right) + (x+1) - 1 = \left\langle\left\langle \text{for } y = \frac{\pi}{2}(x+1) \in \mathbb{R} \implies x \in \mathbb{R}, r = \infty \right\rangle\right\rangle \\
&= -(x+1) \sum_{k=0}^{\infty} (-1)^k \frac{\left[\frac{\pi}{2}(x+1)\right]^{2k}}{(2k)!} + (x+1) - 1 \\
&= \sum_{k=0}^{\infty} (-1)^{k+1} \frac{\pi^{2k}}{2^{2k} (2k)!} (x+1)^{2k+1} + (x+1) - 1 \\
&= \left[(-1)^1 \frac{\pi^0}{2^0 \cdot 0!} (x+1)^1 + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\pi^{2k}}{2^{2k} (2k)!} (x+1)^{2k+1} \right] + (x+1) - 1 \\
&= -1 + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\pi^{2k}}{4^k (2k)!} (x+1)^{2k+1}; \quad x \in \mathbb{R}.
\end{aligned}$$

5 c). We need to get $(x - 1)$ and get rid of x in the numerator.

$$\begin{aligned}
\frac{2x+5}{x+1} &= \frac{2(x+1)+3}{x+1} = 2 + 3 \frac{1}{x+1} = 2 + 3 \frac{1}{2+(x-1)} = 2 + \frac{3}{2} \frac{1}{1+\frac{x-1}{2}} \\
&= 2 + \frac{3}{2} \frac{1}{1-\frac{-(x-1)}{2}} = \left\langle\left\langle \text{for } |y| = \left|-\frac{1}{2}(x-1)\right| < 1 \implies |x-1| < 2, r = 2 \right\rangle\right\rangle \\
&= 2 + \frac{3}{2} \sum_{k=0}^{\infty} \left(\frac{-(x-1)}{2}\right)^k = 2 + \sum_{k=0}^{\infty} \frac{3(-1)^k}{2^{k+1}} (x-1)^k \\
&= 2 + \left[\frac{3(-1)^0}{2^1} (x-1)^0 + \sum_{k=1}^{\infty} \frac{3(-1)^k}{2^{k+1}} (x-1)^k \right] \\
&= \frac{7}{2} + \sum_{k=1}^{\infty} (-1)^k \frac{3}{2^{k+1}} (x-1)^k; \quad x \in (-1, 3).
\end{aligned}$$

5 d). We need to get $(x - 2)$, but first we need to get rid of a power using a trick with derivative. By guessing or using $\int \frac{dx}{(1-x)^2} = \frac{1}{1-x}$:

$$\begin{aligned}
\frac{1}{(1-x)^2} &= \left[\frac{1}{1-x} \right]' = \left[\frac{1}{-1-(x-2)} \right]' = \left[\frac{-1}{1+(x-2)} \right]' = \left[-\frac{1}{1-[-(x-2)]} \right]' \\
&= \left\langle\left\langle \text{for } |y| = |-(x-2)| < 1 \implies |x-2| < 1, r = 1 \right\rangle\right\rangle = \left[-\sum_{k=0}^{\infty} [-(x-2)]^k \right]' \\
&= \left[\sum_{k=0}^{\infty} (-1)^{k+1} (x-2)^k \right]' = \sum_{k=1}^{\infty} k (-1)^{k+1} (x-2)^{k-1} = \left\langle\left\langle \begin{array}{l} k-1 \mapsto k^* \\ k \mapsto k^*+1 \end{array} \right\rangle\right\rangle \\
&= \sum_{k=0}^{\infty} (k+1) (-1)^{k+2} (x-2)^k = \sum_{k=0}^{\infty} (-1)^k (k+1) (x-2)^k; \quad x \in (1, 3).
\end{aligned}$$