Checklist: How to Deal with Limits

Type: Plug in and it works. **Trick:** Plug in the limit point.

Here we often use the following facts (L stands for a limit that converges, that is, exists and is finite):

— $\infty+\infty=\infty$, $\infty\pm L=\infty$,

— $\infty\cdot L=\infty$ pro $L>0$, $\infty\cdot L=-\infty$ for $L<0$, $\infty\cdot\infty=\infty$,

— $\infty^L=\infty$ for $L>0$, $L^\infty=\infty$ for $L>1$, $L^\infty=0$ for $L\in(-1,1)$, L^∞ DNE for $L<-1$,

— $\frac{L}{\infty}=0$, $\frac{L}{0^+}=\infty$, $\frac{L}{0^-}=-\infty$,

— $0^L=0$ for $L>0$, $1^L=1$, $L^0=1$ for $L>0$

— $e^\infty=\infty$, $\ln(\infty)=\infty$, $\ln(0^+)=-\infty$.

Negative exponents are tricky and it is best to handle them using $A^{-b}=\frac{1}{A^b}$.

Example: $\lim\limits_{n\to\infty}(e^{-n})=\lim\limits_{n\to\infty}\big(\frac{1}{e^n}\big)=\frac{1}{\infty}=0$, $\lim\limits_{x\to 0}\big(\frac{\sin(x)\ln(2e^x-1)}{x^3+1}\big)=\frac{0}{1}=0$.

Note: The answer to $\frac{1}{0}$ is not immediately clear, which is after all clear from the fact that $\frac{1}{0^\pm}=\pm\infty$. If we get $\frac{1}{0}$, we have to check out one-sided limits. If they yield equal answers, it will answer the limit question. If not, the limit DNE.

Indeterminate expressions: It is good to know when the substitution fails:

— $0\cdot\infty$: for instance $\lim\limits_{n\to\infty}\big(\frac{1}{n}\cdot n\big)=1$, $\lim\limits_{n\to\infty}\big(\frac{1}{n^2}\cdot n\big)=0$, $\lim\limits_{n\to\infty}\big(\frac{1}{n}\cdot n^2\big)=\infty$,

— $\frac{\infty}{\infty}$: for instance $\lim\limits_{n\to\infty}\big(\frac{n}{n}\big)=1$, $\lim\limits_{n\to\infty}\big(\frac{n}{n^2}\big)=0$, $\lim\limits_{n\to\infty}\big(\frac{n^2}{n}\big)=\infty$,

— $\frac{0}{0}$: for instance $\lim\limits_{x\to 0}\big(\frac{\sin(x)}{\sin(x)}\big)=1$, $\lim\limits_{x\to 0}\big(\frac{\sin(2x)}{\sin(x)}\big)=2$, $\lim\limits_{x\to 0^+}\big(\frac{\sin(x)}{\sin(x^2)}\big)=\infty$,

— $\infty-\infty$: for instance $\lim\limits_{n\to\infty}(n^2-n)=\infty$, $\lim\limits_{n\to\infty}\big((n+13)-n\big)=13$, $\lim\limits_{n\to\infty}(n-n^2)=-\infty$,

— ∞^0, 1^∞, 0^0: for instance $\lim\limits_{n\to\infty}\big(1+\frac{c}{n}\big)^n=e^c$ for arbitrary c.

Below we cover tricks for these expressions

Type: Geometric sequence. **Trick:** Remember that $a^n\to 0$ for $|a|<1$, $\{a^n\}$ does not converge for $|a|>1$, actually, $a^n\to\infty$ for $a>1$.

Example: $\lim\limits_{n\to\infty}\big(\frac{3^{n+1}}{2^{2n}}\big)=\lim\limits_{n\to\infty}\big(\frac{3\cdot 3^n}{(2^2)^n}\big)=3\lim\limits_{n\to\infty}\big(\frac{3^n}{4^n}\big)=3\lim\limits_{n\to\infty}\big((\frac{3}{4})^n\big)=0$.

Type: Limit with polynomials and (general) exponentials. **Trick:** Factor out the highest power.

This way one can see (and prove) that a polynomials behaves at infinity exactly as its leading (highest) power, which is handy for instance when it comes to fractions.

Example: $\lim\limits_{x\to\infty}\big(\frac{x^2-1}{x+2}\big)=\lim\limits_{x\to\infty}\left(\dfrac{x^2(1-\frac{1}{x^2})}{x(1+\frac{2}{x})}\right)=\lim\limits_{x\to\infty}\left(x\dfrac{1-\frac{1}{x^2}}{1+\frac{2}{x}}\right)=\infty\frac{1-0}{1+0}=\infty\cdot 1=\infty$.

Sometimes one can directly cancel: $\lim\limits_{x\to -1}\big(\frac{x^2+3x+2}{x+1}\big)=\lim\limits_{x\to -1}\big(\frac{(x+1)(x+2)}{x+1}\big)=\lim\limits_{x\to -1}(x+2)=1$.

Example: $\lim\limits_{n\to\infty}\big(\frac{(-3)^n+2^{2n+1}}{e^n-2^n}\big)=\lim\limits_{n\to\infty}\big(\frac{(-3)^n+2\cdot 4^n}{e^n-2^n}\big)=\lim\limits_{n\to\infty}\left(\dfrac{4^n\big((-\frac{3}{4})^n+2\big)}{e^n\big(1-(\frac{2}{e})^n\big)}\right)$

$$=\lim_{n\to\infty}\left(\left(\frac{4}{e}\right)^n\frac{\big(-\frac{3}{4}\big)^n+2}{1-\big(\frac{2}{e}\big)^n}\right)=\infty\frac{0+2}{1-0}=\infty.$$

Here we used the knowledge of geometric series and the facts that $\left|\frac{2}{e}\right|<1$, $\left|-\frac{3}{4}\right|<1$, and $\frac{4}{e}>1$.

Factoring out is often used also with roots, as in $\sqrt{x^2+x}=\sqrt{x^2(1+1/x)}=\sqrt{x^2}\sqrt{1+1/x}$. If $x>0$, we can go on: $\sqrt{x^2}\sqrt{1+1/x}=|x|\sqrt{1+1/x}=x\sqrt{1+1/x}$.

Example (note that if $x\to\infty$, then $x>0$):

$$\lim_{x\to\infty}\left(\frac{x-1}{\sqrt[3]{x^3+x+1}+x}\right)=\lim_{x\to\infty}\left(\frac{x(1-1/x)}{\sqrt[3]{x^3}\sqrt[3]{1+\frac{1}{x^2}+1x^3}+x}\right)=\lim_{x\to\infty}\left(\frac{x(1-1/x)}{x(\sqrt[3]{1+\frac{1}{x^2}+\frac{1}{x^3}}+1)}\right)$$

$$=\lim_{x\to\infty}\left(\frac{1-1/x}{\sqrt[3]{1+\frac{1}{x^2}+\frac{1}{x^3}}+1}\right)=\frac{1-0}{\sqrt[3]{1+0+0}+1}=\frac{1}{2}.$$

Type: Expression hidden in a nice function. **Trick:** Move the limit inside.

Example: $\lim\bigl(\sin\sqrt{\frac{e^x-\ln(x)}{1+x^2}}\bigr)=\sin\sqrt{\lim\bigl(\frac{e^x-\ln(x)}{1+x^2}\bigr)}$, tu limitu uvnitř udělám jednodušeji než limitu celé původní funkce.

Type: Limit with an expression that we would like to simplify (most likely appearing at more places).

Trick: Substitution.

Note: the limit must change entirely, that is, when using a substitution $y=g(x)$, all x must disappear from the limit (including down under lim); this replacement is done using the substitution equation $y=g(x)$.

Example:

$$\lim_{x\to 1^+}\Bigl(x\frac{(1+\frac{1}{x})\ln(1+\frac{1}{x})}{\cos(\pi+\frac{\pi}{x})}\Bigr)=\left|\begin{matrix} y=1+\frac{1}{x} \implies x=\frac{1}{y-1} \\ x\to 1^+ \implies y\to 2^- \end{matrix}\right|=\lim_{y\to 2^-}\bigl(\tfrac{1}{y-1}\,\tfrac{y\ln(y)}{\cos(\pi y)}\bigr)=\tfrac{1}{1}\,\tfrac{2\ln(2)}{1}=2\ln(2).$$

Trick: $\lim\limits_{x\to-\infty} f(x)$, we don't want negative infinity, use $y=-x$, so $x=-y$ and we get $\lim\limits_{y\to\infty} f(-y)$.

Type: Limit that we remember. **Trick:** Search your memory, ocassionally change the expression algebraically.

Some limits that are worth remembering: $\lim\limits_{n\to\infty}\bigl((1+\frac{c}{n})^n\bigr)=e^c$, $\lim\limits_{x\to 0}\bigl(\frac{\sin(x)}{x}\bigr)=1$, $\lim\limits_{x\to 0^+}\bigl(x\ln(x)\bigr)=0$, $\lim\limits_{x\to\infty}\bigl(\frac{\ln(x)}{x}\bigr)=0$, $\lim\limits_{x\to\infty}\bigl(\frac{x}{e^x}\bigr)=0$, also $\lim\limits_{x\to 0}\bigl(\frac{e^x-1}{x}\bigr)=1$, $\lim\limits_{x\to 0}\bigl(\frac{\ln(x+1)}{x}\bigr)=1$, $\lim\limits_{x\to 0}\bigl(\frac{\cos(x)-1}{x}\bigr)=0$.

These work only exactly as written, for instance in the problem $\lim\limits_{x\to 0}\bigl(\frac{\sin(2x)}{x}\bigr)$ the substitution $y=2x$ must be used, yielding $\lim\limits_{y\to 0}\bigl(2\frac{\sin(y)}{y}\bigr)=2$.

Example: $\lim\limits_{x\to 0}\bigl(\frac{\tan(x^2)}{x}\bigr)=\lim\limits_{x\to 0}\bigl(\frac{x}{\cos(x)}\,\frac{\sin(x^2)}{x^2}\bigr)$, the nice part $\lim\limits_{x\to 0}\bigl(\frac{x}{\cos(x)}\bigr)$ is done directly, the part $\lim\limits_{x\to 0}\bigl(\frac{\sin(x^2)}{x^2}\bigr)$ is done using the substitution $y=x^2$ and the results are multiplied.

Type: $\sqrt{A}-\sqrt{B}$ and it bothers me. **Trick:** Multiply and divide by the expression $\sqrt{A}+\sqrt{B}$, use $(\sqrt{A}-\sqrt{B})(\sqrt{A}+\sqrt{B})=A-B$.

Similar trick can be used for $\sqrt[3]{A}-\sqrt[3]{B}$, multiply and divide by $\bigl(\sqrt[3]{A}\bigr)^2+\sqrt[3]{A}\sqrt[3]{B}+\bigl(\sqrt[3]{B}\bigr)^2$.

Example:

$$\lim_{x\to 0}\Bigl(\frac{\sqrt{1+x}-\sqrt{1-x}}{x}\Bigr)=\lim_{x\to 0}\Bigl(\frac{(1+x)-(1-x)}{x\bigl(\sqrt{1+x}+\sqrt{1-x}\bigr)}\Bigr)=\lim_{x\to 0}\Bigl(\frac{2x}{x\bigl(\sqrt{1+x}+\sqrt{1-x}\bigr)}\Bigr)=\lim_{x\to 0}\Bigl(\frac{2}{\sqrt{1+x}+\sqrt{1-x}}\Bigr)=1.$$

Type: $\frac{\infty}{\infty}$, $\frac{0}{0}$. **Trick:** L'Hôpital.

Remember that $\lim\limits_{x\to A}\bigl(\frac{f}{g}\bigr)=\lim\limits_{x\to A}\bigl(\frac{f'}{g'}\bigr)$, but only for the types $\frac{0}{0}$, $\frac{\infty}{\infty}$ (in general $\frac{*}{\infty}$), and only if the right hand-side exists.

Example: $\lim\limits_{x\to\infty}\bigl(\frac{x^2}{e^{3x}}\bigr)=\lim\limits_{x\to\infty}\bigl(\frac{2x}{3e^{3x}}\bigr)=\lim\limits_{x\to\infty}\bigl(\frac{2}{9e^{3x}}\bigr)=\frac{2}{\infty}=0$.

Sometimes one can solve the problem by cancelling (and often it is shorter that way), see the *Type: polynomials*, with the type $\sqrt{A}-\sqrt{B}$ the corresponding trick tends to be much shorter than L'Hôpital.

Type: $0\cdot\infty$. **Trick:** L'Hôpital, first the multiplication has to be written as a fraction.

Example:

Trick $0\cdot\infty=\dfrac{1}{\frac{1}{0}}\cdot\infty=\frac{\infty}{\infty}$: $\lim\limits_{x\to 0^+}\bigl(x\ln(x)\bigr)=\lim\limits_{x\to 0^+}\bigl(\frac{\ln(x)}{x^{-1}}\bigr)=\lim\limits_{x\to 0^+}\Bigl(\dfrac{\frac{1}{x}}{-x^{-2}}\Bigr)=\lim\limits_{x\to 0^+}(-x)=0$.

Trick $0\cdot\infty=0\cdot\dfrac{1}{\frac{1}{\infty}}=\frac{0}{0}$: $\lim\limits_{x\to\infty}\bigl(x\sin\bigl(\frac{1}{x}\bigr)\bigr)=\lim\limits_{x\to\infty}\Bigl(\dfrac{\sin\bigl(\frac{1}{x}\bigr)}{\frac{1}{x}}\Bigr)$; this is $\frac{0}{0}$, but a solution by L'H would be rather long (derivative of a composition, the chain rule), it is better to use substitution $y=\frac{1}{x}$, get $\lim\limits_{y\to 0^+}\bigl(\frac{\sin(y)}{y}\bigr)=1$ (which you either remember or use L'Hôpital).

Type: $\infty - \infty$. **Trick:** We hope that it is the *type* $\sqrt{A} - \sqrt{B}$ (see the appropriate trick) or that a common denominator can be made somehow naturally. In general one can use $A - B = A(1 - \frac{B}{A})$, where $\frac{B}{A}$ is then of the *type* $\frac{\infty}{\infty}$ and L'Hôpital does it.

Example: $\lim\limits_{x\to\infty}\big(\sqrt{x} - \ln(x)\big) = \lim\limits_{x\to\infty}\Big(\sqrt{x}\big(1 - \frac{\ln(x)}{\sqrt{x}}\big)\Big) = \infty\big(1-0\big) = \infty.$

In emergency one can use the trick $A - B = \dfrac{1}{\frac{1}{A}} - \dfrac{1}{\frac{1}{B}} = \dfrac{\frac{1}{B} - \frac{1}{A}}{\frac{1}{AB}}$, which is the *type* $\frac{0}{0}$ and L'Hôpital might do it.

Type: General powers. **Trick:** $A^B = e^{B\ln(A)}$. Use this whenever you are not sure about some power. The apply the trick for the *Type: expression inside a nice function* to "pull" e out.

Example:

$\lim\limits_{x\to 0^+}(x^x)$ is of the type 0^0. $\lim\limits_{x\to 0^+}(x^x) = \lim\limits_{x\to 0^+}\big(e^{x\ln(x)}\big) = e^{\lim\limits_{x\to 0^+}\left(x\ln(x)\right)} = e^0 = 1$, for $\lim\limits_{x\to 0^+}\big(x\ln(x)\big)$ use *Type* $0\cdot\infty$: change into a ratio and L'Hôpital.

Type: Limit with oscillating expressions of the type $(-1)^n$, $\sin(\infty)$, $\cos(\infty)$. **Trick:** The Squeeze theorem, sometimes it helps to use the Theorem: Bounded times zero-limit gives zero-limit.

Example: $\lim\limits_{n\to\infty}\big(\frac{n+(-1)^n}{n}\big)$, we squeeze: $\frac{n-1}{n} \le \frac{n+(-1)^n}{n} \le \frac{n+1}{n}$. Since $\frac{n\pm 1}{n} \to 1$ (see polynomials), necessarily $\lim\limits_{n\to\infty}\big(\frac{n+(-1)^n}{n}\big) = 1$.

Example: $\lim\limits_{x\to\infty}\big(\frac{\sin(x)}{\sqrt{x}}\big) = \lim\limits_{x\to\infty}\big(\frac{1}{\sqrt{x}}\sin(x)\big)$. Since $\frac{1}{\sqrt{x}} \to 0$ and $\sin(x)$ is bounded, $\lim\limits_{x\to\infty}\frac{\sin(x)}{\sqrt{x}} = 0$.

A short note on the omnipotence of L'Hôpital: Ha!

L'Hospital's rule cannot remove exponentials: $\lim\limits_{x\to\infty}\big(\frac{e^x - e^{-x}}{e^x + e^{-x}}\big) = \text{l'H} = \lim\limits_{x\to\infty}\big(\frac{e^x + e^{-x}}{e^x - e^{-x}}\big) = \text{l'H} = \lim\limits_{x\to\infty}\big(\frac{e^x - e^{-x}}{e^x + e^{-x}}\big)$.

Here it is best to cancel: $\lim\limits_{x\to\infty}\big(\frac{e^x - e^{-x}}{e^x + e^{-x}}\big) = \lim\limits_{x\to\infty}\big(\frac{1 - e^{-2x}}{1 + e^{-2x}}\big) = 1.$

L'Hospital's rule can't remove roots:

$$\lim_{x\to 1^-}\left(\frac{\sqrt{1-x^2}}{\sqrt{1-x^3}}\right) = \text{l'H} = \lim_{x\to 1^-}\left(\frac{\frac{-2x}{2\sqrt{1-x^2}}}{\frac{-3x^2}{2\sqrt{1-x^3}}}\right) = \lim_{x\to 1^-}\big(\tfrac{2}{3x}\big)\lim_{x\to 1^-}\left(\frac{\sqrt{1-x^3}}{\sqrt{1-x^2}}\right) = \frac{2}{3}\lim_{x\to 1^-}\left(\frac{\sqrt{1-x^3}}{\sqrt{1-x^2}}\right)$$

$$= \text{l'H} = \frac{2}{3}\lim_{x\to 1^-}\left(\frac{\frac{-3x^2}{2\sqrt{1-x^3}}}{\frac{-2x}{2\sqrt{1-x^2}}}\right) = \frac{2}{3}\lim_{x\to 1^-}\big(\tfrac{3x}{2}\big)\lim_{x\to 1^-}\left(\frac{\sqrt{1-x^2}}{\sqrt{1-x^3}}\right) = \lim_{x\to 1^-}\left(\frac{\sqrt{1-x^2}}{\sqrt{1-x^3}}\right).$$

Again, here the best way is to cancel: $\lim\limits_{x\to 1^-}\left(\dfrac{\sqrt{1-x^2}}{\sqrt{1-x^3}}\right) = \lim\limits_{x\to 1^-}\left(\sqrt{\frac{1-x^2}{1-x^3}}\right)$

$$= \sqrt{\lim_{x\to 1^-}\big(\tfrac{1-x^2}{1-x^3}\big)} = \sqrt{\lim_{x\to 1^-}\big(\tfrac{(1-x)(1+x)}{(1-x)(1+x+x^2)}\big)} = \sqrt{\lim_{x\to 1^-}\big(\tfrac{1+x}{1+x+x^2}\big)} = \sqrt{\tfrac{2}{3}}.$$

Bonus: Sometimes it is good to know how limits that do not exist (DNE) work together. We will use N for a limit that does not exist, for instance $\lim\limits_{n\to\infty}(-1)^n$, $\lim\limits_{n\to\infty}\cos(n)$, $\lim\limits_{x\to 0}\frac{1}{x}$, or $\lim\limits_{x\to 0}\sin(\frac{1}{x})$.

Rules:

$L\cdot N = N$ for $L \ne 0$, $L + N = N$, $N/L = N$, $N^\alpha = N$ for $\alpha > 0$.

Indeterminate expressions with N are not as useful as those we had above (most examples are based on $\lim\limits_{x\to 0}\frac{1}{x}$, which DNE):

— $0\cdot N$: for instance $\lim\limits_{x\to 0}\big(x^2\frac{1}{x}\big) = 0$, $\lim\limits_{x\to 0}\big(x\frac{1}{x}\big) = 1$, $\lim\limits_{x\to 0}\big(\sqrt[3]{x}\frac{1}{x}\big) = \infty$ (it is $\frac{1}{0^+}$).

— $\frac{N}{N}$: for instance $\lim\limits_{x\to 0}\big(\frac{1/x}{1/x}\big) = 1$, $\lim\limits_{x\to 0}\big(\frac{1/x}{1/x^3}\big) = 0$, $\lim\limits_{x\to 0}\big(\frac{1/x^3}{1/x}\big) = \infty$, $\lim\limits_{n\to\infty}\big(\frac{\sin(n)}{\cos(n)}\big)$ is N.

— $N \pm N$: for instance $\lim\limits_{x\to 0}\big(\frac{1}{x} - \frac{1}{x}\big) = 0$, $\lim\limits_{x\to 0}\big(\frac{2}{x} - \frac{1}{x}\big)$ is N.

— N^α: for instance $\lim\limits_{x\to 0}\big(\frac{1}{x}\big)^3$ is N, but $\lim\limits_{x\to 0}\big(\frac{1}{x}\big)^2 = \infty$.