MA2: Practice problems—Derivatives, geometry
Brief solutions

1. $\frac{\partial f}{\partial x} = (4x+2y)e^{2x^2+y^2+2xy+2y}$, $\qquad \frac{\partial f}{\partial y} = (2y+2x+2)e^{2x^2+y^2+2xy+2y}$.

2. $\frac{\partial f}{\partial x} = \frac{(x+y^2)-x}{(x+y^2)^2} = \frac{y^2}{(x+y^2)^2}$, $\qquad \frac{\partial f}{\partial y} = -\frac{2xy}{(x+y^2)^2}$.

3. $\frac{\partial f}{\partial x} = \cos(x^3+z)3x^2\ln(z) + 2xy^2z$, $\qquad \frac{\partial f}{\partial y} = x^2 2yz$, $\qquad \frac{\partial f}{\partial z} = \cos(x^3+z)\ln(z) + \sin(x^3+z)\frac{1}{z} + x^2y^2$.

4. $\frac{\partial f}{\partial x} = 2x\frac{e^{5y+3z}}{\sin(z)} + y\,x^{y-1}$, $\qquad \frac{\partial f}{\partial y} = 5e^{5y+3z}\frac{x^2}{\sin(z)} + \ln(x)x^y$, $\qquad \frac{\partial f}{\partial z} = x^2\frac{3e^{5y+3z}\sin(z)-e^{5y+3z}\cos(z)}{\sin^2(z)}$.

5. $\frac{\partial f}{\partial x} = (2x+y)\cos(x^2+xy)$, $\qquad \frac{\partial f}{\partial y} = x\,\cos(x^2+xy)$;
$\frac{\partial^2 f}{\partial x^2} = 2\cos(x^2+xy) - (2x+y)^2\sin(x^2+xy)$, $\qquad \frac{\partial^2 f}{\partial x\partial y} = \cos(x^2+xy) - x(2x+y)\sin(x^2+xy)$,
$\frac{\partial^2 f}{\partial y^2} = -x^2\,\sin(x^2+xy)$.

6. $\frac{\partial f}{\partial x} = \sqrt{y+2z}$, $\qquad \frac{\partial f}{\partial y} = \frac{x}{2\sqrt{y+2z}}$, $\qquad \frac{\partial f}{\partial z} = \frac{x}{\sqrt{y+2z}}$;
$\frac{\partial^2 f}{\partial x^2} = 0$, $\qquad \frac{\partial^2 f}{\partial x\partial y} = \frac{\partial^2 f}{\partial y\partial x} = \frac{1}{2\sqrt{y+2z}}$, $\qquad \frac{\partial^2 f}{\partial x\partial z} = \frac{\partial^2 f}{\partial z\partial x} = \frac{1}{\sqrt{y+2z}}$, $\qquad \frac{\partial^2 f}{\partial y^2} = \frac{-x}{4[\sqrt{y+2z}]^3}$,
$\frac{\partial^2 f}{\partial y\partial z} = \frac{\partial^2 f}{\partial z\partial y} = \frac{-x}{2[\sqrt{y+2z}]^3}$, $\qquad \frac{\partial^2 f}{\partial z^2} = \frac{-x}{[\sqrt{y+2z}]^3}$.
Derivatives are simpler with $\frac{1}{\sqrt{y+2z}} = (y+2z)^{-1/2}$.

7. $\frac{\partial f}{\partial x} = \ln(xy+1) + \frac{xy}{xy+1}$, $\qquad \frac{\partial f}{\partial y} = \frac{x^2}{xy+1}$;
therefore $\nabla f(1,0) = (0,1)$ and $D_{\vec{u}}f(1,0) = \nabla f(1,0) \bullet \vec{u} = \frac{1}{\sqrt{5}}$.

8. a) We need the direction of the maximal descent, which is $-\nabla f(1,2)$. We have
$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \left(\frac{-6x}{(3x^2+y^2+1)^2}, \frac{-2y}{(3x^2+y^2+1)^2}\right)$. Thus $-\nabla f(1,2) = \left(\frac{3}{32}, \frac{2}{32}\right)$, we can take $\vec{d} = (3,2)$.
b) We need the directional derivative $D_{\vec{u}}f(1,2)$, where $\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{5}(-3,4)$, therefore
$D_{\vec{u}}f(1,2) = \nabla f(1,2) \bullet \vec{u} = \frac{1}{32}(3,2) \bullet \frac{1}{5}(-3,4) = \frac{-1}{160}$. The ground goes down in this direction.

9. There are several possible ways to approach this problem.
1) One can place the rectangle so that the upper left corner is at the origin, then the lower right corner is at $(10,-5)$. The area is given by $A(x,y) = -xy$. We want to know the rate of change of A when the point $(10,-5)$ changes in direction (and magnitude) $\vec{v} = (2,2)$.
$\nabla A = (-y,-x) \implies \nabla A(10,-5) = (5,-10)$, therefore $D_{\vec{v}}A(10,-5) = -10$. The area starts getting smaller at the rate 10 cm^2/sec.
2) We may simply consider the area $A(x,y) = xy$ and consider the case when $x = x(t)$, $y = y(t)$ depend on time. We differentiate with respect to t: $A'(x,y) = \frac{\partial A}{\partial x}x'(t) + \frac{\partial A}{\partial y}y'(t) = y\,x'(t) + x\,y'(t)$.
We have $x = 10$, $y = 5$ and the given data are $x' = 2$, $y' = -2$. Thus $A'(10,5) = 5\cdot 2 + 10\cdot(-2) = -10$.
We can also use the total differential, $\mathrm{d}A(10,5) = 5\mathrm{d}x + 10\mathrm{d}y$.

10. We interpret it as a level curve problem for $f(x,y) = \frac{x^2}{4} + y^2 = 1$. We check that the given point P satisfies $f(x,y) = 1$, so it indeed lies on this curve.
$\frac{\partial f}{\partial x} = \frac{1}{2}x$, $\frac{\partial f}{\partial y} = 2y$, therefore $\nabla f\left(\sqrt{3}, -\frac{1}{2}\right) = \left(\frac{1}{2}\sqrt{3}, -1\right)$.
This vector is perpendicular (normal) to the curve, hence also to the tangent line. Thus its equation is
$\nabla f(P) \bullet ((x,y)-P) = 0 \implies \frac{1}{2}\sqrt{3}\left(x-\sqrt{3}\right) - \left(y+\frac{1}{2}\right) = 0 \implies y = \frac{1}{2}\sqrt{3}x - 1$.
To find the normal line, we can use $\nabla f(P)$ as its directional vector, obtaining parametric equations
$x = \sqrt{3} + \frac{1}{2}\sqrt{3}\,t$, $y = -\frac{1}{2}t$. To get classical equations we elliminate t, obtaining $x + \frac{1}{2}\sqrt{3}y = \frac{3}{4}\sqrt{3}$.
One can also find a vector perpendicular to $\nabla f(P)$, for instance vector $\left(1, \frac{1}{2}\sqrt{3}\right)$, and find the equation of the normal line using $\left(1, \frac{1}{2}\sqrt{3}\right) \bullet ((x,y)-P) = 0$, again we end up with $x + \frac{1}{2}\sqrt{3}y = \frac{3}{4}\sqrt{3}$.

11. We interpret it as a level curve problem, $f(x,y,z) = \frac{(x-1)^2}{2} + \frac{y^2}{3} + \frac{z^2}{6} = 1$. We check that the given point P satisfies $f(x,y,z) = 1$, so it indeed lies on this curve.
$\frac{\partial f}{\partial x} = x-1$, $\frac{\partial f}{\partial y} = \frac{2}{3}y$, $\frac{\partial f}{\partial y} = \frac{1}{3}z$, therefore $\nabla f(0,1,-1) = \left(-1, \frac{2}{3}, -\frac{1}{3}\right)$.
This vector is perpendicular (normal) to the curve, hence also to the tangent plane. Thus its equation is
$\nabla f(P) \bullet ((x,y,z)-P) = 0 \implies -x + \frac{2}{3}(y-1) - \frac{1}{3}(z+1) = 0 \implies -x + \frac{2}{3}y - \frac{1}{3}z = 1 \implies 3x - 2y + z + 3 = 0$.

To find the normal line, we can use $\nabla f(P)$ as the directional vector, obtaining parametric equations $x = -t$, $y = 1 + \frac{2}{3}t$, $z = -1 - \frac{1}{3}t$.

12. We need a normal vector. Two possibilities.
1) Level curve approach: $F(x, y, z) = x^2 + y^2 - z = 0$, $\nabla F = (2x, 2y, -1)$, hence a normal vector is $\nabla F(1, 2, 5) = (2, 4, -1)$. From this we get the tangent plane using
$\nabla f(P) \bullet ((x, y, z) - P) = 0 \implies 2(x - 1) + 4(y - 2) - (z - 5) = 0 \implies 2x + 4y - z = 5.$
2) Graph of $f(x, y)$ approach: Theory says that the vector $\bigl(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1\bigr)$ is normal to the graph, it leads to $(2, 4, -1)$ again.