MA2: Practice problems—Extrema
Brief solutions

1. Stationary points:

$$\left.\begin{aligned}\frac{\partial f}{\partial x}&=e^{2x^2+y^2+2xy+2y}(4x+2y)=0\\ \frac{\partial f}{\partial y}&=e^{2x^2+y^2+2xy+2y}(2y+2x+2)=0\end{aligned}\right\}\implies \left.\begin{aligned}2x+y&=0\\ y+x&=-1\end{aligned}\right\}\implies (1,-2).$$

Classification: $\frac{\partial^2 f}{\partial x^2}\Big|_{x=1,\ y=-2}=\big[e^{2x^2+y^2+2xy+2y}(4x+2y)^2+4e^{2x^2+y^2+2xy+2y}\big]\Big|_{x=1,y=-2}=4,$

$\frac{\partial^2 f}{\partial x\partial y}\Big|_{x=1,y=-2}=\big[e^{2x^2+y^2+2xy+2y}(2y+2x+2)(4x+2y)+2e^{2x^2+y^2+2xy+2y}\big]\Big|_{x=1,\ y=-2}=2,$

$\frac{\partial^2 f}{\partial y^2}\Big|_{x=1,y=-2}=\big[e^{2x^2+y^2+2xy+2y}(2y+2x+2)^2+2e^{2x^2+y^2+2xy+2y}\big]\Big|_{x=1,y=-2}=2\implies H=\begin{pmatrix}4&2\\2&2\end{pmatrix}.$

$\Delta_1=h_{11}=4>0$, $\Delta_2=|H|=4>0$, therefore $f(1,-2)=e^{-2}$ is a local minimum.

2. Stationary points:

$$\left.\begin{aligned}\frac{\partial f}{\partial x}&=3x^2-3=0\\ \frac{\partial f}{\partial y}&=3y^2-12=0\end{aligned}\right\}\implies \left.\begin{aligned}x^2&=1\\ y^2&=4\end{aligned}\right\}\implies (\pm1,\pm2).$$

Classification: $\frac{\partial^2 f}{\partial x^2}=6x$, $\frac{\partial^2 f}{\partial x\partial y}=0$, $\frac{\partial^2 f}{\partial y^2}=6y\implies H=\begin{pmatrix}6x&0\\0&6y\end{pmatrix}\implies \Delta_1=6x$, $\Delta_2=36xy$.

$(1,2)$: $\Delta_1=6>0$, $\Delta_2=72>0$, therefore $f(1,2)=-18$ is a local minimum.
$(1,-2)$: $\Delta_1=6>0$, $\Delta_2=-72<0$, therefore $f(1,-2)=14$ is a saddle.
$(-1,2)$: $\Delta_1=-6<0$, $\Delta_2=-72<0$, therefore $f(-1,2)=-14$ is a saddle.
$(-1,-2)$: $\Delta_1=-6<0$, $\Delta_2=72>0$, therefore $f(-1,-2)=18$ is a local maximum.

3. Stationary points:

$$\left.\begin{aligned}\frac{\partial f}{\partial x}&=2x+y^2-16=0\\ \frac{\partial f}{\partial y}&=2xy=0\end{aligned}\right\}\implies \left.\begin{cases}x=0\implies y=\pm4\\ y=0\implies x=8\end{cases}\right\}\implies (0,\pm4),(8,0).$$

Classification: $\frac{\partial^2 f}{\partial x^2}=2$, $\frac{\partial^2 f}{\partial x\partial y}=2y$, $\frac{\partial^2 f}{\partial y^2}=2x\implies H=\begin{pmatrix}2&2y\\2y&2x\end{pmatrix}\implies \Delta_1=2$, $\Delta_2=4x-4y^2$.

$(0,-4)$: $\Delta_1=2>0$, $\Delta_2=-64<0$, therefore $f(0,-4)=0$ is a saddle.
$(0,4)$: $\Delta_1=2>0$, $\Delta_2=-64<0$, therefore $f(0,4)=0$ is a saddle.
$(8,0)$: $\Delta_1=2>0$, $\Delta_2=32>0$, therefore $f(8,0)=-64$ is a local minimum.

4. Stationary points:

$$\left.\begin{aligned}\frac{\partial f}{\partial x}&=-\frac{y^2}{x^2}+1=0\\ \frac{\partial f}{\partial y}&=\frac{2y}{x}-\frac{2z^2}{y^2}=0\\ \frac{\partial f}{\partial z}&=\frac{4z}{y}-\frac{4}{z^2}=0\end{aligned}\right\}\implies y=z^3\implies \left.\begin{aligned}\frac{z^6}{x^2}&=1\\ \frac{2z^3}{x}&=\frac{2}{z^4}\end{aligned}\right\}\implies \left.\begin{aligned}x^2&=z^6\\ x&=z^7\end{aligned}\right\}\implies z^6=z^{14}\implies z=0,z=\pm1.$$

$z=0\implies x=0,y=0$ but this is not in the domain, $z=1\implies x=1,y=1$,
$z=-1\implies x=-1,y=-1$. Two stationary points, $(1,1,1)$ and $(-1,-1,-1)$.

Classification: $\frac{\partial^2 f}{\partial x^2}=\frac{2y^2}{x^3}$, $\frac{\partial^2 f}{\partial x\partial y}=-\frac{2y}{x^2}$, $\frac{\partial^2 f}{\partial x\partial z}=0$, $\frac{\partial^2 f}{\partial y^2}=\frac{2}{x}+\frac{4z^2}{y^3}$, $\frac{\partial^2 f}{\partial y\partial z}=-\frac{4z}{y^2}$, $\frac{\partial^2 f}{\partial z^2}=\frac{4}{y}+\frac{8}{z^3}$.

$$H(1,1,1)=\begin{pmatrix}2&-2&0\\-2&6&-4\\0&-4&12\end{pmatrix}\implies \Delta_1=2>0,\ \Delta_2=\begin{vmatrix}2&-2\\-2&6\end{vmatrix}=8>0,\ \Delta_3=64>0.$$

Therefore $f(1,1,1)=7$ is a local minimum.

$$H(-1,-1,-1)=\begin{pmatrix}-2&2&0\\2&-6&4\\0&4&-12\end{pmatrix}\implies \Delta_1=-2<0,\ \Delta_2=\begin{vmatrix}-2&2\\2&-6\end{vmatrix}=8>0,\ \Delta_3=-64<0.$$

Therefore $f(-1,-1,-1)=-7$ is a local maximum.

5. Stationary points:

$$\left.\begin{aligned}\frac{\partial f}{\partial x}&=3x^2-3=0\\ \frac{\partial f}{\partial y}&=3y^2+3=0\end{aligned}\right\}\text{ no solution, no local extrema.}$$

6. Stationary points:

$$\left.\begin{aligned}\frac{\partial f}{\partial x}&=3x^2+6xy^2-27=0\\ \frac{\partial f}{\partial y}&=6x^2y-6y=0\end{aligned}\right\}\implies \left.\begin{aligned}x^2+2xy^2&=9\\ y(x^2-1)&=0\end{aligned}\right\}\implies \begin{cases}y=0\implies x^2=9\implies x=\pm3\\ x=1\implies 2y^2=8\implies y=\pm2\\ x=-1\implies -2y^2=8\implies\emptyset\end{cases}$$

Stationary points $(\pm 3,0)$, $(1,\pm 2)$.

Classification: $\frac{\partial^2 f}{\partial x^2}=6x+6y^2$, $\frac{\partial^2 f}{\partial x\partial y}=12xy$, $\frac{\partial^2 f}{\partial y^2}=6x^2-6 \implies H=\begin{pmatrix}6x+6y^2 & 12xy\\ 12xy & 6x^2-6\end{pmatrix}$.

$(3,0)$: $H=\begin{pmatrix}18 & 0\\ 0 & 48\end{pmatrix}$. $\Delta_1=18>0$, $\Delta_2=18\cdot 48>0$, therefore $f(3,0)=-54$ is a local minimum.

$(-3,0)$: $H=\begin{pmatrix}-18 & 0\\ 0 & 48\end{pmatrix}$. $\Delta_1=-18<0$, $\Delta_2=-18\cdot 48<0$, therefore $f(-3,0)=54$ is a saddle.

$(1,2)$: $H=\begin{pmatrix}30 & 24\\ 24 & 0\end{pmatrix}$. $\Delta_1=30>0$, $\Delta_2=-24^2<0$, therefore $f(1,2)=-26$ is a saddle.

$(1,-2)$: $H=\begin{pmatrix}30 & -24\\ -24 & 0\end{pmatrix}$. $\Delta_1=30>0$, $\Delta_2=-24^2<0$, therefore $f(1,-2)=-26$ is a saddle.

7. Stationary points:

$$\left.\begin{aligned}\frac{\partial f}{\partial x}&=3x^2-3y=0\\ \frac{\partial f}{\partial y}&=3y^2-3x=0\end{aligned}\right\} \implies \left.\begin{aligned}x^2&=y\\ y^2&=x\end{aligned}\right\} \implies x^4=x \implies \begin{cases}x=0 \implies y=0\\ x=1 \implies y=1\end{cases} \implies (0,0),(1,1).$$

Classification: $\frac{\partial^2 f}{\partial x^2}=6x$, $\frac{\partial^2 f}{\partial x\partial y}=-3$, $\frac{\partial^2 f}{\partial y^2}=6y \implies H=\begin{pmatrix}6x & -3\\ -3 & 6y\end{pmatrix} \implies \Delta_1=6x$, $\Delta_2=36xy-9$.

$(0,0)$: $\Delta_1=0$, $\Delta_2=-9<0$, therefore $f(0,0)=0$ is a saddle.

$(1,1)$: $\Delta_1=6>0$, $\Delta_2=27>0$, therefore $f(1,1)=-1$ is a local minimum.

8. Stationary points:

$$\left.\begin{aligned}\frac{\partial f}{\partial x}&=4x^3-4x=0\\ \frac{\partial f}{\partial y}&=2y=0\end{aligned}\right\} \implies x=0,\ x=\pm 1,\ y=1, \text{ independent equations} \implies (0,0),(1,0),(-1,0).$$

Classification: $\frac{\partial^2 f}{\partial x^2}=12x-4$, $\frac{\partial^2 f}{\partial x\partial y}=0$, $\frac{\partial^2 f}{\partial y^2}=2 \implies H=\begin{pmatrix}12x-4 & 0\\ 0 & 2\end{pmatrix} \implies \Delta_1=12x-4$, $\Delta_2=24x-8$.

$(0,0)$: $\Delta_1=-4<0$, $\Delta_2=-8<0$, therefore $f(0,0)=-2$ is a saddle.

$(1,0)$: $\Delta_1=8>0$, $\Delta_2=16>0$, therefore $f(1,0)=-3$ is a local minimum.

$(-1,0)$: $\Delta_1=-16<0$, $\Delta_2=-32<0$, therefore $f(-1,0)=-3$ is a saddle.

9. Stationary points:

$$\left.\begin{aligned}\frac{\partial f}{\partial x}&=2x+y+3=0\\ \frac{\partial f}{\partial y}&=x+2y=0\end{aligned}\right\} \implies -3y+3=0 \implies y=1, x=-2.$$

Classification: $\frac{\partial^2 f}{\partial x^2}=2$, $\frac{\partial^2 f}{\partial x\partial y}=1$, $\frac{\partial^2 f}{\partial y^2}=2 \implies H=\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix} \implies \Delta_1=2>0$, $\Delta_2=3>0$.

Therefore $f(-2,1)=13$ is a local minimum.

10. Stationary points:

$$\left.\begin{aligned}\frac{\partial f}{\partial x}&=3x^2-3=0\\ \frac{\partial f}{\partial y}&=3y^2-12=0\\ fz&=3z^2-12=0\end{aligned}\right\} \implies x=\pm 1,\ y=\pm 2,\ z=\pm 2.$$

Since the equations were independent, we may combine signs freely and there are 8 stationary points.

Classification: $\frac{\partial^2 f}{\partial x^2}=6x$, $\frac{\partial^2 f}{\partial x\partial y}=0$, $\frac{\partial^2 f}{\partial x\partial z}=0$, $\frac{\partial^2 f}{\partial y^2}=6y$, $\frac{\partial^2 f}{\partial y\partial z}=0$, $\frac{\partial^2 f}{\partial z^2}=6z$. $H(x,y,z)=\begin{pmatrix}6x & 0 & 0\\ 0 & 6y & 0\\ 0 & 0 & 6z\end{pmatrix}$

point:	$\Delta_1=6x$	$\Delta_2=36xy$	$\Delta_3=216xyz$	
$f(1,2,2)=-34$	+	+	+	loc. minimum
$f(1,2,-2)=-2$	+	+	−	unidentified
$f(1,-2,2)=-2$	+	−	−	saddle
$f(1,-2,-2)=30$	+	−	+	saddle
$f(-1,2,2)=-30$	−	−	−	saddle
$f(-1,2,-2)=2$	−	−	+	saddle
$f(-1,-2,2)=2$	−	+	+	unidentified
$f(-1,-2,-2)=34$	−	+	−	loc. maximum

11. Stationary points:

$$\left.\begin{array}{l}\frac{\partial f}{\partial x}=3x^2+3y^2-12y=0\\ \frac{\partial f}{\partial y}=6xy-12x=0\end{array}\right\}\implies \left.\begin{array}{l}x^2+y^2-4y=0\\ x(y-2)=0\end{array}\right\}\implies \left\{\begin{array}{l}x=0\implies y^2=4y\implies y=0,4\\ y=2\implies x^2-4=0\implies x=\pm 2\end{array}\right.$$

Stationary points: $(0,0),(0,4),(\pm 2,2)$.

Classification: $\frac{\partial^2 f}{\partial x^2}=6x$, $\frac{\partial^2 f}{\partial x\partial y}=6y-12$, $\frac{\partial^2 f}{\partial y^2}=6x \implies H=\begin{pmatrix}6x & 6y-12\\ 6y-12 & 6x\end{pmatrix}\implies \Delta_1=6x$, $\Delta_2=36x^2-36(y-2)^2$.

$(0,0)$: $\Delta_1=0$, $\Delta_2=-36\cdot 4<0$, therefore $f(0,0)=0$ is a saddle.
$(0,4)$: $\Delta_1=0$, $\Delta_2=-36\cdot 4<0$, therefore $f(0,4)=0$ is a saddle.
$(2,2)$: $\Delta_1=12\cdot 4>0$, $\Delta_2=36\cdot 4>0$, therefore $f(2,2)=-16$ is a local minimum.
$(-2,2)$: $\Delta_1=-12\cdot 4<0$, $\Delta_2=36\cdot 4>0$, therefore $f(-2,2)=16$ is a local maximum.

12. Stationary points:

$$\left.\begin{array}{l}\frac{\partial f}{\partial x}=6x^2+18x-12=0\\ \frac{\partial f}{\partial y}=6y^2-6=0\end{array}\right\}\implies \left.\begin{array}{l}x^2+3x-2=0\\ y^2=1\end{array}\right\}\implies x=1,2,\ y=\pm 1.$$

Independent equations $\implies (1,1),(1,-1),(2,1),(2,-1)$.

Classification: $\frac{\partial^2 f}{\partial x^2}=12x+18$, $\frac{\partial^2 f}{\partial x\partial y}=0$, $\frac{\partial^2 f}{\partial y^2}=12y \implies H=\begin{pmatrix}12x+18 & 0\\ 0 & 12y\end{pmatrix}\implies \Delta_1=12x+18$, $\Delta_2=12(12x+18)y$.

$(1,1)$: $\Delta_1=30>0$, $\Delta_2=12\cdot 30>0$, therefore $f(1,1)=-5$ is a local minimum.
$(1,-1)$: $\Delta_1=30>0$, $\Delta_2=-12\cdot 30<0$, therefore $f(1,-1)=3$ is a saddle.
$(2,1)$: $\Delta_1=42>0$, $\Delta_2=12\cdot 42>0$, therefore $f(2,1)=24$ is a local minimum.
$(2,-1)$: $\Delta_1=42>0$, $\Delta_2=-12\cdot 42<0$, therefore $f(2,-1)=32$ is a saddle.

13. Stationary points:

$$\left.\begin{array}{l}\frac{\partial f}{\partial x}=-2x+6=0\\ \frac{\partial f}{\partial y}=-2y=0\end{array}\right\}\implies x=3,y=0\implies (3,0).$$

Classification: $\frac{\partial^2 f}{\partial x^2}=-2$, $\frac{\partial^2 f}{\partial x\partial y}=0$, $\frac{\partial^2 f}{\partial y^2}=-2 \implies H=\begin{pmatrix}-2 & 0\\ 0 & -2\end{pmatrix}\implies \Delta_1=-2<0$, $\Delta_2=4>0$.

Therefore $f(3,0)=12$ is a local maximum.

14. Stationary points:

$$\left.\begin{array}{l}\frac{\partial f}{\partial x}=-4x+4y+4=0\\ \frac{\partial f}{\partial y}=4x-6y-2=0\end{array}\right\}\implies x=2,y=1\implies (2,1).$$

Classification: $\frac{\partial^2 f}{\partial x^2}=-4$, $\frac{\partial^2 f}{\partial x\partial y}=4$, $\frac{\partial^2 f}{\partial y^2}=-6 \implies H=\begin{pmatrix}-4 & 4\\ 4 & -6\end{pmatrix}\implies \Delta_1=-4$, $\Delta_2=8$.

Therefore $f(2,1)=8$ is a local maximum.

15. Stationary points:

$$\left.\begin{array}{l}\frac{\partial f}{\partial x}=-\frac{y^2}{x^2}+\frac{1}{4}=0\\ \frac{\partial f}{\partial y}=\frac{2y}{x}-\frac{z^2}{y^2}=0\\ \frac{\partial f}{\partial z}=\frac{2z}{y}-\frac{2}{z^2}=0\end{array}\right\}\implies y=z^3\implies \left.\begin{array}{l}\frac{z^6}{x^2}=\frac{1}{4}\\ \frac{2z^3}{x}=\frac{1}{z^4}\end{array}\right\}\implies \left.\begin{array}{l}x^2=4z^6\\ x=2z^7\end{array}\right\}\implies 4z^{14}=4z^6\implies z=0,z=\pm 1.$$

$z=0\implies x=0,y=0$ but this is not in the domain, $z=1\implies x=2,y=1$, $z=-1\implies x=-2,y=-1$.

Two stationary points, $(2,1,1)$ and $(-2,-1,-1)$.

Classification: $\frac{\partial^2 f}{\partial x^2}=\frac{2y^2}{x^3}$, $\frac{\partial^2 f}{\partial x\partial y}=-\frac{2y}{x^2}$, $\frac{\partial^2 f}{\partial x\partial z}=0$, $\frac{\partial^2 f}{\partial y^2}=\frac{2}{x}+\frac{2z^2}{y^3}$, $\frac{\partial^2 f}{\partial y\partial z}=-\frac{2z}{y^2}$, $\frac{\partial^2 f}{\partial z^2}=\frac{2}{y}+\frac{4}{z^3}$.

$$H(2,1,1)=\begin{pmatrix}\frac{1}{4} & -\frac{1}{2} & 0\\ -\frac{1}{2} & 3 & -2\\ 0 & -2 & 6\end{pmatrix}\implies \Delta_1=\frac{1}{4}>0,\ \Delta_2=\begin{vmatrix}\frac{1}{4} & -\frac{1}{2}\\ -\frac{1}{2} & 3\end{vmatrix}=\frac{1}{2}>0,\ \Delta_3=\frac{1}{4}>0.$$

Therefore $f(2,1,1)=4$ is a local minimum.

$$H(-2,-1,-1)=\begin{pmatrix}-\frac{1}{4} & \frac{1}{2} & 0\\ \frac{1}{2} & -3 & 2\\ 0 & 2 & -6\end{pmatrix}\implies \Delta_1=-\frac{1}{4}<0,\ \Delta_2=\begin{vmatrix}-\frac{1}{4} & \frac{1}{2}\\ \frac{1}{2} & -3\end{vmatrix}=\frac{1}{2}>0,\ \Delta_3=-\frac{4}{<}0.$$

Therefore $f(-2,-1,-1)=-4$ is a local maximum.

16. Stationary points:

$$\left.\begin{aligned}\frac{\partial f}{\partial x} &= 6x-6y=0\\ \frac{\partial f}{\partial y} &= -6x+6y^2=0\end{aligned}\right\} \implies y^2-y=0 \implies \left\{\begin{aligned} y=0 &\implies x=0\\ y=1 &\implies y=1\end{aligned}\right\} \implies (0,0),(1,1).$$

Classification: $\frac{\partial^2 f}{\partial x^2}=6$, $\frac{\partial^2 f}{\partial x\partial y}=-6$, $\frac{\partial^2 f}{\partial y^2}=12y \implies H=\begin{pmatrix}6 & -6\\ -6 & 12y\end{pmatrix} \implies \Delta_1=6,\ \Delta_2=72y-36.$

$(0,0)$: $\Delta_1=6>0$, $\Delta_2=-36<0$, therefore $f(0,0)=0$ is a saddle.
$(1,1)$: $\Delta_1=6>0$, $\Delta_2=36>0$, therefore $f(1,1)=-1$ is a local minimum.

17. Stationary points:

$$\left.\begin{aligned}\frac{\partial f}{\partial x} &= 3x^2-6y=0\\ \frac{\partial f}{\partial y} &= 24y^2-6x=0\end{aligned}\right\} \implies \left.\begin{aligned}x^2&=2y\\ x&=4y^2\end{aligned}\right\} \implies 16y^4=2y \implies \left\{\begin{aligned} y=0 &\implies x=0\\ y=\tfrac12 &\implies x=1\end{aligned}\right\} \implies (0,0),\big(1,\tfrac12\big).$$

Classification: $\frac{\partial^2 f}{\partial x^2}=6x$, $\frac{\partial^2 f}{\partial x\partial y}=-6$, $\frac{\partial^2 f}{\partial y^2}=48y \implies H=\begin{pmatrix}6x & -6\\ -6 & 48y\end{pmatrix} \implies \Delta_1=6x,\ \Delta_2=288xy-36.$

$(0,0)$: $\Delta_1=0$, $\Delta_2=-36<0$, therefore $f(0,0)=5$ is a saddle.
$\big(1,\frac12\big)$: $\Delta_1=6>0$, $\Delta_2=144-36=108>0$, therefore $f\big(1,\frac12\big)=4$ is a local minimum.

18. Stationary points:

$$\left.\begin{aligned}\frac{\partial f}{\partial x} &= 2x-4x\,e^{-x^2}=0\\ \frac{\partial f}{\partial y} &= -2y=0\end{aligned}\right\} \implies y=0, x(1-2e^{-x^2})=0 \implies x=0, x=\pm\sqrt{\ln(2)} \implies (0,0),(\pm\sqrt{\ln(2)},0).$$

Classification: $\frac{\partial^2 f}{\partial x^2}=2+8x^2e^{-x^2}$, $\frac{\partial^2 f}{\partial x\partial y}=0$, $\frac{\partial^2 f}{\partial y^2}=-2 \implies H=\begin{pmatrix}2+8x^2e^{-x^2} & 0\\ 0 & -2\end{pmatrix}$

$\implies \Delta_1=2+8x^2e^{-x^2}$, $\Delta_2=-4-16x^2e^{-x^2}$.
$(0,0)$: $\Delta_1=2>0$, $\Delta_2=-2<0$, therefore $f(0,0)=2$ is a saddle.
$(\sqrt{\ln(2)},0)$: $\Delta_1=2+4\ln(2)>0$, $\Delta_2=-2-4\ln(2)<0$, therefore $f(\sqrt{\ln(2)},0)=\ln(2)+1$ is a saddle.
$(-\sqrt{\ln(2)},0)$: $\Delta_1=2+4\ln(2)>0$, $\Delta_2=-2-4\ln(2)<0$, therefore $f(-\sqrt{\ln(2)},0)=\ln(2)+1$ is a saddle.

19. $\nabla f=\lambda\nabla g$, $g(x,y)=x^2+y^2=4 \implies \left|\begin{aligned} &y^3=\lambda 2x\\ &3xy^2=\lambda 2y\\ &x^2+y^2=4\end{aligned}\right.$

If $x=0$ then $y=0$ by (#1) but then (#3) won't work, so $x=0$ not possible. Then from (#1) $\lambda=\frac{y^3}{2x}$, into (#2), $3x^2y^2=y^4$.
If $y=0$, then $x=\pm2$ and $\lambda=0$ solves all equations.
If $y\neq0$, then $3x^2=y^2$, into (#3), $4x^2=4 \implies x=\pm1$. For $x=\pm1$, $y=\pm\sqrt3$.
Possible local extrema are $f(2,0)=0$, $f(-2,0)=0$, $f(1,\sqrt3)=3\sqrt3$. $f(1,-\sqrt3)=-3\sqrt3$. $f(-1,\sqrt3)=-3\sqrt3$. $f(-1,-\sqrt3)=3\sqrt3$.
The condition $g=4$ defines a circle, which is a bounded closed set. Thus f must attain global extrema, they happen in endpoints (but a circle has none) or at pts of local extrema. Thus the only candidates are the six values above.
Conclusion: Local/global minima are $f(-1,\sqrt3)=f(1,-\sqrt3)=-3\sqrt3$. Local/global maxima are $f(1,\sqrt3)=f(-1,-\sqrt3)=3\sqrt3$.

20. $\nabla f=\lambda\nabla g$, $g(x,y)=x^4+y^4-8x-8y=0 \implies \left|\begin{aligned} 1&=\lambda(4x^3-8)\\ 1&=\lambda(4y^3-8)\\ x^4+y^4-8x&-8y=0\end{aligned}\right.$

Obviously $\lambda=0$ not possible, so first two equations give $4x^3-8=\frac1\lambda=4y^3-8 \implies y=x$. Put it into (#3), $x^4-8x=8 \implies x=0,2$.
Possible local extrema are $f(0,0)=0$, $f(2,2)=4$.
Is the set defined by $g=0$ bounded? Yes, since for large values of x,y one can ignore $8x$ and $8y$ and the condition becomes essentially $x^4+y^4=0$, which makes it impossible for x or y to grow arbitrarily. Thus one can guess that global extrema will be attained.
Conclusion: Local/global minimum is $f(0,0)=0$. Local/global maximum is $f(2,2)=4$.

21. $\nabla f=\lambda\nabla g$, $g(x,y)=x^4-y^4-8x-8y=2 \implies \left|\begin{aligned} 1&=\lambda(4x^3-8)\\ 1&=\lambda(-4y^3-8)\\ x^4-y^4-8x&-8y=2\end{aligned}\right.$

Obviously $\lambda=0$ not possible, so first two equations give $4x^3-8=\frac1\lambda=-4y^3-8 \implies y=-x$. Put it into (#3), $2x^4=2 \implies x=\pm1$.

Possible local extrema are $f(1,-1)=0$, $f(-1,1)=0$.
Is the set bounded? The condition $x^4-8x=y^4+8y+2$ makes it possible for x,y to simultaneously go to infinity, then also $f\to\infty$.
Conclusion: Local/global minimum is $f(1,-1)=f(-1,1)=0$. Global maximum is not attained, supremum is infinity.

22. $\nabla f=\lambda\nabla g$, $g(x,y)=\frac{x^2}{8}+\frac{y^2}{2}=1 \implies \left|\begin{array}{c} y=\lambda\frac{x}{4} \\ x=\lambda y \\ \frac{x^2}{8}+\frac{y^2}{2}=1 \end{array}\right.$

Put y from (#1) into (#2), $4x=\lambda^2 x$. If $x=0$, then by (#1) also $y=0$, contradiction with (#3). So $x\neq 0$, therefore $\lambda=\pm 2$.
If $\lambda=2$ then $x=2y$, into (#3), $y^2=1$, $y=\pm 1$, points $(2,1)$ and $(-2,-1)$.
If $\lambda=-2$ then $x=-2y$, into (#3), $y^2=1$, $y=\pm 1$, points $(-2,1)$ and $(2,-1)$.
Possible local extrema are $f(2,1)=2$, $f(2,-1)=-2$, $f(-2,1)=-2$, $f(-2,-1)=2$.
The condition $g=4$ defines an ellipse, which is a bounded closed set. Thus f must attain global extrema, they happen in endpoints (but an ellipse has none) or at pts of local extrema. Thus the only candidates are the four values above.
Conclusion: Local/global minima are $f(2,-1)=f(-2,1)=-2$. Local/global maxima are $f(2,1)=f(-2,-1)=2$.

23. $\nabla f=\lambda\nabla g$, $g(x,y)=x^2-2x+y^2-4y=0 \implies \left|\begin{array}{c} 2x=\lambda(2x-2) \\ 2y=\lambda(2y-4) \\ x^2-2x+y^2-4y=0 \end{array}\right.$

Cannot have $x=1$ (then (#1) reads $2=0$) and similarly $y\neq 1$, so $\frac{x}{x-1}=\lambda=\frac{y}{y-2} \implies y=2x$. Put into (#3), get $5x^2-10x=0 \implies x=0,2$.
Possible local extrema are $f(0,0)=0$, $f(2,4)=20$.
The condition $g=0$, that is, $(x-1)^2+(y-2)^2=5$ defines a circle, which is a bounded closed set. The only candidates for global extrema are the four values above.
Conclusion: Local/global minimum is $f(0,0)=0$. Local/global maximum is $f(2,4)=20$.

24. $\nabla f=\lambda\nabla g$, $g(x,y)=x^3+y^3=2 \implies \left|\begin{array}{c} 1=\lambda 3x^2 \\ 1=\lambda 3y^2 \\ x^3+y^3=2 \end{array}\right.$

Cannot have $\lambda=0$, therefore $x^3=\frac{1}{3\lambda}=y^2$.
If $y=x$, then (#3) gives $x^3=1$, $x=1$. If $y=-x$, then (#3) gives $0=1$, not possible.
Possible local extreme is $f(1,1)=2$.
The condition $g=2$ defines a set that is not bounded, it allows for simultaneous $x\to\infty$, $y\to-\infty$, but x and $-y$ must be stay and so f cannot grow large. In fact, if we let $x\to\infty$, then $x^3+y^3=2$ means that $x+y\to 0$. One way to see it: If $x\to\infty$ then $y=\sqrt[3]{2-x^3}\to-\infty$, therefore $x^2-xy+y^2\to\infty$ and $2=x^3+y^3=(x+y)(x^2-xy+y^2)$, that is, $x+y=\frac{2}{x^2-xy+y^2}\to 0$.
Conclusion: Local/global maximum is $f(1,1)=2$. Global minimum does not exist, global infimum is 0.

25. $\nabla f=\lambda\nabla g$, $g(x,y,z)=x^2+y^2+z^2=14 \implies \left|\begin{array}{c} 1=\lambda 2x \\ 2=\lambda 2y \\ 3=\lambda 2z \\ x^2+y^2+z^2=14 \end{array}\right.$

Cannot have $\lambda=0$, so $x=\frac{1}{2\lambda}$, $y=\frac{1}{\lambda}$, $z=\frac{3}{2\lambda}$, put into (#3), $\frac{14}{4\lambda^2}=14 \implies \lambda=\pm\frac{1}{2}$.
Possible local extrema are $f(1,2,3)=14$, $f(-1,-2,-3)=-14$.
The condition $g=0$ defines a sphere, which is a bounded closed set. The only candidates for global extrema are the two values above.
Conclusion: Local/global minimum is $f(-1,-2,-3)=-14$. Local/global maximum is $f(1,2,3)=14$.

26. $\nabla f=\lambda\nabla g$, $g_1(x,y,z)=x^2+y^2=2$, $g_2(x,y,z)=x+y+z=0 \implies \left|\begin{array}{l} 1=\lambda 2x+\mu 1 \\ 1=\lambda 2y+\mu 1 \\ 0=\lambda\cdot 0+\mu 1 \\ \;x^2+y^2=2 \\ x+y+z=0 \end{array}\right.$

From (#3): $\mu=0$, into the first two, get $y=x$. Put into last two, $2x^2=2$, $2x+z=0$. Then $x=\pm 1$ and $z=-2x$.
Possible local extrema are $f(1,1,-2)=2$, $f(-1,-1,2)=-2$.

The set defined by conditions is bounded. Indeed, the first one gives $|x| \le 2$ and $|y| \le 2$, then the second gives $|z| \le 4$. The only candidates for global extrema are the two values above.
Conclusion: Local/global minimum is $f(-1,-1,2) = -2$. Local/global maximum is $f(1,1,-2) = 2$.

27. $\nabla f = \lambda \nabla g$, $g_1(x,y,z) = x^2 + y^2 = 5$, $g_2(x,y,z) = x + 2y + z = 0 \implies \left| \begin{array}{l} 4 = \lambda 2x + \mu 1 \\ 0 = \lambda 2y + \mu 2 \\ 2 = \lambda \cdot 0 + \mu 1 \\ x^2 + y^2 = 5 \\ x + 2y + z = 0 \end{array} \right.$

From (#3): $\mu = 2$, into the first two, get $\lambda x = 1$, $\lambda y = -2$. This gives $y = -2x$, put into last two, $5x^2 = 5$, $-3x + z = 0$. Then $x = \pm 1$ and $z = 3x$.
Possible local extrema are $f(1,-2,3) = 10$, $f(-1,2,-3) = -10$.
The set defined by conditions is bounded. Indeed, the first one gives $|x| \le \sqrt{5}$ and $|y| \le \sqrt{5}$, then the second gives $|z| \le 3\sqrt{5}$. The only candidates for global extrema are the two values above.
Conclusion: Local/global minimum is $f(-1,2,-3) = -10$. Local/global maximum is $f(1,-2,3) = 10$.

28. $\nabla f = \lambda \nabla g$, $g_1(x,y,z) = x + y = 0$, $g_2(x,y,z) = x^2 - z = 0 \implies \left| \begin{array}{l} 2xy = \lambda 1 + \mu 2x \\ x^2 = \lambda 1 + \mu \cdot 0 \\ 3 = \lambda \cdot 0 + \mu(-1) \\ x + y = 0 \\ x^2 - z = 0 \end{array} \right.$

From (#3): $\mu = -3$, into the first two, get $2xy = \lambda - 6x$, $x^2 = \lambda$. This gives $2xy = x^2 - 6x$. From (#3) $y = -x$, so the last one reads $0 = 3x^3 - 6x \implies x = 0, 2$.
Possible local extrema are $f(0,0,0) = 0$, $f(2,-2,4) = 4$.
The set defined by conditions is not bounded. One can easily send $x \to \infty$, then by the first condition $y \to -\infty$ and by the second one $z \to \infty$. Using the second condition we get that $f(x,y,z) = x^2(y+3)$, so $f \to -\infty$. Similarly, if we let $x \to -\infty$, then $y \to \infty$ and $f \to \infty$.
Conclusion: Local minimum is $f(0,0,0) = 0$. Local maximum is $f(2,-2,4) = 4$. There is no global maximum or minimum. Global supremum is infinity, global infimum is negative infinity.

29. First we find local extrema with respect to the given set. $\nabla f = \lambda \nabla g$, $g(x,y) = xy = 32 \implies$
$\left| \begin{array}{l} 2(x-2) = \lambda y \\ 2(y-16) = \lambda x \\ xy = 32 \end{array} \right.$

Since $x, y \ne 0$, we can do $\frac{x-2}{y} = \frac{\lambda}{2} = \frac{y-16}{x} \implies x(x-2) = y(y-16)$. From (#3) use $y = \frac{32}{x}$, obtain $x(x-2) = \frac{32 \cdot 16}{x^2}(2-x)$. If $x \ne 2$ then cancel, $x^3 = -32 \cdot 16 \implies x = -8$. This is not in the range, we are not interested.
If $x = 2$, then $y = 16$, get a suspicious point $f(2,16) = 0$.
Other suspicious points are the endpoints: $f(1,32) = 257$, $f\big(12, \frac{8}{3}\big) = 277 + \frac{7}{9}$.
Conclusion: Global minimum is $f(2,16) = 0$, global maximum is $f\big(12, \frac{8}{3}\big) = 277 + \frac{7}{9}$.

30. First we find local extrema with respect to the given set. $\nabla f = \lambda \nabla g$, $g(x,y) = xy = 4 \implies$
$\left| \begin{array}{l} 2(x+2) = \lambda y \\ 2(y+2) = \lambda x \\ xy = 4 \end{array} \right.$

Since $x, y \ne 0$, we can do $\frac{x+2}{y} = \frac{\lambda}{2} = \frac{y+2}{x} \implies x(x+2) = y(y+2)$. From (#3) use $y = \frac{4}{x}$, obtain $x(x+2) = \frac{8}{x^2}(2+x)$. The range is $x \ge 1$, so $x \ne -2$ and we cancel, $x^3 = 8 \implies x = 2$. Suspicious point $f(2,2) = 32$.
Other suspicious points are the endpoints: $f(1,4) = 45$, $\lim\limits_{x \to \infty} \big(f\big(x, \frac{4}{x}\big)\big) = \infty$.
Conclusion: Global minimum is $f(2,2) = 32$, global maximum does not exist, supremum is infinity.

31. First we find local extrema with respect to the given set. $\nabla f = \lambda \nabla g$, $g(x,y,z) = x + y + z^2 = 5 \implies$
$\left| \begin{array}{l} yz = \lambda 1 \\ xz = \lambda 1 \\ xy = \lambda 2z \\ x + y + z^2 = 5 \end{array} \right.$

From the first two $yz = xz$, since $z > 0$ we have $x = y$. We get $xz = \lambda$, $x^2 = 2\lambda z$, so $x^2 = 2xz^2$, that is, $x = 2z^2$. Put into the constraint, $2z^2 + 2z^2 + z^2 = 5 \implies z = \pm 1$, but $z > 0$ so $z = 1$.
Suspicious point $f(2,2,1) = 4$.

What do we know about M? It is a bounded set, since from $x,y,z>0$ and the constraint we have $x,y,z<5$. Thus f cannot run away to infinity. We can have $x\to 0^+$ (with for instance $z=2$, $y=1-x$), then $f\to 0$. However, we cannot have $f=0$ with $x,y,z>0$.
Conclusion: Global maximum is $f(2,2,1)=4$, global minimum does not exist, infimum is 0.

32. M is a triangle with vertices $(0,0)$, $(3,0)$, and $(0,3)$.
1) Interior. Stationary points:
$$\left.\begin{aligned}\frac{\partial f}{\partial x}&=4x-2y-2=0\\ \frac{\partial f}{\partial y}&=-2x+2y=0\end{aligned}\right\}\implies x=y=1.\ (1,1)\in M\text{, hence the first candidate: } f(1,1)=-1.$$
2) Boundary. Corners: $f(0,0)=0$, $f(3,0)=12$, $f(0,3)=9$.
Now parts:
2a) $x=0$, $y\in[0,3]$ (the vertical side). Consider $\varphi(y)=f(0,y)=y^2$ on $[0,3]$, endpoints see 2), $\varphi'(y)=0\implies y=0$, $(0,0)$ already checked.
2b) $y=0$, $x\in[0,3]$ (the horizontal side). Consider $\varphi(x)=f(x,0)=2x^2-2x$ on $[0,3]$, endpoints see 2), $\varphi'(x)=0\implies x=\frac{1}{2}$. Candidate $f\big(\frac{1}{2},0\big)=-\frac{1}{2}$.
2c) $y=3-x$, $x\in[0,3]$ (the oblique side). Lagrange multipliers with $g(x,y)=x+y=3$: $\left|\begin{aligned}4x-2y-2&=\lambda 1\\ -2x+2y&=\lambda 1\\ x+y&=3\end{aligned}\right.$
From the first two $4x-2y-2=-2x+2y\implies 6x-4y=2$, add with four times the constraint: $10x=14\implies x=\frac{7}{5}$, then $y=\frac{8}{5}$. Candidate $f\big(\frac{7}{5},\frac{8}{5}\big)=\frac{16}{5}$.
Alternative: Consider $\varphi(x)=f(x,3-x)=5x^2-14x+9$ on $[0,3]$. Endpoints see 2), $\varphi'(x)=0\implies x=\frac{7}{5}$.
Comparing candidates gives: $f(1,1)=-1$ is the minimum on M, $f(3,0)=12$ is the maximum on M.

33. $f(x,y,z)=x-xy-z^2$, $\quad M=\{(x,y,z)\in\mathbb{R}^3;\ x^2+y^2+z^2\le 3\}$. M is a ball with radius $\sqrt{5}$.
1) Interior. Stationary points:
$$\left.\begin{aligned}\frac{\partial f}{\partial x}&=1-y=0\\ \frac{\partial f}{\partial y}&=-x=0\\ \frac{\partial f}{\partial z}&=-2z=0\end{aligned}\right\}\implies x=z=0,y=1.\ (0,1,0)\in M\text{, hence the first candidate: } f(0,1,0)=0.$$
2) Boundary:
$$\nabla f=\lambda\nabla g,\ g(x,y,z)=x^2+y^2+z^2=3\implies\left|\begin{aligned}1-y&=\lambda 2x\\ -x&=\lambda 2y\\ -2z&=\lambda 2z\\ x^2+y^2+z^2&=3\end{aligned}\right.$$
The third equation offers two possibilities.
a) Case $z=0$. Then we have the system $\left|\begin{aligned}1-y&=\lambda 2x\\ -x&=\lambda 2y\\ x^2+y^2&=3\end{aligned}\right.$
If $x=0$ then from (#1) $y=1$, but that contradicts the constraint, similarly $y=0$ not possible. Thus from (#1), (#2) $\frac{1-y}{x}=2\lambda=\frac{-x}{y}\implies y-y^2=-x^2\implies x^2-y^2+y=0$. Subtract from constraint, $2y^2-y=3$. Thus $y=-1,\frac{3}{2}$. Two candidates $f\big(\sqrt{2},-1,0\big)=2\sqrt{2}$, $f\big(\frac{\sqrt{3}}{2},\frac{3}{2},0\big)=-\frac{\sqrt{3}}{4}$.
b) Case $\lambda=-1$. The first two equations then read $y=2x+1$, $2y=x$, hence $x=-\frac{2}{3}$, $y=-\frac{1}{3}$. The constraint then gives $z=\pm\frac{\sqrt{22}}{3}$. We have two more candidates: $f\big(-\frac{2}{3},-\frac{1}{3},\frac{\sqrt{22}}{3}\big)=-\frac{26}{9}$, $f\big(-\frac{2}{3},-\frac{1}{3},-\frac{\sqrt{22}}{3}\big)=-\frac{26}{9}$.
Comparing candidates gives: $f\big(\sqrt{2},-1,0\big)=2\sqrt{2}$ is the maximum on M, $f\big(-\frac{2}{3},-\frac{1}{3},\pm\frac{\sqrt{22}}{3}\big)=-\frac{26}{9}$ give the minimum on M.

34. M is a tip of hyperbola cut off by a line, intersections are $(5,4)$, $(5,-4)$.
1) Interior. Stationary points:
$$\left.\begin{aligned}\frac{\partial f}{\partial x}&=2x-8=0\\ \frac{\partial f}{\partial y}&=2y=0\end{aligned}\right\}\implies x=4,y=0.\ (4,0)\in M\text{, hence the first candidate: } f(4,0)=-16.$$
2) Boundary. Corners: $f(5,\pm 4)=1$.
Now parts:
2a) $x=5$, $y\in[-4,4]$ (the vertical side). Consider $\varphi(y)=f(5,y)=y^2-15$ on $[-4,4]$, endpoints see 2), $\varphi'(y)=0\implies y=0$, another candidate $f(5,0)=-15$.
2b) $x^2-y^2=9$, $x\in[3,5]$, $y\in[-4,4]$ (the hyperbolic side). Lagrange multipliers: $\left|\begin{aligned}2x-8&=\lambda 2x\\ 2y&=\lambda(-2y)\\ x^2-y^2&=9\end{aligned}\right.$

One possibility $y = 0$, then $x = 3$, candidate $f(3, 0) = -15$.
If $y \neq 0$ then from (#2) $\lambda = -1$, (#1) gives $x = 2$, not in M.
Comparing candidates gives: $f(4, 0) = -16$ is the minimum on M, $f(5, \pm 4) = 1$ give the maximum on M.

35. $f(x, y) = x^2 + 2x + y^2$, M is three quarters of a circle, with a little sliced off by a line. Intersections are $(-2, \sqrt{5})$, $(2, \sqrt{5})$.
1) Interior. Stationary points:

$$\left.\begin{array}{l}\frac{\partial f}{\partial x} = 2x + 2 = 0 \\ \frac{\partial f}{\partial y} = 2y = 0\end{array}\right\} \implies x = -1, y = 0.\ (-1, 0) \in M, \text{ hence the first candidate: } f(-1, 0) = -1.$$

2) Boundary. Corners: $f\big(-2, \pm\sqrt{5}\big) = 5$.
Now parts:
2a) $x = -2$, $y \in \big[-\sqrt{5}, \sqrt{5}\big]$ (the vertical side). Consider $\varphi(y) = f(-2, y) = y^2$ on $\big[-\sqrt{5}, \sqrt{5}\big]$, endpoints see 2),
$\varphi'(y) = 0 \implies y = 0$, another candidate $f(-2, 0) = 0$.

2b) $x^2 + y^2 = 9$, $x \in [-2, 3]$, $y \in [-3, 3]$ (the circle side). Lagrange multipliers: $\left|\begin{array}{c} 2x + 2 = \lambda 2x \\ 2y = \lambda 2y \\ x^2 + y^2 = 9 \end{array}\right.$

One possibility $y = 0$, then $x = 3$, candidate $f(3, 0) = 15$.
If $y \neq 0$ then from (#2) $\lambda = 1$, (#1) gives $2 = 0$, not possible.
Comparing candidates gives: $f(-1, 0) = -1$ is the minimum on M, $f(3, 0) = 15$ give the maximum on M.

36. Take any point $Q(x, y, z)$ from the sheet, we want to minimize $f(x, y, z) = \text{dist}(0, Q)^2 = x^2 + y^2 + z^2$.
Constraint: $g(x, y, z) = x^2 - z^2 = 1$.

Lagrange multipliers: $\left|\begin{array}{c} 2x = \lambda 2x \\ 2y = \lambda \cdot 0 \\ 2z = \lambda(-2z) \\ x^2 - z^2 = 1 \end{array}\right.$

Second gives $y = 0$. First offers two possibilities.
Case $x = 0$: Then $-z^2 = 1$, not possible.
Case $\lambda = 1$: then (#3) gives $z = 0$, constraint gives $x = \pm 1$. Candidates $Q(\pm 1, 0, 0)$.
Since the distance can grow to infinity as we let $y \to \infty$, we found minimizing points. Their distance to the origin is 1.
Answer: The nearest points are $Q(\pm 1, 0, 0)$.

Remark: Often one prefers to use the constraint to reduce number of variables.
a) We express $z^2 = x^2 - 1$. Substitute into f, we want to minimize the function $h(x, y) = 2x^2 + y^2 - 1$.
Local extrema:
$\frac{\partial f}{\partial x} = 2x = 0$, $\frac{\partial f}{\partial y} = 2y = 0 \implies x = y = 0$, we minimized h, but unfortunately this is not a point that lies on the given hyperbolic sheet! So this solution does not work.
To make it work we would have to notice that the sheet forces x to satisfy $|x| \geq 1$, so one can try to minimize $h(x, y)$ under such condition.
b) We express $x^2 = z^2 + 1$. Substitute into f, we want to minimize the function $h(y, z) = y^2 + 2z^2 + 1$.
Local extrema: Here we do find $y = z = 0$ that give the right answer.

37. Take any point $Q(x, y, z)$ from the surface, we want to minimize $f(x, y, z) = \text{dist}(0, Q)^2 = x^2 + y^2 + z^2$.
Constraint: $g(x, y, z) = xy - z = -1$.

Lagrange multipliers: $\left|\begin{array}{c} 2x = \lambda y \\ 2y = \lambda x \\ 2z = \lambda(-1) \\ xy - z = -1 \end{array}\right.$

Put y from the second into the first, $4x = \lambda^2 x$. Two cases.
a) Case $x = 0$. Then also $y = 0$ and constraint gives $(0, 0, 1)$.
b) Case $\lambda = 2$. Then $y = x$ and (#3) gives $z = -1$, constraint yields $x^2 = -2$, not possible.
There is one candidate, its distance from the origin is 1. We can make the distance go to infinity wih points from the given surface, so we found the minimum.
Answer: The nearest point is $Q(0, 0, 1)$.

38. We need to take a point (s, t) from one curve and a point (u, v) from the second curve and minimize their mutual distance, for convenience we will be minimizing the square of their distance, $f(s, t, u, v) = (s - u)^2 + (t - v)^2$. Constraints: $g_1(s, t, u, v) = t - s = 1$ and $g_2(s, t, u, v) = u - v^2 = 0$.

$$\left|\begin{array}{c} 2(s-u)=\lambda(-1)+\mu\cdot 0 \\ 2(t-v)=\lambda\cdot 1+\mu\cdot 0 \\ -2(s-u)=\lambda\cdot 0+\mu\cdot 1 \\ -2(t-v)=\lambda\cdot 0+\mu(-2v) \\ t-s=1 \\ u-v^2=0 \end{array}\right. \implies \left|\begin{array}{c} 2(s-u)=-\lambda \\ 2(t-v)=\lambda \\ 2(s-u)=-\mu \\ 2(t-v)=2\mu v \\ t-s=1 \\ u-v^2=0 \end{array}\right.$$

Compare (#1) and (#3), $\lambda=\mu$, from (#2) and (#4) $\lambda=2\mu v$, so $\mu=2\mu v$. Two possibilities.
a) Case $\mu=0$: $s=u$, $t=v$, so $t-s=1$ and $s=t^2$, $t^2-t+1=0$ no solution. So $\mu\neq 0$.
b) Case $\mu\neq 0$. Then $v=\frac{1}{2}$, hence $u=\frac{1}{4}$. We get $2\big(s-\frac{1}{4}\big)=-\lambda$, $2\big(t-\frac{1}{2}\big)=\lambda$, hence $s=-\frac{1}{2}\lambda+\frac{1}{4}$, $t=\frac{1}{2}\lambda+\frac{1}{2}$. Put into the constraint, $\frac{1}{2}\lambda+\frac{1}{2}+\frac{1}{2}\lambda-\frac{1}{4}=1$, hence $\lambda=\frac{3}{4}$, $s=-\frac{1}{8}$, $t=\frac{7}{8}$.
We found points $\big(-\frac{1}{8},\frac{7}{8}\big)$ and $\big(\frac{1}{4},\frac{1}{2}\big)$. Their distance is $\frac{3}{8}\sqrt{2}$.