

**MA2: Practice problems—Functions of more variables: Integrals**  
**Brief solutions**

$$1. \int_0^1 \int_0^2 6xy + 2x + 2y \, dy \, dx = \int_0^1 [3xy^2 + 2xy + y^2]_0^2 \, dx = \int_0^1 16x + 4 \, dx = [8x^2 + 4x]_0^1 = 12.$$

$$\text{Alternative: } \int_0^2 \int_0^1 6xy + 2x + 2y \, dx \, dy = \int_0^2 [3x^2y + x^2 + 2xy]_0^1 \, dy = \int_0^2 5y + 1 \, dy = \left[\frac{5}{2}y^2 + y\right]_0^2 = 12.$$

$$2. \int_0^\pi \int_0^{2\pi} 3 \sin(3x + y) \, dy \, dx = \left| \begin{array}{l} z = 3x + y \\ dz = \frac{\partial}{\partial y}[3x + y] \, dy \\ dz = \frac{\partial}{\partial x}[3x + y] \, dx \end{array} \right| = \int_0^\pi [-3 \cos(3x + y)]_0^{2\pi} \, dx$$

$$= \int_0^\pi -3 \cos(3x + 2\pi) + 3 \cos(3x) \, dx = [-\sin(3x + 2\pi) + \sin(3x)]_0^\pi = 0.$$

Remark: Actually,  $\cos(3x + 2\pi) - \cos(3x) = \cos(3x) - \cos(3x) = 0$ .

$$\text{Alternative: } \int_0^{2\pi} \int_0^\pi 3 \sin(3x + y) \, dx \, dy = \left| \begin{array}{l} z = 3x + y \\ dz = \frac{\partial}{\partial x}[3x + y] \, dx \\ dz = \frac{\partial}{\partial y}[3x + y] \, dy \end{array} \right| = \int_0^{2\pi} [-\cos(3x + y)]_0^\pi \, dy$$

$$= \int_0^{2\pi} -\cos(3\pi + y) + \cos(y) \, dy = [-\sin(3\pi + y) + \sin(y)]_0^{2\pi} = 0.$$

$$3. \int_0^1 \int_0^{2\pi} x \sin(xy) \, dy \, dx = \left| \begin{array}{l} z = xy \\ dz = \frac{\partial}{\partial y}[xy] \, dy \\ dz = x \, dx \end{array} \right| = \int_0^1 [-\cos(xy)]_0^{2\pi} \, dx = \int_0^1 -\cos(2\pi x) + \cos(0) \, dx$$

$$= \int_0^1 1 - \cos(2\pi x) \, dx = \left[x - \frac{1}{2\pi} \sin(2\pi x)\right]_0^1 = 1.$$

$$\text{Alternative: } \int_0^{2\pi} \int_0^1 x \sin(xy) \, dx \, dy = \left| \begin{array}{l} f = x \quad g' = \sin(xy) \\ f' = 1 \quad g = -\frac{1}{y} \cos(xy) \end{array} \right| = \int_0^{2\pi} \left[-\frac{1}{y} x \cos(xy) + \frac{1}{y^2} \sin(xy)\right]_0^1 \, dy$$

$$= \int_0^{2\pi} \frac{1}{y^2} \sin(y) - \frac{1}{y} \cos(y) \, dy,$$

and we have a problem,  $\int \frac{1}{y} \cos(y) \, dy$  is one of those integrals that we can't express using a formula with elementary functions.

$$4. \int_0^1 \int_0^2 x^2 y e^{xy} \, dy \, dx = \left| \begin{array}{l} f = xy \quad g' = x e^{xy} \\ f' = x \quad g = e^{xy} \end{array} \right| \left| \begin{array}{l} z = xy \\ dz = \frac{\partial}{\partial y}[xy] \, dy \\ dz = x \, dx \end{array} \right| = \int_0^1 [xy e^{xy} - e^{xy}]_0^2 \, dx$$

$$= \int_0^1 2x e^{2x} - e^{2x} + 1 \, dx = \int_0^1 2x e^{2x} \, dx + \int_0^1 1 - e^{2x} \, dx = [x e^{2x}]_0^1 - \int_0^1 2e^{2x} \, dx + \left[x - \frac{1}{2}e^{2x}\right]_0^1$$

$$= e^2 - \left[\frac{1}{2}e^{2x}\right]_0^1 + 1 - \frac{1}{2}e^2 + \frac{1}{2} = 2.$$

$$\text{Alternative: } \int_0^2 \int_0^1 x^2 y e^{xy} \, dx \, dy = \left| \begin{array}{l} f = x^2 \quad g' = y e^{xy} \\ f' = 2x \quad g = e^{xy} \end{array} \right| = \int_0^2 \left([x^2 e^{xy}]_0^1 - \int_0^1 2x e^{xy} \, dx\right) \, dy$$

$$= \left| \begin{array}{l} f = 2x \quad g' = e^{xy} \\ f' = 2 \quad g = \frac{1}{y} e^{xy} \end{array} \right| = \int_0^2 e^y - \left[2x \frac{1}{y} e^{xy} - \frac{2}{y^2} e^{xy}\right]_0^1 \, dy = \int_0^2 e^y - \frac{2}{y} e^y + \frac{2}{y^2} e^y - \frac{2}{y^2} \, dy.$$

This is not a nice integral, the first way is preferable.

$$5. \int_0^1 \int_0^1 \frac{y}{(1 + x^2 + y^2)^2} \, dy \, dx = \left| \begin{array}{l} z = y^2 + x^2 + 1 \\ dz = \frac{\partial}{\partial y}[y^2 + x^2 + 1] \, dy \\ dz = 2y \, dy \end{array} \right| = \int_0^1 \left[-\frac{1}{2} \frac{1}{1 + x^2 + y^2}\right]_0^1 \, dx$$

$$= \int_0^1 \frac{1}{2} \frac{1}{1+x^2} - \frac{1}{2} \frac{1}{2+x^2} dx = \left[ \frac{1}{2} \arctan(x) - \frac{1}{2\sqrt{2}} \arctan\left(\frac{x}{\sqrt{2}}\right) \right]_0^1 = \frac{\pi}{8} - \frac{\sqrt{2}}{4} \arctan\left(\frac{1}{\sqrt{2}}\right).$$

Alternative:  $\int_0^1 \int_0^1 \frac{y}{(1+x^2+y^2)^2} dx dy$

The integral  $\int \frac{A}{(x^2+B)^2} dx$  is very unpleasant, the first approach is better.

$$\begin{aligned} 6. \int_0^1 \int_0^\infty \frac{1}{1+x^2+y^2+x^2y^2} dy dx &= \int_0^1 \int_0^\infty \frac{1}{(1+x^2)y^2+(1+x^2)} dy dx = \int_0^1 \frac{1}{x^2+1} \int_0^\infty \frac{1}{y^2+1} dy dx \\ &= \int_0^1 \frac{1}{x^2+1} [\arctan(y)]_0^\infty dx = \int_0^1 \frac{1}{x^2+1} \frac{\pi}{2} dx = \left[ \frac{\pi}{2} \arctan(x) \right]_0^1 = \frac{\pi^2}{8}. \end{aligned}$$

$$7. \int_0^\infty \int_0^\infty e^{-x-y} dy dx = \int_0^\infty e^{-x} \int_0^\infty e^{-y} dy dx = \int_0^\infty e^{-x} [-e^{-y}]_0^\infty dx = \int_0^\infty e^{-x} \cdot 1 dx = [-e^{-x}]_0^\infty = 1.$$

Bonus: What if we had  $\int_0^\infty \int_0^\infty e^{-x^2-y^2} dy dx$ ? We run into trouble, since we know that the antiderivative to  $e^{-y^2}$  cannot be expressed as a formula using elementary functions. However, there is an interesting trick.

It is known that  $\int_{-\infty}^\infty e^{-z^2/2} dz = \sqrt{2\pi}$ . Therefore  $\int_{-\infty}^\infty e^{-y^2} dy = \left| \begin{matrix} z = \sqrt{2}y \\ dz = \sqrt{2} dy \end{matrix} \right| = \frac{1}{\sqrt{2}} \int_{-\infty}^\infty e^{-z^2/2} dz = \sqrt{\pi}$ .

By symmetry,  $\int_0^\infty e^{-y^2} dy = \frac{\sqrt{\pi}}{2}$ . Thus

$$\int_0^\infty \int_0^\infty e^{-x^2-y^2} dy dx = \int_0^\infty e^{-x^2} \int_0^\infty e^{-y^2} dy dx = \int_0^\infty \frac{\sqrt{\pi}}{2} e^{-x^2} dx = \frac{\sqrt{\pi}}{2} \int_0^\infty e^{-x^2} dx = \frac{\pi}{4}.$$

For pictures of  $\Omega$ s from 8–15 see below.

8. Vertical slicing seems natural.

$$\int_{-1}^1 \int_{x^2}^1 3x^3 e^{xy} dy dx = \int_{-1}^1 [3x^2 e^{xy}]_{x^2}^1 dx = \int_{-1}^1 3x^2 e^x - 3x^2 e^{x^3} dx = [3x^2 e^x - 6x e^x + 6 e^x - e^{x^3}]_{-1}^1 = 2e - 14e^{-1}.$$

9. Vertical slicing seems better.

$$\begin{aligned} \int_1^2 \int_{\pi x}^{2\pi x} x^2 \sin(xy) dy dx &= \int_1^2 [-x \cos(xy)]_{\pi x}^{2\pi x} dx = \int_1^2 -x \cos(2\pi x^2) + x \cos(\pi x^2) dx \\ &= \left[ -\frac{1}{4\pi} \sin(2\pi x^2) + \frac{1}{2\pi} \sin(\pi x^2) \right]_1^2 = 0. \end{aligned}$$

10. Intersections: (1, -2), (2, 1). Horizontal slicing is the best.

$$\int_{-2}^1 \int_{(y+5)/3}^{\sqrt{5-y^2}} 9y dx dy = \int_{-2}^1 [9xy]_{(y+5)/3}^{\sqrt{5-y^2}} dy = \int_{-2}^1 9y\sqrt{5-y^2} - 3y^2 - 15y dx = \left[ -3(5-y^2)^{3/2} - y^3 - \frac{15}{2}y^2 \right]_{-2}^1 = -\frac{15}{2}.$$

11. Intersections: (1, 1), (3, 3), (4, 2). It is a triangle.

$$\begin{aligned} \int_1^3 \int_{(x+2)/3}^x 9x dy dx + \int_3^4 \int_{(x+2)/3}^{6-x} 9x dy dx &= \int_1^3 [9xy]_{(x+2)/3}^x dx + \int_3^4 [9xy]_{(x+2)/3}^{6-x} dx \\ &= \int_1^3 9x^2 - 3x(x+2) dx + \int_3^4 9x(6-x) - 3x(x+2) dx = \int_1^3 6x^2 - 6x dx + \int_3^4 48x - 12x^2 dx \\ &= \left[ 2x^3 - 3x^2 \right]_1^3 + \left[ 24x^2 - 4x^3 \right]_3^4 = 48. \end{aligned}$$

Alternativa:

$$\begin{aligned} \int_1^2 \int_y^{3y-2} 9x \, dx \, dy + \int_2^3 \int_y^{6-y} 9x \, dx \, dy &= \int_1^2 \left[ \frac{9}{2}x^2 \right]_y^{3y-2} dy + \int_2^3 \left[ \frac{9}{2}x^2 \right]_y^{6-y} dy \\ &= \int_1^2 36y^2 - 54y + 18 \, dy + \int_2^3 162 - 54y \, dy = \left[ 12y^3 - 27y^2 + 18y \right]_1^2 + \left[ 162y - 27y^2 \right]_2^3 = 48. \end{aligned}$$

**12.** Triangle with corners  $(0, 0)$ ,  $(0, 2)$ ,  $(1, 2)$ .

Vertical slicing: 
$$\int_0^1 \int_{2x}^2 (4e^{y^2} + 8xy) \, dy \, dx.$$

However, we know that the antiderivative to  $e^{y^2}$  cannot be expressed using formulas with elementary functions, so this is not a good idea.

Horizontal slicing:

$$\int_0^2 \int_0^{y/2} 4e^{y^2} + 8xy \, dx \, dy = \int_0^2 \left[ 4e^{y^2}x + 4x^2y \right]_{x=0}^{x=y/2} dy = \int_0^2 2y e^{y^2} + y^3 \, dy = \left[ e^{y^2} + \frac{1}{4}y^4 \right]_0^2 = e^4 + 3.$$

**13.** Triangle, corners  $(0, 0)$ ,  $(3, 0)$ ,  $(0, 3)$ .

Vertical slicing:

$$\int_0^3 \int_0^{3-x} \frac{y}{x} \, dy \, dx = \int_0^3 \left[ \frac{y^2}{2x} \right]_0^{3-x} dx = \int_0^3 \frac{(3-x)^2}{2x} dx = \int_0^3 \frac{9}{2} \frac{1}{x} - 3 + \frac{1}{2}x \, dx = \left[ \frac{9}{2} \ln|x| - 3x + \frac{1}{4}x^2 \right]_0^3 = \infty.$$

Horizontal slicing:

$$\int_0^3 \int_0^{3-y} \frac{y}{x} \, dx \, dy = \int_0^3 \left[ y \ln|x| \right]_0^{3-y} dy = \int_0^3 \infty \, dy. \text{ This way it will not work.}$$

**14.** Vertical slicing:

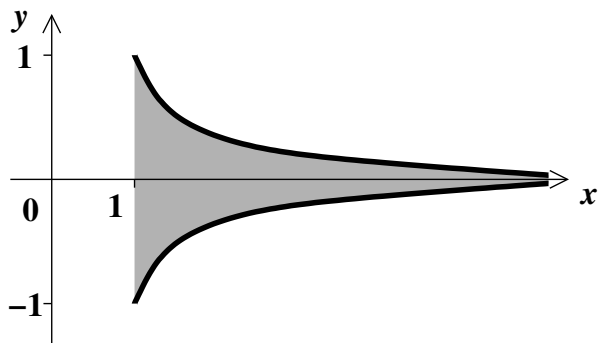
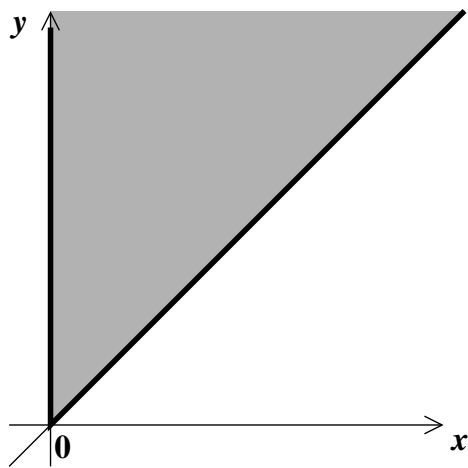
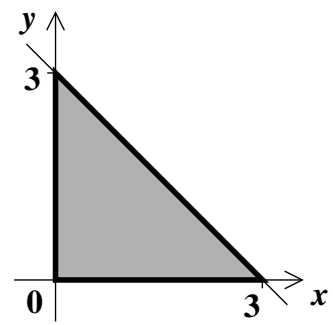
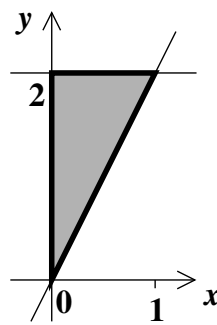
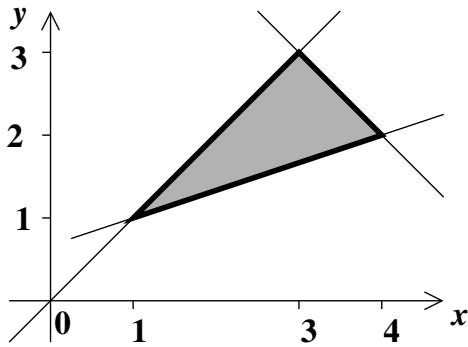
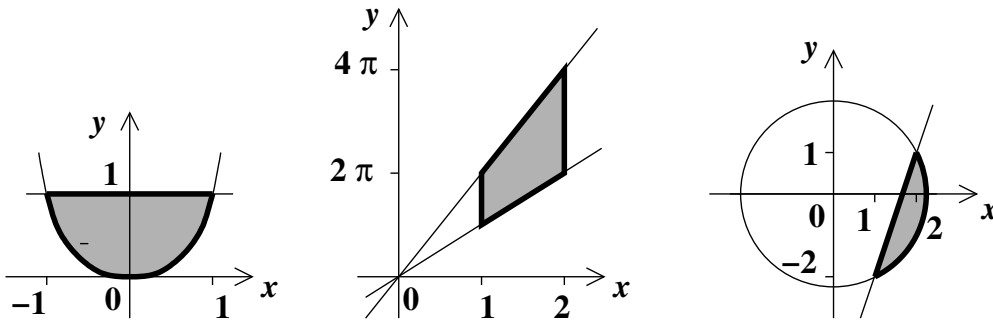
$$\int_0^\infty \int_x^\infty e^{-x-y} \, dy \, dx = \int_0^\infty \left[ -e^{-x-y} \right]_x^\infty dx = \int_0^\infty -0 + e^{-2x} \, dx = \left[ -\frac{1}{2}e^{-2x} \right]_0^\infty = \frac{1}{2} - 0 = \frac{1}{2}.$$

Horizontal slicing:

$$\int_0^\infty \int_0^y e^{-x-y} \, dx \, dy = \int_0^\infty \left[ -e^{-x-y} \right]_0^y dy = \int_0^\infty -e^{-2y} + e^{-y} \, dy = \left[ \frac{1}{2}e^{-2y} - e^{-y} \right]_0^\infty = 0 - \frac{1}{2} + 1 = \frac{1}{2}.$$

**15.** Vertical slicing:

$$\int_1^\infty \int_{-1/\sqrt{x}}^{1/\sqrt{x}} \frac{1}{x} \, dy \, dx = \int_1^\infty \left[ \frac{y}{x} \right]_{-1/\sqrt{x}}^{1/\sqrt{x}} dx = \int_1^\infty \frac{2}{x^{3/2}} \, dx = \left[ -\frac{4}{\sqrt{x}} \right]_1^\infty = -0 + 4 = 4.$$



For pictures of  $\Omega$ s from 16–21 see below.

$$16. \int_0^1 \int_{x^3}^{x^2} xy \, dy \, dx = \int_0^1 \left[ \frac{1}{2}xy^2 \right]_{x^3}^{x^2} dx = \int_0^1 \frac{1}{2}x^5 - \frac{1}{2}x^7 \, dx = \left[ \frac{1}{12}x^6 - \frac{1}{16}x^8 \right]_0^1 = \frac{1}{12} - \frac{1}{16} = \frac{1}{48}.$$

$$\int_0^1 \int_{\sqrt{y}}^{y^{1/3}} xy \, dx \, dy = \int_0^1 \left[ \frac{1}{2}x^2y \right]_{\sqrt{y}}^{y^{1/3}} dy = \int_0^1 \frac{1}{2}y^{5/3} - \frac{1}{2}y^2 \, dy = \left[ \frac{3}{16}y^{8/3} - \frac{1}{6}y^3 \right]_0^1 = \frac{1}{48}.$$

$$17. \int_0^{\pi/2} \int_{2x/\pi}^{\sin(x)} f(x, y) \, dy \, dx.$$

$$18. \int_0^1 \int_1^{1/y} f(x, y) \, dx \, dy.$$

$$19. \int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x, y) \, dx \, dy.$$

$$20. \int_0^1 \int_y^{2-y} f(x, y) \, dx \, dy.$$

$$21. \int_0^3 \int_{\sqrt{y/3}}^{y+1} f(x, y) \, dx \, dy.$$

