MA2: Solved problems—Functions of more variables: $D(f)$, graph, limit

1. Find and sketch the domain of the function $f(x,y)=\dfrac{\sqrt{2x-x^2-y^2}}{x-y}$.

2. Find and sketch the domain of the function $f(x,y)=\dfrac{\ln(y)}{2\ln(x)}$.
Find and sketch its level curves for values $c=0,\ \pm\frac{1}{4},\ \pm\frac{1}{2},\ \pm1,\ \pm2$.

3. Find and sketch the domain of the function $f(x,y)=3-\sqrt{4-x^2-y^2}$.
Identify the shape of the graph and sketch it.

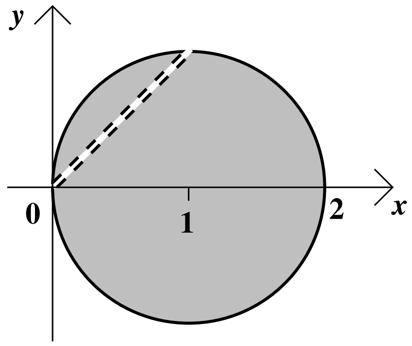
4. Evaluate limits of the function $f(x,y)=\dfrac{x^2y}{x^2+2y^2}$ at points $(1,2)$ and $(0,0)$.

5. Evaluate limits of the function $f(x,y)=\dfrac{x^3+y^3}{x^2-y^2}$ at points $(1,1)$, $(1,-1)$ and $(0,0)$.

Solutions:

1. Conditions for existence are $2x-x^2-y^2\geq 0$ and $x-y\neq 0$.
The first condition reads $x^2-2x+y^2\leq 0$. What is the object that it describes? Experience tells us that we can transform it into a standard equation of a circle by completing the square there. The completion goes $x^2-2\cdot 1\cdot x+1^2+y^2\leq 1$, that is, $(x-1)^2+y^2\leq 1$, which describes the circle of radius 1 and center $(1,0)$ including the perimeter.
The condition $y-x\neq 0$ means that the line $y=x$ is excluded. Thus the domain is the circle without the coresponding piece of the line.

$$D(f)=\{(x,y)\in I\!R^2;\ (x-1)^2+y^2\leq 1 \text{ and } y\neq x\}.$$



2. Three conditions for existence here: $y>0$, $x>0$, and $\ln(x)\neq 0$. The last condition reads $x\neq 1$. Thus we get the domain

$$D(f)=\{(x,y)\in I\!R^2;\ x,y>0 \text{ and } x\neq 1\}.$$

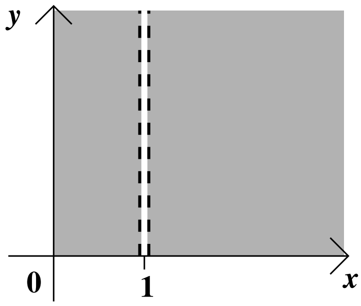
This is the first quadrant (without border) with the line $x=1$ removed (see picture below).
Level curves: We try it in general first: $f(x,y)=c$ reads

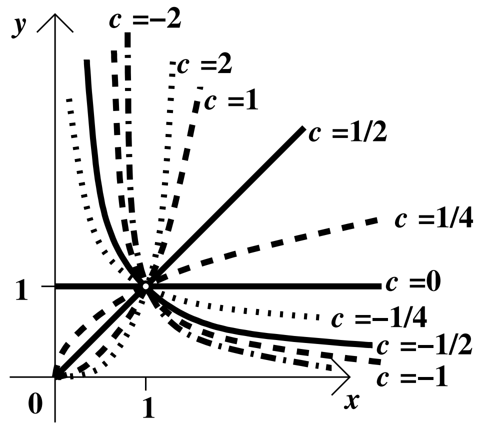
$$\frac{\ln(y)}{2\ln(x)}=c \quad\Longrightarrow\quad \ln(y)=2c\ln(x)=\ln(x^{2c}) \quad\Longrightarrow\quad y=x^{2c}.$$

So we get:

c:	-2	-1	$-\frac{1}{2}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$	1	2
level curve:	$y=\frac{1}{x^4}$	$y=\frac{1}{x^2}$	$y=\frac{1}{x}$	$y=\frac{1}{\sqrt{x}}$	$y=1$	$y=\sqrt{x}$	$y=x$	$y=x^2$	$y=x^4$

Thus the pictures of the domain and the level curves are



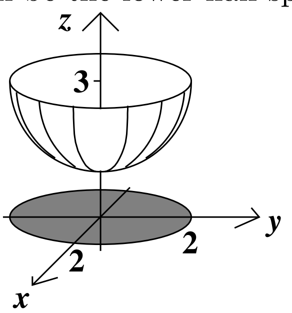


3. Condition for existence is $4 - x^2 - y^2 \geq 0$. This reads $x^2 + y^2 \leq 4$, that is, the domain is the disc centered at the origin and with radius 2:

$$D(f) = \{(x,y) \in I\!R^2;\ x^2 + y^2 \leq 4\}.$$

The graph of f is some surface given by the equation $z = f(x,y)$. When we substitute, we get $z = 3 - \sqrt{4 - x^2 - y^2}$, that is, $x^2 + y^2 + (z-3)^2 = 2^2$. This equation describes the sphere with radius 2 and center $(0,0,3)$.

However, every function has only one value per point in a domain, which means that we have to decide whether to consider the top or bottom part of this sphere. Since the values of our function are obtained by subtracting something positive from 3, we are interested in the part of the sphere that lies below 3, that is, the graph will be the lower half-sphere:



4. We always start with domain. It is determined by the condition $x^2 + 2y^2 \neq 0$, which reads $(x,y) \neq (0,0)$. Thus we have

$$D(f) = I\!R^2 - \{(0,0)\}.$$

a) Limit at $(1,2)$.

This point lies in the domain, thus it suffices to substitute it in.

$$\lim_{(x,y)\to(1,2)} \Big(\frac{x^2 y}{x^2 + 2y^2}\Big) = \frac{2}{9}.$$

b) Limit at $(0,0)$. This point does not lie in $D(f)$, but it is on domain's boundary, so the limit makes sense. After substituting we get

$$\lim_{(x,y)\to(0,0)} \Big(\frac{x^2 y}{x^2 + 2y^2}\Big) \overset{\frac{0}{0}}{=} ?$$

We have an indeterminate expression and no l'Hospital's rule for more variables.

The traditional approach calls for simplification of situation, we will try to approach the point $(0,0)$ along some simple paths. The simplest paths are along lines parallel to axes. We start parallel to the y-axis, which is done by fixing $x = 0$ and letting points of the type $(0,y)$ go towards $(0,0)$, so actually we do $y \to 0$.

$$\lim_{\substack{(x,y)\to(0,0)\\ x=0}} \Big(\frac{x^2y}{x^2+2y^2}\Big) = \lim_{y\to 0}\Big(\frac{0}{2y^2}\Big) = \lim_{y\to 0}(0) = 0.$$

We did not get an indeterminate expression, since we first substitute $x = 0$, then rearrange using algebra and only then we process the limit $y \to 0$.
Now parallel to x-axis, we fix $y = 0$.

$$\lim_{\substack{(x,y)\to(0,0)\\ y=0}} \Big(\frac{x^2y}{x^2+2y^2}\Big) = \lim_{x\to 0}(0) = 0.$$

We obtained the same answer, but this happens also with limits that do not exist, since direction along coordinate axes are often unusually good. We therefore try to approach $(0, 0)$ along general lines with slopes k, which is a very traditional thing to do. If points are to approach $(0, 0)$ along a line with slope k, then they have to satisfy the equation $y = kx$. After substituting we obtain

$$\lim_{\substack{(x,y)\to(0,0)\\ y=kx}} \Big(\frac{x^2y}{x^2+2y^2}\Big) = \lim_{x\to 0}\Big(\frac{kx^3}{x^2+2k^2x^2}\Big) = \lim_{x\to 0}\Big(\frac{kx}{1+2k^2}\Big) = 0.$$

So now we have a suspicion that the limit could be 0, but unfortunately what we did is not a proof, since we can approach the origin also along other paths. Parabolas $y = kx^2$ are also quite popular, but check that they also lead to limit zero.
It is obvious that we can never try **all** possible paths in this way, we need another idea. The given function cannot be reasonably transformed into a different expression, which rules out another popular trick. What is left? Let's have a closer look at the limit. What is actually happening when $(x, y) \to (0, 0)$? By definition this means that $\|(x, y)\| \to 0$. One possible approach is to compare parts of the fraction with the norm of the point.
In the denominator comparison is obvious: $x^2 + y^2 \le x^2 + 2y^2 \le 2x^2 + 2y^2$, therefore we have $\|(x, y)\|^2 \le x^2 + 2y^2 \le \sqrt{2}\|(x, y)\|^2$.
How about the numerator? We cannot really find a lower estimate by norm, even for points with a relatively large norm the expression x^2y can get rather small (just take $x = 0$). But we do have an upper estimate, if a norm is small, then also that expression must be small:

$$|x^2y| = x^2\sqrt{y^2} \le (x^2+y^2)\sqrt{y^2} \le (x^2+y^2)\sqrt{x^2+y^2} = \|(x,y)\|^3.$$

Thus we can estimate

$$\Big|\frac{x^2y}{x^2+2y^2}\Big| = \frac{|x^2y|}{x^2+2y^2} \le \frac{\|(x,y)\|^3}{\|(x,y)\|^2} = \|(x,y)\|.$$

Therefore, if we let $\|(x, y)\| \to 0$, then by the comparison theorem also $\frac{x^2y}{x^2+2y^2} \to 0$. Conclusion:

$$\lim_{(x,y)\to(0,0)} \Big(\frac{x^2y}{x^2+2y^2}\Big) = 0.$$

5. The domain is given by the condition $x^2 - y^2 \neq 0$, we have

$$D(f) = \{(x, y) \in I\!R^2;\ |x| \neq |y|\}.$$

That is, we take diagonals out of the plane.
a) Limit at $(1, 1)$. This point does not lie in $D(f)$, but it is on domain's boundary, so the limit makes sense. After substituting we get

$$\lim_{(x,y)\to(1,1)} \Big(\frac{x^3+y^3}{x^2-y^2}\Big) \stackrel{\frac{2}{0}}{=} \text{diverges}.$$

Is it possible that the limit diverges but exists, that is, could it be ∞ or $-\infty$? That depends on the sign of that 0 in the denominator. However, here it is clear that when the point (x, y) approaches $(1, 1)$ in the plane, then we can do it in such a way that $x > y > 1$, then the denominator becomes $\frac{2}{0^+} = \infty$, but it is also possible to go to $(1,1)$ so that $1 < x < y$, the the fraction goes $\frac{2}{0^-} = -\infty$. Conclusion:

$$\lim_{(x,y)\to(1,1)}\Big(\frac{x^3+y^3}{x^2-y^2}\Big) \text{ DNE.}$$

b) Limit at $(1,-1)$. This point does not lie in $D(f)$, but it is on domain's boundary, so the limit makes sense. After substituting we get

$$\lim_{(x,y)\to(1,-1)}\Big(\frac{x^3+y^3}{x^2-y^2}\Big) \overset{\frac{0}{0}}{=\!=} ?$$

An indeterminate expression, we therefore try the traditional approach and approach the point $(1,-1)$ along lines, first those parallel with axes. We start with $x=1$ ans $y\to -1$.

$$\lim_{\substack{(x,y)\to(1,-1)\\ x=1}}\Big(\frac{x^3+y^3}{x^2-y^2}\Big) = \lim_{y\to -1}\Big(\frac{1+y^3}{1-y^2}\Big) \underset{\text{l'H}}{\overset{\frac{0}{0}}{=\!=}} \lim_{y\to -1}\Big(\frac{3y^2}{-2y}\Big) = \lim_{y\to -1}\Big(\frac{3y}{-2}\Big) = \frac{3}{2}.$$

Now parallel with the x-axis, so we set $y=-1$.

$$\lim_{\substack{(x,y)\to(1,-1)\\ y=-1}}\Big(\frac{x^3+y^3}{x^2-y^2}\Big) = \lim_{x\to 1}\Big(\frac{x^3-1}{x^2-1}\Big) \underset{\text{l'H}}{\overset{\frac{0}{0}}{=\!=}} \lim_{x\to 1}\Big(\frac{3x^2}{2x}\Big) = \lim_{x\to 1}\Big(\frac{3x}{2}\Big) = \frac{3}{2}.$$

We have the same answer, but that need not mean anything. We therefore try to approach $(1,-1)$ along general lines with slopes k. Note that we can move (x,y) only through the domain, therefore we cannot use the line with slope $k=-1$. If points (x,y) approach $(1,-1)$ along the line N_k with slope k, then they satisfy the equation $y-(-1) = k(x-1)$. We therefore consider the situation when $x\to 1$ and we consider points $(x,y) = (x, kx-k-1)$ with $k\neq -1$. After substituting we get

$$\begin{aligned}\lim_{\substack{(x,y)\to(1,-1)\\ (x,y)\in N_k}}\Big(\frac{x^3+y^3}{x^2-y^2}\Big) &= \lim_{x\to 1}\Big(\frac{x^3+(kx-(k+1))^3}{x^2-(kx-(k+1))^2}\Big)\\ &= \lim_{x\to 1}\Big(\frac{x^3+k^3x^3-3k^2x^2(k+1)+3kx(k+1)^2-(k+1)^3}{x^2-k^2x^2-(k+1)^2+2kx(k+1)}\Big)\\ &= \frac{1+k^3-3k^2(k+1)+3k(k+1)^2-(k+1)^3}{1-k^2-(k+1)^2+2k(k+1)} \overset{\frac{0}{0}}{=\!=} ?\end{aligned}$$

Oh well, can we try something else? How about algebra? After all, both the numerator and the denominator can be factored.

$$\lim_{(x,y)\to(1,-1)}\Big(\frac{x^3+y^3}{x^2-y^2}\Big) = \lim_{(x,y)\to(1,-1)}\Big(\frac{(x+y)(x^2-xy+y^2)}{(x+y)(x-y)}\Big) = \lim_{(x,y)\to(1,-1)}\Big(\frac{x^2-xy+y^2}{x-y}\Big) = \frac{3}{2}.$$

So this limit converges.

c) Limit at $(0,0)$. This point does not lie in $D(f)$, but it is on domain's boundary, so the limit makes sense. After substituting we get

$$\lim_{(x,y)\to(0,0)}\Big(\frac{x^3+y^3}{x^2-y^2}\Big) \overset{\frac{0}{0}}{=\!=} ?$$

An indeterminate expression again. Will cancelling as above help?

$$\lim_{(x,y)\to(0,0)}\Big(\frac{x^3+y^3}{x^2-y^2}\Big) = \lim_{(x,y)\to(0,0)}\Big(\frac{x^2-xy+y^2}{x-y}\Big) \overset{\frac{0}{0}}{=\!=} ?$$

No. We try the traditional approach, we will go to the origin $(0,0)$ along lines. If points (x,y) approach $(0,0)$ along a line with slope k, then they satisfy the equation $y=kx$. Note that we can move points (x,y) only through the domain, so we cannot use lines with $k=\pm 1$. After substituting we get

$$\lim_{(x,y)\to(0,0)}\Big(\frac{x^3+y^3}{x^2-y^2}\Big) = \lim_{x\to 0}\Big(\frac{x^3+(kx)^3}{x^2-(kx)^2}\Big) = \lim_{x\to 0}\Big(\frac{x^3(1+k^3)}{x^2(1-k^2)}\Big) = \lim_{x\to 0}\Big(x\frac{1+k^3}{1-k^2}\Big) = 0.$$

We suspect that the limit is 0, but unfortunately our calculations do not constitute a proof, since we can approach the origin also along other paths. The parabolas $y = kx^2$ are also quite popular, but they also lead to limit zero (try it).
Can we rewrite the limit somehow?

$$\begin{aligned}\lim_{(x,y)\to(0,0)}\Big(\frac{x^3+y^3}{x^2-y^2}\Big) &= \lim_{(x,y)\to(0,0)}\Big(\frac{x^2-xy+y^2}{x-y}\Big) = \lim_{(x,y)\to(0,0)}\Big(\frac{(x-y)^2+xy}{x-y}\Big)\\ &= \lim_{(x,y)\to(0,0)}\Big(x-y+\frac{xy}{x-y}\Big) = \lim_{(x,y)\to(0,0)}(x-y) + \lim_{(x,y)\to(0,0)}\Big(\frac{xy}{x-y}\Big)\\ &= 0 + \lim_{(x,y)\to(0,0)}\Big(\frac{xy}{x-y}\Big) = \lim_{(x,y)\to(0,0)}\Big(\frac{xy}{x-y}\Big) \overset{\frac{0}{0}}{=} ?\end{aligned}$$

So far we have a suspicion that the limit could be zero. Is it really true that if we let (x, y) apporach the origin, then the expression xy gets small faster than the expression $x - y$? Actually, no. But for that we need to recall how the definition works. Let's say that we restrict ourselves to points from some δ-neighborhood of the origin, that is, $\|(x, y)\| < \delta$. The the coordinates x, y can be relatively large, for instance for points near the diagonal, x, y are close to $\frac{1}{\sqrt{2}}\delta$, then also their product is relatively large, namely about $\frac{1}{2}\delta^2$, while we have no restriction on the difference $x - y$, we can make it as small as we want (and then the whole fraction is arbitrarily large) by taking points really close to the diagonal. We cannot prevent this from happening by any condition of the form ,,let a point be close to the origin". Thus it seems that even if we restrict ourselves to a small neighborhood of the origin, we can still make the fraction ,,blow up".
So it is actually possible that the limit does not exist. To see that we need to design a path that would on its way to the origin quickly approach the diagonal, preferably much faster than it apporaches the origin. After some experimanting one can come up with the path $y = x + kx^2$, that it, $y - x$ approaches zero significantly faster than x and y themselves. Let's try it.

$$\begin{aligned}\lim_{\substack{(x,y)\to(0,0)\\ y=x+kx^2}}\Big(\frac{x^3+y^3}{x^2-y^2}\Big) &= \lim_{x\to 0}\Big(\frac{x^3+(x+kx^2)^3}{x^2-(x+kx^2)^2}\Big) = \lim_{x\to 0}\Big(\frac{x^3+x^3+3kx^4+3k^2x^5+k^3x^6}{x^2-x^2-2kx^3-k^2x^4}\Big)\\ &= \lim_{x\to 0}\Big(\frac{2x^3+3kx^4+3k^2x^5+k^3x^6}{-2kx^3-k^2x^4}\Big) = \lim_{x\to 0}\Big(\frac{2+3kx+3k^2x^2+k^3x^3}{-2k-k^2x}\Big) = -\frac{1}{k}.\end{aligned}$$

The result depends on the choice of k, therefore the limit $\lim\limits_{(x,y)\to(0,0)}\Big(\dfrac{x^3+y^3}{x^2-y^2}\Big)$ cannot exist.

By the way, the path $y = x + x^3$ is an interesting choice, now $y - x$ approaches zero even faster. Then

$$\begin{aligned}\lim_{\substack{(x,y)\to(0,0)\\ y=x+x^3}}\Big(\frac{x^3+y^3}{x^2-y^2}\Big) &= \lim_{x\to 0}\Big(\frac{x^3+(x+x^3)^3}{x^2-(x+x^3)^2}\Big) = \lim_{x\to 0}\Big(\frac{x^3+x^3+3x^5+3x^7+x^9}{x^2-x^2-2x^4-x^6}\Big)\\ &= \lim_{x\to 0}\Big(\frac{2x^3+3x^5+3x^7+x^9}{-2x^4-x^6}\Big) = \lim_{x\to 0}\Big(\frac{2+3x+3x^3+x^5}{-2x-x^3}\Big) \overset{\frac{2}{0}}{=} \text{DNE}.\end{aligned}$$