MA2: Solved problems—Functions of more variables: Derivative and geometry

1. Find the domain and all first order partial derivatives of the function

$$f(x,y,z)=\arcsin(xy)e^{3z}+(x+z)^{y+z}.$$

2. Find the domain and all second order partial derivatives of the function

$$f(x,y)=x^2y+(3x+5y)^2.$$

Find all derivatives directly, do not use any theorems.

3. Consider the function $f(x,y,z)=\dfrac{xyz}{\sqrt{x+y}}+e^{2x-z}$.
Find the gradient $\nabla f(1,0,2)$ and the total differential at $(1,0,2)$.

4. Consider the function $f(x,y)=xy^2+x^2y+2x-y$. We are at the point $(1,2)$.
a) In which direction does the function f decrease fastest? At what rate?
b) If we move towards the point $(3,2)$, at what rate will the values of function f start changing?

5. Find the equation of the tangent plane and the normal line at the point $(1,1,2)$ to the surface given by

$$z^2=7-x^2-2y^2.$$

6. Find the equation of the tangent line and the normal line at $(0,5\pi/6)$ to the curve given by

$$2\sin(y)-\sin(x)=1.$$

Write the equations in the classical form.

7. Use total differential to approximate $\ln(1.1)\cos(0.3)$.

8. Find the third order Taylor polynomial at $\vec{a}=(1,-1,1)$ of

$$f(x,y,z)=2xy^2+xz^3.$$

9. Let $f(x,y)$ be some function. Consider the transformation of variables $x=\sqrt{s}$, $y=t/s$.
a) Find $\dfrac{\partial f}{\partial s}$ and $\dfrac{\partial f}{\partial t}$.
b) Transform the expressions $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.
(That is, express them using partial derivatives of f with respect to s and t.)
c) Transform the expression $\dfrac{1}{2}\dfrac{\partial^2 f}{\partial x\partial y}$.

10. Consider the equation $\sin(xy)+x^2+y^2=1$.
a) Prove that on a neighborhood of $(0,1)$, this equation defines an implicit function $y(x)$.
b) Find the equation of the tangent line to the graph of this y at $(0,1)$.
c) Find $y''(0)$.

11. Consider the equation $\sin(xz)+\sin(yz)=\sin(xy)$.
a) Prove that on a neighborhood of $(0,1,\pi)$, this equation defines an implicit function $z(x,y)$.
b) Find the equation of the tangent plane to the graph of this z at $(0,1,\pi)$.
c) Find $\dfrac{\partial^2 z}{\partial x\partial y}(0,1,\pi)$.

Solutions:

1. To see the domain we have to first modify the general power in the second term:

$$f(x,y,z)=\arcsin(xy)e^{3z}+e^{(y+z)\ln(x+z)}.$$

Thus the conditions for existence are $-1 \le xy \le 1$ and $x+z>0$:

$$D(f)=\{(x,y,z)\in I\!R^3;\ -1\le xy\le 1 \text{ and } z>-x\}.$$

To get the partial derivative by x, we pretend that y,z are constants. What does it imply? In the first term we have a product, but its second part involves z only, therefore the whole exponential part is a constant and we can pull it out of the derivative. To differentiate the remaining arctangent we use the chain rule (composed function). In the second term we have the variable x only in the base, so we do not have to use the general power trick, the formula for $[x^a]'$ applies. However, it is not exactly this, we have a function in the base, so again we have to use the chain rule. We recall that the derivative of a constant is zero and $[Ax]'=A$.

$$\begin{aligned}\frac{\partial f}{\partial x}&=e^{3z}\frac{\partial}{\partial x}\big[\arcsin(xy)\big]+(y+z)(x+z)^{y+z-1}\frac{\partial}{\partial x}[x+z]\\&=\frac{e^{3z}}{\sqrt{1-(xy)^2}}\cdot\frac{\partial}{\partial x}[xy]+(y+z)(x+z)^{y+z-1}\cdot[1+0]=\frac{y\,e^{3z}}{\sqrt{1-x^2y^2}}+(y+z)(x+z)^{y+z-1}.\end{aligned}$$

Now we pretend that x,z are constants. The first term is handled similarly, but the second term is now of the form a^{y+b}, so the rule for derivative of general exponential must be used. We get

$$\begin{aligned}\frac{\partial f}{\partial y}&=e^{3z}\frac{\partial}{\partial y}\big[\arcsin(xy)\big]+\ln(x+z)(x+z)^{y+z}\frac{\partial}{\partial y}[y+z]\\&=\frac{e^{3z}}{\sqrt{1-(xy)^2}}\cdot\frac{\partial}{\partial y}[xy]+\ln(x+z)(x+z)^{y+z}\cdot[1+0]=\frac{x\,e^{3z}}{\sqrt{1-x^2y^2}}+\ln(x+z)(x+z)^{y+z}.\end{aligned}$$

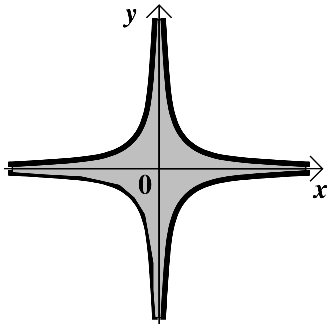
Finally we pretend that x,y are constants. In the second term now z is both in the base and the exponent of the power, so we have to use the general power transcription we used earlier in the domain part and differentiate the composed function $\frac{\partial}{\partial z}[e^{\varphi(z)}]=e^{\varphi(z)}\frac{\partial}{\partial z}[\varphi(z)]$:

$$\begin{aligned}\frac{\partial f}{\partial z}&=\arcsin(xy)\frac{\partial}{\partial z}(e^{3z})+e^{(y+z)\ln(x+z)}\cdot\frac{\partial}{\partial z}[(y+z)\ln(x+z)]\\&=\arcsin(xy)3e^{3z}+(x+z)^{y+z}\Big([1+0]\ln(x+z)+\frac{1}{x+z}\frac{\partial}{\partial z}[y+z]\Big)\\&=3\arcsin(xy)e^{3z}+(x+z)^{y+z}\Big(\ln(x+z)+\frac{y+z}{x+z}\Big).\end{aligned}$$

Remark concerning the domain (for inquisitive students):
What si the shape of the set $D(f)=\{(x,y,z)\in I\!R^3;\ -1\le xy\le 1 \text{ and } z>-x\}$?
Fortunately, there is no z in the first condition, so we can start by investigating what kind of shape it defines in the xy-plane. When deciphering the condition $-1\le xy\le 1$ we have to consider three possibilities:

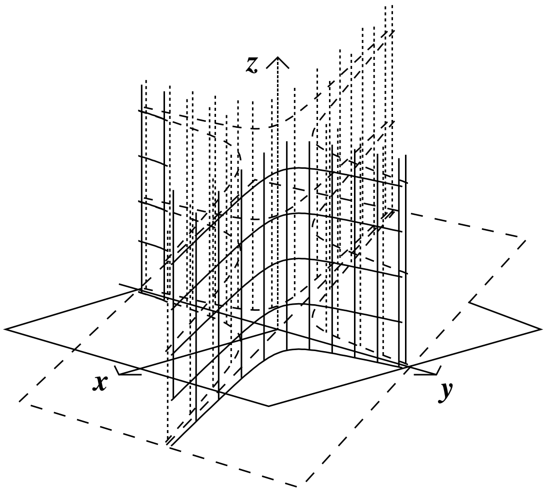
$$\begin{aligned}x=0:&\qquad -1\le 0\le 1 \text{ always} \implies \text{all } y \text{ work.}\\x>0:&\qquad -\tfrac{1}{x}\le y\le\tfrac{1}{x}\\x<0:&\qquad -\tfrac{1}{x}\ge y\ge\tfrac{1}{x}\implies\tfrac{1}{x}\le y\le-\tfrac{1}{x}\end{aligned}$$

Thus we get the following shape in $I\!R^2$:



However, the domain is an object in three dimensions, now we know that it is something that is erected over the set in the picture above. If there were no other condition, then z could be anything and we would obtain an infinite vertical "cylinder" with cross-section given by the shape above.

However, there is a condition, namely $z > -x$. This condition limits z from below depending on the position in the xy-plane. The equation $z = -x$ describes a plane tilted at 45 degrees, parallel with and going through the y-axis. Therefore the domain $D(f)$ is the part of the "infinite cylinder" that is above this plane.



2. Since there are no troubles in the formula, we get $D(f) = I\!R^2$.

To get the second order partial derivatives we first calculate the first order partial derivatives:

$$\frac{\partial f}{\partial x} = y\frac{\partial f}{\partial x}[x^2] + 2(3x+5y)\frac{\partial f}{\partial x}[3x+5y] = 2xy + 2(3x+5y)\cdot 3 = 2xy + 18x + 30y,$$
$$\frac{\partial f}{\partial y} = x^2\frac{\partial f}{\partial x}[y] + 2(3x+5y)\frac{\partial f}{\partial y}[3x+5y] = x^2 + 2(3x+5y)\cdot 5 = x^2 + 30x + 50y.$$

Thus

$$\frac{\partial^2 f}{\partial x^2} = \tfrac{\partial}{\partial x}\left[\tfrac{\partial f}{\partial x}\right] = \tfrac{\partial}{\partial x}[2xy + 18x + 30y] = 2y + 18,$$
$$\frac{\partial^2 f}{\partial y \partial x} = \tfrac{\partial}{\partial y}\left[\tfrac{\partial f}{\partial x}\right] = \tfrac{\partial}{\partial y}[2xy + 18x + 30y] = 2x + 30,$$
$$\frac{\partial^2 f}{\partial x \partial y} = \tfrac{\partial}{\partial x}\left[\tfrac{\partial f}{\partial y}\right] = \tfrac{\partial}{\partial x}[x^2 + 30x + 50y] = 2x + 30,$$
$$\frac{\partial^2 f}{\partial y^2} = \tfrac{\partial}{\partial y}\left[\tfrac{\partial f}{\partial y}\right] = \tfrac{\partial}{\partial y}[x^2 + 30x + 50y] = 50.$$

Note that this function has partial derivatives of all orders and they are continuous, so a theorem says that we must have $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$.

3. This is easy. First we find partial derivatives and then substitute into them the given point:

$$\frac{\partial f}{\partial x} = \frac{yz\sqrt{x+y} - \frac{xyz}{2\sqrt{x+y}}}{x+y} + 2e^{2x-z} \implies \frac{\partial f}{\partial x}(1,0,2) = 2,$$
$$\frac{\partial f}{\partial y} = \frac{xz\sqrt{x+y} - \frac{xyz}{2\sqrt{x+y}}}{x+y} \implies \frac{\partial f}{\partial y}(1,0,2) = 2,$$
$$\frac{\partial f}{\partial z} = \frac{xy}{\sqrt{x+y}} - e^{2x-z} \implies \frac{\partial f}{\partial z}(1,0,2) = -1.$$

Now we can form the gradient and the total differential:

$$\nabla f(0,2,0) = \operatorname{grad}(f)(0,2,0) = (2,2,-1),$$
$$\mathrm{d}f(1,0,2) = 2\mathrm{d}x + 2\mathrm{d}y - \mathrm{d}z \quad \text{or} \quad \mathrm{d}f(1,0,2)[\vec{h}] = 2h_1 + 2h_2 - h_3.$$

4. Both questions will require the knowledge of gradient, so we find the appropriate partial derivatives and work it out:

$$\left.\begin{aligned} \tfrac{\partial f}{\partial x} &= y^2 + 2xy + 2 \\ \tfrac{\partial f}{\partial y} &= 2xy + x^2 - 1 \end{aligned}\right\} \quad \Longrightarrow \quad \left.\begin{aligned} \tfrac{\partial f}{\partial x}(1,2) &= 10 \\ \tfrac{\partial f}{\partial y}(1,2) &= 4 \end{aligned}\right\} \quad \Longrightarrow \quad \nabla f(1,2) = (10,4).$$

The direction of the fastest decrease is $-\nabla f(1,2)$, that is, $(-10,-4)$. Since direction does not change if we multiply (or divide) a vector by a positive number, we can also give the direction $\frac{1}{2}(-10,-4) = (-5,-2)$ as the answer.
The rate of change in the direction of the gradient is $\|\nabla f(1,2)\| = \sqrt{10^2 + 4^2} = \sqrt{116} = 2\sqrt{29}$. In the opposite direction the rate of change will be the same in magnitude, but negative. So the rate of change in the direction $(-5,-2)$ is $-2\sqrt{29}$.
b) First we determine the vector describing our movement ("displacement") when we go from $(1,2)$ to $(3,2)$: $\vec{v} = (3,2) - (1,2) = (2,0)$. Next we find a vector in the same direction but of norm equal to one:

$$\vec{u} = \tfrac{\vec{v}}{\|\vec{v}\|} = \tfrac{1}{2}(2,0) = (1,0).$$

The answer to question b) is given by the appropriate directional derivative. The rate of change in the direction $\vec{u}$ is

$$f'_{\vec{u}}(1,2) = D_{\vec{u}}f(1,2) = \nabla f(1,2) \bullet \vec{u} = (10,4) \bullet (1,0) = 10.$$

5. We check that the given point $P = (1,1,2)$ satisfies the given equation, and so the question makes sense, the point indeed lies on the given curve. To find the given objects we first need to find a normal vector to the given surface at $(1,1,2)$. There are two approaches:
1) We can consider the surface to be a level curve $x^2 + 2y^2 + z^2 = 7$ of the function

$$f(x,y,z) = x^2 + 2y^2 + z^2.$$

We know that the gradient then yields a normal vector to a level curve. So we find the appropriate partial derivatives and substitute:

$$\tfrac{\partial f}{\partial x} = 2x, \qquad \tfrac{\partial f}{\partial y} = 4y, \qquad \tfrac{\partial f}{\partial z} = 2z$$
$$\Longrightarrow \qquad \tfrac{\partial f}{\partial x}(1,1,2) = 2, \qquad \tfrac{\partial f}{\partial y}(1,1,2) = 4, \qquad \tfrac{\partial f}{\partial z}(1,1,2) = 4.$$

Thus $\nabla f(1,1,2) = (2,4,4)$. Since for a normal vector we can take any multiple of the gradient, for simplicity we prefer to take $\vec{n} = \frac{1}{2}\nabla f(1,1,2) = (1,2,2)$.
2) We can also consider the surface to be the graph of one of the functions

$$z = -\sqrt{7 - x^2 - 2y^2}, \qquad z = +\sqrt{7 - x^2 - 2y^2}.$$

We have to determine which of the two functions applies to our problem, but this is easy, we quickly find that the given point $(1,1,2)$ does not satisfy the first equation, but the second is fine. So we consider the function $g(x,y) = \sqrt{7 - x^2 - 2y^2}$.
We know that a normal vector to the graph can be found as $\bigl(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, -1\bigr)$ evaluated at the point $(1,1)$. So we calculate:

$$\tfrac{\partial g}{\partial x} = \tfrac{-x}{\sqrt{7-x^2-2y^2}}, \qquad \tfrac{\partial g}{\partial y} = \tfrac{-2y}{\sqrt{7-x^2-2y^2}}$$
$$\Longrightarrow \qquad \bigl(\tfrac{\partial g}{\partial x}(1,1), \tfrac{\partial g}{\partial y}(1,1), -1\bigr) = \bigl(-\tfrac{1}{2}, -1, -1\bigr).$$

Since we can take for $\vec{n}$ any multiple of this vector (including negative now, opposite direction to a perpendicular vector is still perpendicular!), we prefer to multiply it by -2 and get $\vec{n} = (1,2,2)$

as in 1). Note that this way it was not so easy, which is typical. The level curve approach is usually more natural, sometimes the only possible.

Either way, we have the normal vector $\vec{n} = (1,2,2)$. The *normal line* is given by the parametric equation

$$\vec{z}(t) = P + t\vec{n} = (1,1,2) + t(1,2,2) = (1+t, 1+2t, 2+2t),\ t \in \mathbb{R}.$$

Since we are in three dimensions, the alternative way of determining a line is to express it as an intersection of two planes. It is not as elegant as the parametric expression, so we choose not to do it.

The *tangent plane* given by the normal vector $(1,2,2)$ must satisfy the equation

$$1 \cdot x + 2 \cdot y + 2 \cdot z = d$$

for some d. We find it by substituting the point $(1,1,2)$ into this equation, it is $d = 7$, so the equation is

$$x + 2y + 2z = 7.$$

Another possibility (my favourite) is to remember that that the equation of the plane perpendicular to $\vec{n}$ and going through the point P is given by

$$0 = \vec{n} \bullet [(x,y,z) - P] = 1 \cdot (x-1) + 2 \cdot (y-1) + 2 \cdot (z-2),$$

which of course leads to the same answer.

6. We check that the point $P = \big(0, \frac{5\pi}{6}\big)$ satisfies the given equality and therefore the question makes sense. As usual, first we need to find a normal vector $\vec{n}$. The curve is given by an equation with two variables, so we can consider it as a problem of finding $\vec{n}$ to a level curve given by the function

$$f(x,y) = 2\sin(y) - \sin(x).$$

From the theory we know that then one candidate for such a vector is the gradient of f, so we find partial derivatives and then substitute the point:

$$\frac{\partial f}{\partial x} = -\cos(x), \qquad \frac{\partial f}{\partial y} = 2\cos(y) \qquad \Longrightarrow \qquad \frac{\partial f}{\partial x}\big(0, \tfrac{5\pi}{6}\big) = -1, \qquad \frac{\partial f}{\partial y}\big(0, \tfrac{5\pi}{6}\big) = -2.$$

We get the gradient $\nabla f\big(0, \frac{5\pi}{6}\big) = (-1,-2)$. Since for the normal vector we can take any multiple of this gradient, we prefer for simplicity to take $\vec{n} = -\nabla f\big(0, \frac{5\pi}{6}\big) = (1,2)$.

Now it is easy to write the parametric equation of the *normal line*:

$$\vec{z}(t) = P + t\vec{n} = \big(0, \tfrac{5\pi}{6}\big) + t(1,2) = \big(t, \tfrac{5\pi}{6} + 2t\big).$$

To get the classical form, we eliminate t from the resulting equations $x = t$, $y = \frac{5\pi}{6} + 2t$, obtaining first $t = x$ and then

$$y = 2x + \tfrac{5\pi}{6}.$$

There are two ways to get the *tangent line*.

1) One approach is to see it as the line that is perpendicular to $\vec{n}$ and going through the point P. Such a line satisfies the equation

$$0 = \vec{n} \bullet [(x,y) - P] = 1 \cdot (x-0) + 2 \cdot \big(y - \tfrac{5\pi}{6}\big) \implies x + 2y = \tfrac{5\pi}{3}.$$

Some people remember that in this situation one has $1 \cdot x + 2 \cdot y = c$ for some c. This they identify by substituting the point $\big(0, \frac{5\pi}{6}\big)$ into this equation, obtaining $c = \frac{5\pi}{3}$.

2) The other approach is to start with a parametric equation of the tangent line. For this we first need to find some tangent vector at $\big(0, \frac{5\pi}{6}\big)$. Since we know a normal vector $\vec{n} = (1,2)$, we use the popular two-dimensional trick with switching coordinates and putting one minus somewhere, thus obtaining for instance $\vec{d} = (2,-1)$. Now we can write

$$\vec{z}(t) = P + t\vec{d} = \big(0, \tfrac{5\pi}{6}\big) + t(2,-1) = \big(2t, \tfrac{5\pi}{6} - t\big).$$

To get the classical form, we eliminate t from the resulting equations $x = 2t$, $y = \frac{5\pi}{6} - t$, obtaining first $t = x/2$ and then

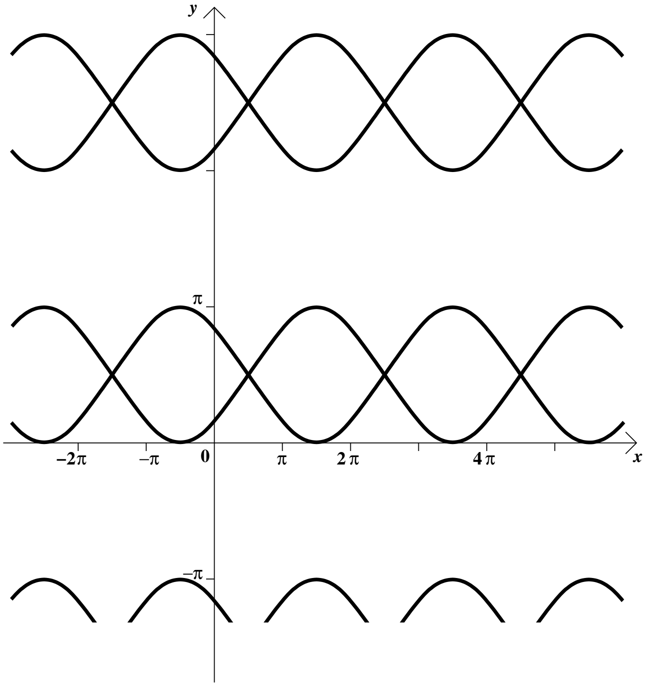
$$y = -\tfrac{1}{2}x + \tfrac{5\pi}{6}.$$

Remark: Check that the point $(\pi/2, \pi/2)$ also lies on the curve. What if we try to find the normal vector there?

Substituting into partial derivatives as above yields $\vec{n} = (0,0)$ and we cannot solve the problem. Why is this? It is caused by the fact that the curve described by the equation $2\sin(y) - \sin(x) = 1$ is very complicated, which is something that the above solution mercifully hid from us. It would become apparent if we tried to transform the problem from the "tangent to a level curve" type to the type "tangent to a graph of a function". When we try to isolate y from the equation, we get

$$\sin(y) = \tfrac{\sin(x)+1}{2}.$$

Given some x, this equation has infinitely many solutions for y (or perhaps no solution, if the number on the right did not fall into $[-1,1]$, but this does not happen here). In fact, from the fact that possible solutions x and y repeat with shifts by 2π it follows that the curve given by this equation is a certain specific shape which is repeated over and over in the x-direction and also in the y-direction with period 2π, and even the mirror image of it is included! This all follows from the question of how many solutions to the problem $\sin(y) = A$ we find and how they look like. The graph is like this:

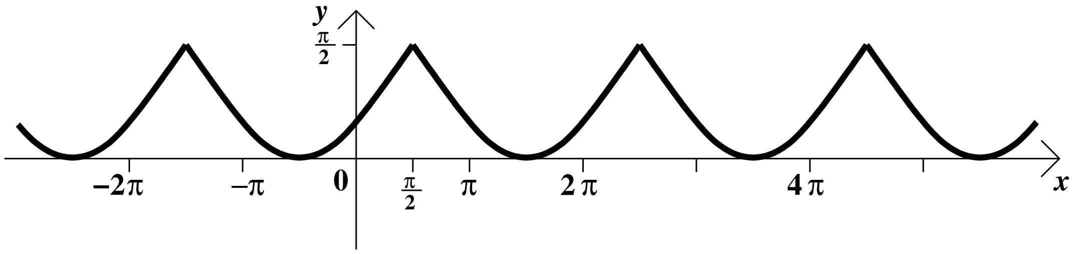


You immediately see that at the point $(0, 5\pi/6)$ the given question makes sense and can be solved, but at the point $(\pi/2, \pi/2)$ this curve passes in two directions, so there is no unique perpendicular direction and asking about tangent there does not make sense.

Note also that if you naively tried to express y using arcsine:

$$y = \arcsin\left(\tfrac{\sin(x)+1}{2}\right) = f(x),$$

you would get the "basic" curve that looks like a sea and whose copies and reflections compose the complete graph above:



This curve does not even contain the given point $(0, 5\pi/6)$, so the given problem cannot be solved using this f.

7. We set up the function $f(x,y)=\ln(x)\cos(y)$. The number $\ln(1.1)\cos(0.3)$ that we want to approximate can be now expressed as $f(1.1,0.3)$. We note that $(1.1,0.3)$ is very close to $(1,0)$ and the function f is much nicer at $(1,0)$, so we base our approaximation on this observation. The notion of total differential allows us to estimate

$$f(1.1,0.3)=f\big((1,0)+(0.1,0.3)\big)\sim f(1,0)+\mathrm{d}f(1,0)[(0.1,0.3)]=f(1,0)+\nabla f(1,0)\bullet(0.1,0.3).$$

We calculate

$$\nabla f(1,0)=\big(\tfrac{\cos(y)}{x},-\ln(x)\sin(y)\big)\big|_{(1,0)}=(1,0).$$

Thus

$$\ln(1.1)\cos(0.3)\sim\ln(1)\cos(0)+(1,0)\bullet(0.1,0.3)=0.1.$$

By the way, the precise value is $\ln(1.1)\cos(0.3)=0.091...$, so our approximation is pretty close.

8. We know that $T(\vec{x})=\sum\limits_{k=0}^{3}\frac{1}{k!}\mathrm{d}^kf(\vec{a})[\vec{x}-\vec{a}]=\sum\limits_{k=0}^{3}\frac{1}{k!}((\vec{x}-\vec{a})\bullet\nabla)^kf(\vec{a})$.

So we need to calculate those differential expressions, for simplicity we will first write $\vec{h}$ in place of $\vec{x}-\vec{a}$. We know that $\mathrm{d}^0f=f$, and

$$\begin{aligned}
\mathrm{d}f[\vec{h}]&=(\vec{h}\bullet\nabla)f=\big(h_x\tfrac{\partial}{\partial x}+h_y\tfrac{\partial}{\partial y}+h_z\tfrac{\partial}{\partial z}\big)f=\tfrac{\partial f}{\partial x}h_x+\tfrac{\partial f}{\partial y}h_y+\tfrac{\partial f}{\partial z}h_z\\
\mathrm{d}^2f[\vec{h}]&=(\vec{h}\bullet\nabla)^2f=\big(h_x\tfrac{\partial}{\partial x}+h_y\tfrac{\partial}{\partial y}+h_z\tfrac{\partial}{\partial z}\big)^2f\\
&=\big(h_x^2\tfrac{\partial^2}{\partial x^2}+h_y^2\tfrac{\partial^2}{\partial y^2}+h_z^2\tfrac{\partial^2}{\partial z^2}+2h_xh_y\tfrac{\partial^2}{\partial x\partial y}+2h_xh_z\tfrac{\partial^2}{\partial x\partial z}+2h_yh_z\tfrac{\partial^2}{\partial y\partial z}\big)f\\
&=\tfrac{\partial^2 f}{\partial x^2}h_x^2+\tfrac{\partial^2 f}{\partial y^2}h_y^2+\tfrac{\partial^2 f}{\partial z^2}h_z^2+2\tfrac{\partial^2 f}{\partial x\partial y}h_xh_y+2\tfrac{\partial^2 f}{\partial x\partial z}h_xh_z+2\tfrac{\partial^2 f}{\partial y\partial z}h_yh_z\\
\mathrm{d}^3f[\vec{h}]&=(\vec{h}\bullet\nabla)^3f=\big(h_x\tfrac{\partial}{\partial x}+h_y\tfrac{\partial}{\partial y}+h_z\tfrac{\partial}{\partial z}\big)^3f\\
&=\big(h_x^3\tfrac{\partial^3}{\partial x^3}+h_y^3\tfrac{\partial^3}{\partial y^3}+h_z^3\tfrac{\partial^3}{\partial z^3}+3h_x^2h_y\tfrac{\partial^3}{\partial x^2\partial y}+3h_xh_y^2\tfrac{\partial^3}{\partial x\partial y^2}+3h_x^2h_z\tfrac{\partial^3}{\partial x^2\partial z}\\
&\qquad+3h_xh_z^2\tfrac{\partial^3}{\partial x\partial z^2}+3h_y^2h_z\tfrac{\partial^3}{\partial y^2\partial z}+3h_yh_z^2\tfrac{\partial^3}{\partial y\partial z^2}+6h_xh_yh_z\tfrac{\partial^3}{\partial x\partial y\partial z}\big)f\\
&=\tfrac{\partial^3 f}{\partial x^3}h_x^3+\tfrac{\partial^3 f}{\partial y^3}h_y^3+\tfrac{\partial^3 f}{\partial z^3}h_z^3+3\tfrac{\partial^3 f}{\partial x^2\partial y}h_x^2h_y+3\tfrac{\partial^3 f}{\partial x\partial y^2}h_xh_y^2+3\tfrac{\partial^3 f}{\partial x^2\partial z}h_x^2h_z\\
&\qquad+3\tfrac{\partial^3 f}{\partial x\partial z^2}h_xh_z^2+3\tfrac{\partial^3 f}{\partial y^2\partial z}h_y^2h_z+3\tfrac{\partial^3 f}{\partial y\partial z^2}h_yh_z^2+6\tfrac{\partial^3 f}{\partial x\partial y\partial z}h_xh_yh_z
\end{aligned}$$

Now we calculate the necessary partial derivatives:

$$\tfrac{\partial f}{\partial x}=2y^2+z^3,\qquad \tfrac{\partial f}{\partial y}=4xy,\qquad \tfrac{\partial f}{\partial z}=3xz^2;$$

$$\tfrac{\partial^2 f}{\partial x^2}=0,\qquad \tfrac{\partial^2 f}{\partial y^2}=4x,\qquad \tfrac{\partial^2 f}{\partial z^2}=6xz,\qquad \tfrac{\partial^2 f}{\partial x\partial y}=4y,\qquad \tfrac{\partial^2 f}{\partial x\partial z}=3z^2,\qquad \tfrac{\partial^2 f}{\partial y\partial z}=0,$$

$$\tfrac{\partial^3 f}{\partial x^3}=\tfrac{\partial^3 f}{\partial x^2\partial y}=\tfrac{\partial^3 f}{\partial x^2\partial z}=\tfrac{\partial^3 f}{\partial x\partial y\partial z}=0,\qquad \tfrac{\partial^3 f}{\partial y^2\partial x}=4,\qquad \tfrac{\partial^3 f}{\partial y^3}=\tfrac{\partial^3 f}{\partial y^2\partial z}=0,$$

$$\tfrac{\partial^3 f}{\partial z^2\partial x}=6z,\qquad \tfrac{\partial^3 f}{\partial z^2\partial y}=0,\qquad \tfrac{\partial^3 f}{\partial z^3}=6x.$$

We substitute the point $\vec{a}=(1,-1,1)$ into these partial derivatives and then put them into the above formulas for total differentials:

$$\begin{aligned}
\mathrm{d}^0 f(1,-1,1)[\vec{h}] &= f(1,-1,1) = 3,\\
\mathrm{d} f(1,-1,1)[\vec{h}] &= 3h_x - 4h_y + 3h_z,\\
\mathrm{d}^2 f(1,-1,1)[\vec{h}] &= 4h_y^2 + 6h_z^2 - 8h_x h_y + 6h_x h_z,\\
\mathrm{d}^3 f(1,-1,1)[\vec{h}] &= 6h_z^3 + 18h_x h_z^2 + 12h_y^2 h_z.
\end{aligned}$$

Finally, we substitute $\vec{h} = (x-1, y+1, z-1)$ and form the Taylor polynomial:

$$\begin{aligned}
T(x,y,z) = 3 + 3(x-1) - 4(y+1) + 3(z-1) + 2(y+1)^2 + 3(z-1)^2 - 4(x-1)(y+1)\\
+ 3(x-1)(z-1) + (z-1)^3 + 3(x-1)(z-1)^2 + 2(y+1)^2(z-1).
\end{aligned}$$

9. After transformation we obtain a new function $f(s,t)$.

a) In order to differentiate with respect to s, we have to realize that in the expression $f(x,y)$, the variable s is included in both x and y after we do the transformation, that is, we have $f(x(s,t), y(s,t))$. The general chain rule says that we have to explore both possibilities of getting to s. Similarly we differentiate with respect to t.

$$\begin{aligned}
\frac{\partial}{\partial s} f(x,y) &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} = \frac{\partial f}{\partial x}\frac{1}{2\sqrt{s}} + \frac{\partial f}{\partial y}\frac{-t}{s^2},\\
\frac{\partial}{\partial t} f(x,y) &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = 0 + \frac{\partial f}{\partial y}\frac{1}{s}.
\end{aligned}$$

b) There are two possibilities to solve this question.

1) Both derivatives that we look for are included in the above equations

$$\begin{aligned}
\frac{\partial f}{\partial s} &= \frac{1}{2\sqrt{s}}\frac{\partial f}{\partial x} - \frac{t}{s^2}\frac{\partial f}{\partial y}\\
\frac{\partial f}{\partial t} &= \frac{1}{s}\frac{\partial f}{\partial y}
\end{aligned}$$

and we can solve this system for them, obtaining

$$\frac{\partial f}{\partial y} = s\frac{\partial f}{\partial t}, \qquad \frac{\partial f}{\partial x} = 2\sqrt{s}\frac{\partial f}{\partial s} + \frac{2t}{\sqrt{s}}\frac{\partial f}{\partial t}.$$

2) The second approach goes by reversing the procedure from part a). We start with function f depending on s,t and we transform them into x,y. Mathematically, we consider the inverse transformation to the one we were given, instead of $(s,t) \mapsto (x,y)$ we will want to go $(s,t) \mapsto (x,y)$. This can be done easily by solving the system $x = \sqrt{s}$, $y = t/s$ for s and t, we obtain the formulas $s = x^2$, $t = x^2 y$.

Now we proceed as before using the chain rule.

$$\begin{aligned}
\frac{\partial}{\partial x} f(s,t) &= \frac{\partial f}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial f}{\partial t}\frac{\partial t}{\partial x} = \frac{\partial f}{\partial s}2x + \frac{\partial f}{\partial t}2xy\\
\frac{\partial}{\partial y} f(s,t) &= \frac{\partial f}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial f}{\partial t}\frac{\partial t}{\partial y} = 0 + \frac{\partial f}{\partial t}x^2.
\end{aligned}$$

People often prefer to have only s and t on the right in this situation, but this is easy using the given transformations.

$$\begin{aligned}
\frac{\partial}{\partial x} f &= 2\sqrt{s}\frac{\partial f}{\partial s} + \frac{2t}{\sqrt{s}}\frac{\partial f}{\partial t}\\
\frac{\partial}{\partial y} f &= s\frac{\partial f}{\partial t}.
\end{aligned}$$

We got the same answer as before.

The first approach seems easier, but the second approach will come handy in part c). However, note that it depends on us being able to solve the two equations $x = x(s,t)$, $y = y(s,t)$ for s,t, which is not always possible.

c) Again, there are two possibilities corresponding to those in part b). One possibility is to take the deduced derivative by y and differentiate it with respect to x. We know that

$$\frac{\partial f}{\partial y}=x^2\frac{\partial f}{\partial t}$$

(note that we chose the version with x^2, it makes the next calculation easier) and now we differentiate by x. Note that $\frac{\partial f}{\partial t}$ depends on s,t, which in turn depend on x,y. Thus we have a product of two functions with x in them and we have to start with the product rule. Then we apply the chain rule to $\frac{\partial f}{\partial t}(s,t)$:

$$\begin{aligned}\frac{\partial^2 f}{\partial x\partial y}&=\frac{\partial}{\partial x}\Big(\frac{\partial f}{\partial y}\Big)=\frac{\partial}{\partial x}\Big(x^2\cdot\frac{\partial f}{\partial t}\Big)=\frac{\partial}{\partial x}\big(x^2\big)\cdot\frac{\partial f}{\partial t}+x^2\cdot\frac{\partial}{\partial x}\Big(\tfrac{\partial f}{\partial t}(s,t)\Big)\\&=2x\frac{\partial f}{\partial t}+x^2\Big[\frac{\partial}{\partial s}\Big(\frac{\partial f}{\partial t}\Big)\cdot\frac{\partial s}{\partial x}+\frac{\partial}{\partial t}\Big(\frac{\partial f}{\partial t}\Big)\cdot\frac{\partial t}{\partial x}\Big]\\&=2x\frac{\partial f}{\partial t}+x^2\Big[\frac{\partial^2 f}{\partial s\partial t}2x+\frac{\partial^2 f}{\partial t^2}2xy\Big]=2x\frac{\partial f}{\partial t}+2x^3\frac{\partial^2 f}{\partial s\partial t}+2x^3y\frac{\partial^2 f}{\partial t^2}.\end{aligned}$$

To complete the question we have to substitute for x and y and get the transformation

$$\frac{1}{2}\frac{\partial^2 f}{\partial x\partial y}=\sqrt{s}\frac{\partial f}{\partial t}+\big(\sqrt{s}\big)^3\frac{\partial^2 f}{\partial s\partial t}+t\sqrt{s}\frac{\partial^2 f}{\partial t^2}.$$

2) What is the alternative? An approach that is longer but safer (we may not be able to find those inverse transsforms) is similar to b1). We start with the equations deduced in part a), in fact we may use just the second one,

$$\frac{\partial f}{\partial t}=\frac{1}{s}\frac{\partial f}{\partial y}$$

and now we differentiate it with respect to s and also with respect to t, obtaining two equations.

$$\begin{aligned}\frac{\partial^2 f}{\partial s\partial t}&=\frac{-1}{s^2}\frac{\partial f}{\partial y}+\frac{1}{s}\frac{\partial^2 f}{\partial x\partial y}\frac{\partial x}{\partial s}+\frac{1}{s}\frac{\partial^2 f}{\partial y^2}\frac{\partial y}{\partial s}\\\frac{\partial^2 f}{\partial t^2}&=\frac{1}{s}\frac{\partial^2 f}{\partial x\partial y}\frac{\partial x}{\partial t}+\frac{1}{s}\frac{\partial^2 f}{\partial y^2}\frac{\partial y}{\partial t}\end{aligned}$$

that is, after simplification and substitution for $\frac{\partial f}{\partial y}$ from part 1),

$$\begin{aligned}\frac{\partial^2 f}{\partial s\partial t}&=\frac{-1}{s^2}s\frac{\partial f}{\partial t}+\frac{1}{s}\frac{\partial^2 f}{\partial x\partial y}\frac{1}{2\sqrt{s}}+\frac{1}{s}\frac{\partial^2 f}{\partial y^2}\frac{-t}{s^2}\\\frac{\partial^2 f}{\partial t^2}&=\frac{1}{s}\frac{\partial^2 f}{\partial y^2}\frac{1}{s}\end{aligned}$$

We solve this system for $\frac{\partial^2 f}{\partial x\partial y}$, there is another unknown derivative, namely $\frac{\partial^2 f}{\partial y^2}$, but we have two equations, so this is no problem, we eliminate. We get

$$\frac{\partial^2 f}{\partial x\partial y}=2\sqrt{s}\frac{\partial f}{\partial t}+2\sqrt{s^3}\frac{\partial^2 f}{\partial s\partial t}+t2\sqrt{s}\frac{\partial^2 f}{\partial t^2}$$

Dividing by 2 we get exactly the same answer as before.

Note that we were lucky that in the equation for $\frac{\partial f}{\partial t}$, only one partial derivative remained on the right. In general we can expect both equations in part 1) to feature derivatives by x and y, like the first equation does. The we would need to differentiate both equations by both s and t, obtaining four equations with four unknown partial derivatives $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$, $\frac{\partial^2 f}{\partial x\partial y}$, and $\frac{\partial^2 f}{\partial y\partial x}$, the system then would have to be solved. So indeed, this procedure is longer, but as we remarked before, more general.

Bonus: We will find $\frac{\partial^2 f}{\partial y\partial x}$. We start with

$$\frac{\partial f}{\partial x} = 2x\frac{\partial f}{\partial s} + 2xy\frac{\partial f}{\partial t}.$$

and then calculate (similarly as above)

$$\begin{aligned}\frac{\partial^2 f}{\partial y\partial x} &= \frac{\partial}{\partial y}\Big(2x\frac{\partial f}{\partial s} + 2xy\frac{\partial f}{\partial t}\Big) = 2x\frac{\partial}{\partial y}\Big(\frac{\partial f}{\partial s}\Big) + 2x\frac{\partial f}{\partial t} + 2xy\frac{\partial}{\partial y}\Big(\frac{\partial f}{\partial t}\Big)\\ &= 2x\Big[\frac{\partial^2 f}{\partial s^2}\frac{\partial s}{\partial y} + \frac{\partial^2 f}{\partial t\partial s}\frac{\partial t}{\partial y}\Big] + 2x\frac{\partial f}{\partial t} + 2xy\Big[\frac{\partial^2 f}{\partial s\partial t}\frac{\partial s}{\partial y} + \frac{\partial^2 f}{\partial t^2}\frac{\partial t}{\partial y}\Big]\\ &= 2x^3\frac{\partial^2 f}{\partial t\partial s} + 2x\frac{\partial f}{\partial t} + 2x^3 y\frac{\partial^2 f}{\partial t^2}.\end{aligned}$$

Thus

$$\frac{1}{2}\frac{\partial^2 f}{\partial y\partial x} = \sqrt{s}\frac{\partial f}{\partial t} + \big(\sqrt{s}\big)^3\frac{\partial^2 f}{\partial t\partial s} + t\sqrt{s}\frac{\partial^2 f}{\partial t^2}.$$

If f is “nice”, the two mixed derivatives should be the same.

10. We have $F(x,y) = \sin(xy) + x^2 + y^2$. We check that $F(0,1) = 1$, so the question makes sense.
a) $\frac{\partial F}{\partial y}(0,1) = x\cos(xy) + 2y\big|_{x=0,y=1} = 2$. Since $\frac{\partial F}{\partial y}(0,1) \neq 0$, by the Implicit Function Theorem there is a function $y(x)$ defined on some neighborhood of $x = 0$ such that $y(0) = 1$ and $F(x,y(x)) = 0$.
b) We can use the Implicit Function Theorem to find $y'(x)$, but it is easier just to differentiate the given equation, keeping in mind that now y is a function of x. We get

$$\begin{aligned}\sin(xy) + x^2 + y^2 &= 1\\ [\sin(xy(x))]' + [x^2]' + [y(x)^2]' &= [1]'\\ \cos(xy)\cdot[y + xy'] + 2x + 2y\cdot y' &= 0. \qquad (\star)\end{aligned}$$

Substituting $(0,1)$ into it we get $\cos(0)\cdot[1 + 0\cdot y'(0)] + 0 + 2\cdot y'(0) = 0$, solving for $y'(0)$ we get $y'(0) = -\frac{1}{2}$.
We have the point $(0,1)$ and the slope $y'(0) = -\frac{1}{2}$, consequently the equation of the tangent line is $y = -\frac{1}{2}(x - 0) + 1$, that is, $x + 2y = 2$.
c) There are two possibilities. One is to take another derivative of the equation $(\star)$:

$$-\sin(xy)[y + xy']\cdot[y + xy'] + \cos(xy)[y' + y' + xy''] + 2 + 2y'y' + 2yy'' = 0.$$

Substituting $x = 0$, $y = 1$, and $y' = -\frac{1}{2}$, then solving for $y''(0)$ we get $y''(0) = -\frac{3}{4}$.
Alternative solution: We can solve $(\star)$ in general for y':

$$y'(x) = -\frac{2x + \cos(xy)y}{2y + \cos(xy)x}$$

and then take the derivative:

$$\begin{aligned}y''(x) &= -\frac{[2x + \cos(xy)y]'\cdot[2y + \cos(xy)x] - [2x + \cos(xy)y]\cdot[2y + \cos(xy)x]'}{[2y + \cos(xy)x]^2}\\ &= -\frac{2 - \sin(xy)(y + xy')y + \cos(xy)y'}{2y + \cos(xy)x} + \frac{(2x + \cos(xy)y)(2y' - \sin(xy)(y + xy')x + \cos(xy))}{[2y + \cos(xy)x]^2}.\end{aligned}$$

Then we put in $x = 0$, $y = 1$, $y' = -\frac{1}{2}$ and it is done.
Obviously the first solution is preferable.

11. The equation can be written as $F(x,y,z) = 0$, where

$$F(x,y,z) = \sin(xz) + \sin(yz) - \sin(xy).$$

a) For the function $z(x,y)$ to exist, we need $\frac{\partial F}{\partial z}(0,1,\pi)\neq 0$. Here $\frac{\partial F}{\partial z}=x\cos(xz)+y\cos(yz)$, so $\frac{\partial F}{\partial z}(0,1,\pi)=-1\neq 0$, and the Implicit Function Theorem does the rest.
b) The given equation defines a level surface of F, so the normal vector can be found using gradient of F. We have $\nabla F=\big(-y\cos(xy)+z\cos(yz),-x\cos(xy)+z\cos(yz),x\cos(xz)+y\cos(yz)\big)$, so $\vec{n}=\nabla F(0,1,\pi)=(\pi-1,-\pi,-1)$.
The tangent plane is given by $(\pi-1)x-\pi(y-1)-(z-\pi)=0$, that is,

$$(\pi-1)x-\pi y-z+2\pi=0.$$

Alternative solution: We will treat it as the question of finding the tangent plane to the graph of $z=z(x,y)$. The normal vector is then given as $(z_x,z_y,-1)$. To find the partial derivatives, we differentiate the given equation by x and by y, remembering that now $z=z(x,y)$:

$$\begin{aligned}\cos(xz)[z+xz_x]+\cos(yz)yz_x&=\cos(xy)y\\ \cos(xz)xz_y+\cos(yz)[z+yz_y]&=\cos(xy)x\end{aligned}$$

We substitute in $(0,1,\pi)$ and get $z_x(0,1)=\pi-1$, $z_y(0,1)=-\pi$, thus $\vec{n}=(\pi-1,-\pi,-1)$ as we had before.
Note that we used the handy shortcut $z_x=\frac{\partial z}{\partial x}$, $z_y=\frac{\partial z}{\partial y}$ to simplify the writing.
c) To find the second partial derivative, it is easiest to use the alternative solution of b). There we differentiated the original equation by x, so we now differentiate that result by y to get the desired z_{xy}. Again, we have to remember that $z=z(x,y)$, but now also $z_x=z_x(x,y)$.

$$\big[\cos(xz)[z+xz_x]+\cos(yz)yz_x\big]_y=\big[\cos(xy)y\big]_y \quad\Longrightarrow$$
$$\begin{aligned}-\sin(xz)xz_y[z+xz_x]+\cos(xz)[z_y+xz_{xy}]-\sin(yz)[z+yz_y]yz_x+\cos(yz)z_x+\cos(yz)yz_{xy}\\ =-\sin(xy)xy+\cos(xy).\end{aligned}$$

We substitute in the point $(0,1,\pi)$ and also $z_x(0,1)=\pi-1$, $z_y(0,1)=-\pi$ and solve the resulting equation to obtain $z_{xy}(0,1)=2-2\pi$.
Of course, we can also take the equation we obtained in b) by differentiating the given equation with respect to y and then differentiate it by x, obtaining the same answer.