MA2: Solved problems—Functions of more variables: Extrema

1. Find and identify local extrema of $f(x,y)=2x^3+9xy^2+15x^2+27y^2$.

2. Find and identify local extrema of $f(x,y,z)=x^3-2x^2+y^2+z^2-2xy+xz-yz+3z$.

3. Find the global extrema of $f(x,y)=x^2+2y^2$ given the condition

$$x^2-2x+2y^2+4y=0.$$

4. Find the point on the plane given by $x+y-z=1$ that is closest to the point $P=(0,-3,2)$ and calculate their distance. Use Lagrange multipliers.

5. A certain line in 3D is given by the equations

$$x+y+z=1, \qquad 2x-y+z=3.$$

Find the distance between this line and the point $P=(1,2,-1)$.

6. Find the global extrema of $f(x,y)=x^2+4y^2$ on the finite region M bounded by the curves $x^2+(y+1)^2=4$, $y=-1$ and $y=x+1$.

7. Find the global extrema of $f(x,y)=x^2+y^2-6x+6y$ on the disk of radius 2, centred at the origin.

8. The equation $y^2+2xy=2x-4x^2$ defines an implicit function $y(x)$. Find and classify its local extrema.

Solutions:

1. First we find stationary points. Partial derivatives:

$$\frac{\partial f}{\partial x}=6x^2+9y^2+30x, \qquad \frac{\partial f}{\partial y}=18xy+54y.$$

We have to make them equal to zero. We get the system

$$2x^2+3y^2+10x=0 \qquad xy+3y=0.$$

The second equation looks promising, since we can write it as $y(x+3)=0$. Thus there are two possibilities:
1) $y=0$. Then the first equation reads $x^2+5x=0$, which yields $x=0$ and $x=-5$. This possibility therefore leads to points $(0,0)$ and $(-5,0)$
2) $x=-3$. Then the first equation reads $y^2=4$, which yields $y=\pm 2$ and points $(-3,\pm 2)$.
Thus we obtain four stationary points: $(0,0)$, $(-5,0)$, $(-3,2)$, and $(-3,-2)$.
To classify them we need to find second order partial derivatives and form the Hess matrix:

$$\frac{\partial^2 f}{\partial x^2}=12x+30, \qquad \frac{\partial^2 f}{\partial x\partial y}=18y, \qquad \frac{\partial^2 f}{\partial y^2}=18x+54$$

$$H=\begin{pmatrix}12x+30 & 18y\\ 18y & 18x+54\end{pmatrix}$$

Now the classification.
For $(0,0)$ we get $H=\begin{pmatrix}30 & 0\\ 0 & 54\end{pmatrix}$. Determinants of principal minors (subdeterminants) are $\Delta_1=a_{11}=30$ and $\Delta_2=\det(H)=30\cdot 54=1620$. Their signs are $\Delta_1>0$, $\Delta_2>0$, which shows that the point $f(0,0)=0$ is a local minimum.
For $(-5,0)$ we get $H=\begin{pmatrix}-30 & 0\\ 0 & -26\end{pmatrix}$. Subdeterminants are $\Delta_1=-30$ and $\Delta_2=780$. Their signs are $\Delta_1<0$, $\Delta_2>0$, which shows that the point $f(-5,0)=125$ is a local maximum.
For $(-3,-2)$ we get $H=\begin{pmatrix}-6 & -36\\ -36 & 0\end{pmatrix}$. Subdeterminants are $\Delta_1=-6$ and $\Delta_2=-(-36)^2$. Their signs are $\Delta_1<0$, $\Delta_2<0$, this does not follow pattern for any local extreme. But from $\Delta_2<0$ we conclude that the point $f(-3,2)=81$ is a saddle point.

For $(-3,2)$ we get $H=\begin{pmatrix}-6 & -36\\ -36 & 0\end{pmatrix}$. Subdeterminants are $\Delta_1=-6$ and $\Delta_2=-36^2$. As above, from $\Delta_2<0$ we conclude that the point $f(-3,-2)=81$ is a saddle point.

Some people prefer a different approach that might be simpler if the derivatives are not too bad, it is also somewhat more organized.
First we evaluate those subdeterminants in general, we obtain $\Delta_1=12x+30$ and $\Delta_2=(12x+30)(18x+54)-(18y)^2=36(6x^2+23x+45-9y^2)$. Then we substitute the stationary points and reach conclusions:

point:	$(0,0)$	$(-5,0)$	$(-3,2)$	$(-3,-2)$
Δ_1:	$+$	$-$	$-$	$-$
Δ_2:	$+$	$+$	$-$	$-$
conclusion:	loc. min.	loc. max.	saddle	saddle

Then one has to write the answer: $f(0,0)=0$ is a local minimum, $f(-5,0)=125$ is a local maximum, $f(-3,2)=f(-3,-2)=81$ are saddle points.

2. First we find stationary points. Partial derivatives:
$\frac{\partial f}{\partial x}=3x^2-4x-2y+z,\qquad \frac{\partial f}{\partial y}=2y-2x-z,\qquad \frac{\partial f}{\partial z}=2z+x-y+3.$
We have to solve the system

$$\begin{gathered}3x^2-4x-2y+z=0\\ 2y-2x-z=0\\ 2z+x-y+3=0\end{gathered}$$

Now noe of the equations has the convenient form of a product, so the method used in the previous problem does not help. Another popular method is elimination.
Since there is x^2 in the first equation, we will try to use the others to get rid of y and z in this first equation and then apply the quadratic rule. We can express $z=2y-2x$ from the second equation and put into the first and the third, obtaining $3x^2-6x=0$ and $3y-3x=-3$. What a piece of luck, the first one already features only x, the third one will also come handy when we express $y=x-1$.
The equation $3x^2-6x=0$ has two solutions: $x=0$ and $x=2$.
If $x=0$, then $y=-1$ and $z=-2$. If $x=2$, then $y=1$ and $z=-2$. Thus we have two stationary points, $(0,-1,-2)$ and $(2,1,-2)$.
Now we use the second derivative test. First we need second partial derivatives arranged into the Hess matrix.

$$H=\begin{pmatrix}6x-4 & -2 & 1\\ -2 & 2 & -1\\ 1 & -1 & 2\end{pmatrix}.$$

Calculating subdeterminant in general does not sound very appealing (but you can try this approach), we handle each point separately.
For $(0,-1,-2)$ we get

$$H=\begin{pmatrix}-4 & -2 & 1\\ -2 & 2 & -1\\ 1 & -1 & 2\end{pmatrix}\implies \Delta_1=-4,\ \Delta_2=\begin{vmatrix}-4 & -2\\ -2 & 2\end{vmatrix}=-12,\ \Delta_3=|H|=-26.$$

Since $\Delta_2<0$, at the stationary point $(0,-1,-2)$ there is no local extreme but a saddle point.
Better answer: $f(0,-1,2)=13$ is a saddle point (we give more information this way).
(Some authors do not use the notion of saddle in cases of more than two variables, they would just say that this point is not a local extreme.)

For $(2,1,-2)$ we get

$$H = \begin{pmatrix} 8 & -2 & 1 \\ -2 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} \quad \Longrightarrow \quad \Delta_1 = 8,\ \Delta_2 = \begin{vmatrix} 8 & -2 \\ -2 & 2 \end{vmatrix} = 12,\ \Delta_3 = |H| = 28.$$

Since always $\Delta_i > 0$, we conclude that $f(2,1,-2) = -7$ is a local minimum.
Recall that for a local maximum we need $\Delta_1 < 0$, $\Delta_2 > 0$, and $\Delta_3 < 0$.

3. Since expressing y from the constraint would be messy, this calls for Lagrange multipliers with $g(x,y) = x^2 - 2x + 2y^2 + 4y$. Equations to solve are $\nabla f = \lambda \nabla g$ and $g = 0$, that is,

$$\left.\begin{array}{l} \frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x} \\ \frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y} \\ g = 0 \end{array}\right\} \Longrightarrow \left.\begin{array}{l} 2x = \lambda(2x-2) \\ 4y = \lambda(4y+4) \\ x^2 - 2x + 2y^2 + 4y = 0 \end{array}\right\} \Longrightarrow \begin{array}{l} x = \lambda(x-1) \\ y = \lambda(y+1) \\ x^2 - 2x + 2y^2 + 4y = 0 \end{array}$$

A typical strategy is to eliminate λ from the first two equations in order to obtain some relationship between the variables x, y, this is then used with condition $g = 0$ to find the desired points.
We would like to isolate λ from the first equation. Can we have $x = 1$? The first equation then reads $1 = 0$, which is not true. Thus for sure $x \neq 1$ and we can write $\lambda = \frac{x}{x-1}$. Putting it into the second equation and multiplying out we get $y = -x$. Now this can be put into the constraint, we obtain $3x^2 - 6x = 0$ and two solutions, $x = 0$ and $x = 2$. Thus there are two suspicious points: $(0,0)$ and $(2,-2)$. We substitute them into f: $f(0,0) = 0$, $f(2,-2) = 12$. Comparing values we guess that the former is a local minimum and the latter is a local maximum.
Determining global extrema usually involves some analysis of the situation. We have two local extrema, but we do not know whether they give global extrema. In general, we find global extrema by comparing values at local extrema and also values at "borders" of the set. Thus we need to know more about M, the set determined by the given condition where we look at f.
A frequent trouble arises when the given set is not bounded, since then we have to ask what happens to f when points of M run away to some infinity. Could it happen that x tends to infinity within this set? Since points from M satisfy $2y^2 + 4y = 2x - x^2$, this would force the expression $2y^2 + 4y$ to tend to minus infinity, but that is not possible. Similarly we argue that also y cannot go to infinity and we thus have a bounded set M.
Another source of trouble is if the set M is a curve that has some endpoints, then we would have to check on those. How does M actually look like? In fact, rewriting the condition as

$$(x-1)^2 + 2(y+1)^2 = 3$$

we see that M is an ellipse. This is a close curve without any end, so whatever important happens to values of f on it, it must happen at one of the points we found earlier. Thus we can conclude that $f(0,0) = 0$ is a minimum and $f(2,-2) = 12$ is a maximum of f on the given set.

4. The unknown point $Q = (x,y,z)$ satisfies $x + y - z = 1$, that would be the constraint with $g(x,y,z) = x + y - z$. The function to minimize should be the distance between P and Q, but that would mean a square root. It will be easier to minimize the distance squared, which is equivalent (think about it). Thus we have $f(x,y,z) = \text{dist}(P,Q)^2 = x^2 + (y+3)^2 + (z-2)^2$. We use Lagrange multipliers, the equations $\nabla f = \lambda \nabla g$ and $g = 1$ now give

$$\left.\begin{array}{l} \frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x} \\ \frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y} \\ \frac{\partial f}{\partial z} = \lambda \frac{\partial g}{\partial z} \\ g = 1 \end{array}\right\} \Longrightarrow \left.\begin{array}{l} 2x = \lambda \cdot 1 \\ 2(y+3) = \lambda \cdot 1 \\ 2(z-2) = \lambda \cdot (-1) \\ x + y - z = 1 \end{array}\right\} \Longrightarrow \begin{array}{l} x = \frac{1}{2}\lambda \\ y + 3 = \frac{1}{2}\lambda \\ z - 2 = -\frac{1}{2}\lambda \\ x + y - z = 1 \end{array}$$

Again, we start by eliminating λ from the first three equations, for instance by substituting for $\frac{1}{2}\lambda$ from the first equation into the next two. Then

$$\left.\begin{aligned} y+3&=x\\ 2-z&=x\\ x+y-z&=1 \end{aligned}\right\} \implies \left.\begin{aligned} y&=x-3\\ z&=2-x\\ x+y-z&=1 \end{aligned}\right\} \implies x+(x-3)+(2-x)=1 \implies x=2.$$

We easily find the other unknowns and obtain a suspicious point $Q=(2,-1,0)$. Is the function f and hence the distance really minimal, not for instance maximal at Q? We try another point from the plane. Say, the point $R=(0,1,0)$ has $\text{dist}(R,P)=\sqrt{4^2+2^2}=\sqrt{20}$. On the other hand, the distance from Q to P is $\text{dist}(Q,P)=\sqrt{2^2+2^2+2^2}=\sqrt{12}$, so it looks like the desired minimum.

We can also argue that it is possible to go to infinity within the given plane, we can easily let $x\to\infty$ and the other coordinates adjust, then also the distance goes to infinity (this is obvious when we imagine the situation) and thus the value we found cannot be the maximum.

Alternative: The first three equations offer the possibility of easily expressing all variables using λ (say, $z=2-\frac{1}{2}\lambda$). When we do it and substitute into the constrain, we get an equation with one unknown λ, namely $\frac{3}{2}\lambda=6$. For this we have $\lambda=4$ and we now have exactly the same $x=2$ etc. as before.

Note: Instead of Lagrange multipliers one could use the constraint to get $x=1-y+z$ and substitute into f, obtaining $F(y,z)=(1-y+z)^2+(y+3)^2+(z-2)^2$. We find its local extrema:

$$\left.\begin{aligned} \tfrac{\partial F}{\partial y}&=0\\ \tfrac{\partial F}{\partial z}&=0 \end{aligned}\right\} \implies \left.\begin{aligned} -2(1-y+z)+2(y+3)&=0\\ 2(1-y+z)+2(z-2)&=0 \end{aligned}\right\} \implies y=-1,\ z=0.$$

5. The distance between a point and a line is given as the distance between the given point and the closest point of the line, so we have to find that point.

We have two constraints, one given by $g(x,y,z)=x+y+z=1$, the other by $h(x,y,z)=2x-y+z=3$. We want a point $Q=(x,y,z)$ satisfying these constraints such that its distance from P is minimal, we will minimize the distance squared $f(x,y,z)=(x-1)^2+(y-2)^2+(z+1)^2$. Now there will be two Lagrange multipliers, we call them λ and μ (it is easier to write g,h and λ,μ rather then g_1,g_2 and λ_1,λ_2 as in the theorem). The equations are $\nabla f=\lambda\nabla g+\mu\nabla h$, $g=1$ and $h=3$, that is,

$$\left.\begin{aligned} \tfrac{\partial f}{\partial x}&=\lambda\tfrac{\partial g}{\partial x}+\mu\tfrac{\partial h}{\partial x}\\ \tfrac{\partial f}{\partial y}&=\lambda\tfrac{\partial g}{\partial y}+\mu\tfrac{\partial h}{\partial y}\\ \tfrac{\partial f}{\partial z}&=\lambda\tfrac{\partial g}{\partial z}+\mu\tfrac{\partial h}{\partial z}\\ g&=1\\ h&=3 \end{aligned}\right\} \implies \left.\begin{aligned} 2(x-1)&=\lambda\cdot1+\mu\cdot2\\ 2(y-2)&=\lambda\cdot1+\mu\cdot(-1)\\ 2(z+1)&=\lambda\cdot1+\mu\cdot1\\ x+y+z&=1\\ 2x-y+z&=3 \end{aligned}\right\} \implies \begin{aligned} 2(x-1)&=\lambda+2\mu\\ 2(y-2)&=\lambda-\mu\\ 2(z+1)&=\lambda+\mu\\ x+y+z&=1\\ 2x-y+z&=3 \end{aligned}$$

We will again try to eliminate the multipliers from the first three equations. For instance, we add the second and the third equation, get $\lambda=y+z-1$, putting it back into the third equation we get $\mu=z-y+3$. Substituting λ,μ into the first equation we get $2x+y-3z=6$. This is typical, we had 3 equations with 5 unknowns, so after using up two equations we end up with one and only three unknowns.

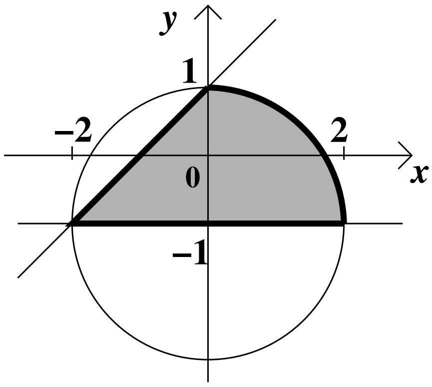
Now we also take into account the two constraints, so we get

$$\left.\begin{aligned} 2x+y-3z&=6\\ x+y+z&=1\\ 2x-y+z&=3 \end{aligned}\right\} \implies x=2,\ y=0,\ z=-1.$$

We then calculate the distance from $Q=(2,0,-1)$ to P: $\text{dist}(P,Q)=\sqrt{5}$. Just to make sure that we have the minimum, we pick another point from the line. For instance, $R = (0, -1, 2)$ satisfies both given equations and $\text{dist}(P, R) = \sqrt{19}$.

Note: If the question did not ask for Lagrange multipliers, we could have also used the given conditions to conclude that, say, $y = \frac{x}{2} + 1$, $z = 2 - \frac{3}{2}x$, substitute into f and then minimize it, obtaining $x = 2$.

6. First we sketch the set M. There are two straight lines and a circle, they split the plane into several parts. However, only one of them is both finite and bounded by all three of these curves.



We have a bounded closed set with non-empty interior and a boundary (border) consisting of three parts. Theory guarantees that global extrema happen either inside, then they have to be local extrema, or on the boundary. This determines the classical algorithm, we will gather all possible candidates and at the end compare values.
1) First we check on interior of M, where extrema happen at points of local extrema. We need not classify them, it is enough to check on all candidates, that is, on stationary points.

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= 0 \\ \frac{\partial f}{\partial y} &= 0 \end{aligned}\right\} \implies \left.\begin{aligned} 2x &= 0 \\ 8y &= 0 \end{aligned}\right\} \implies x = y = 0.$$

Since $(0, 0) \in M$, we get the first candidate: $f(0, 0) = 0$.

2) Now we need to check on the boundary. There are three parts and each of them has the same form, it is a curve that is cut off at ends. For each part the procedure is again the same, global extrema can be attained either at endpoints or at points that give local extrema with respect to the considered curve. This guides our steps.
2a) We start with the quarter-circle. It is a curve determined by $g(x, y) = x^2 + (y + 1)^2 = 4$ and by conditions $0 \le x \le 2$, $-1 \le y \le 1$. It has two endpoints, so we have to check on them: $f(2, -1) = 8$ and $f(0, 1) = 4$ are candidates.
Then we need to find local extrema of f with respect to the constraint $g(x, y) = 4$, this should be possible using Lagrange multipliers. The equations $\nabla f = \lambda \nabla g$ and $g = 4$ read

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= \lambda \frac{\partial g}{\partial x} \\ \frac{\partial f}{\partial y} &= \lambda \frac{\partial g}{\partial g} \\ g &= 4 \end{aligned}\right\} \implies \begin{aligned} 2x &= \lambda 2x \\ 8y &= \lambda 2(y + 1) \\ x^2 + (y + 1)^2 &= 4 \end{aligned}$$

The first equation invites us to cancel, but we have to be careful. Could it happen that $x = 0$? Then the third equation gives $y = 1$ and the second one $\lambda = 2$. We are getting a candidate $(0, 1)$, but we already checked on it, so it is on the list.
To find other points we can assume that $x \neq 0$, then the first equation givess $\lambda = 1$, putting it into the second one we get $y = \frac{1}{3}$ and the third equation yields $x = \frac{2}{3}\sqrt{5}$. We have another candidate, $f\big(\frac{2}{3}\sqrt{5}, \frac{1}{3}\big) = \frac{8}{3}$.

By the way, comparing values at the three points we can guess that $f\big(\frac{2}{3}\sqrt{5},\frac{1}{3}\big)=\frac{8}{3}$ is a local minimum of f with respect to the arc we investigate here.
2b) The next part to explore is the oblique segment given by $y=x+1$, $-2\le x\le 0$. It has two endpoints, $(0,1)$ we already listed and the other is $f(-2,-1)=8$, another candidate for extrema on M.
To see what happens in the middle of this curve we can again use Lagrange multipliers, this time applied to $g(x,y)=y-x=1$:

$$\left.\begin{aligned}2x&=-\lambda\\ 8y&=\lambda\\ y-x&=1\end{aligned}\right\}\implies\left.\begin{aligned}4y&=-x\\ y-x&=1\end{aligned}\right\}\implies 5y=1\implies y=\tfrac{1}{5},$$

then $x=-\frac{4}{5}$. We have another candidate, $f\big(-\frac{4}{5},\frac{1}{5}\big)=\frac{4}{5}$ (which seems to be a local minimum of f with respect to that oblique line).
Since the condition is so simple, one might be tempted to simply substitute the expression $y=x+1$ into f, obtaining

$$\varphi(x)=f(x,x+1)=x^2+4(x+1)^2=5x^2+8x+4.$$

The one can use the usual tools of one-variable calculus to find candidates for extrema of this function over the interval $[-2,0]$, arriving at the same results as above.
2c) Finally we check on the horizontal segment, which is given by $y=-1$, $-2\le x\le 2$. Again, candidates come as endpoints, but we already included them above, and local extrema from the middle. Here the simplest way is to substitute, we are interested in extrema of $\varphi(x)=f(x,-1)=x^2+4$ over $-2\le x\le 2$. Endpoints are already done, local extrema are given by the condition $\varphi'(x)=0$, which gives $x=0$. We have another candidate to consider, $f(0,-1)=4$.

Now we put it all together. Candidates are $f(0,0)=0$, $f(2,-1)=8$, $f(0,1)=4$, $f(-2,-1)=8$, $f\big(\frac{2}{3}\sqrt{5},\frac{1}{3}\big)=\frac{8}{3}$, $f\big(-\frac{4}{5},\frac{1}{5}\big)=\frac{4}{5}$, and $f(0,-1)=4$. Comparing values we arrive at the answer: Global maximum of f over M is $f(-2,-1)=f(2,-1)=8$, global minimum is $f(0,0)=0$.
Just out of curiosity, global minimum with respect to the whole border is $f\big(-\frac{4}{5},\frac{1}{5}\big)=\frac{4}{5}$.

Remark: If we wanted to express M using set notation, we could start with the disc given by the first condition and intersect it with two half-planes given by the lines:

$$M=\{(x,y)\in\mathbb{R}^2;\ x^2+(y+1)^2\le 4\ \text{ and }\ y\ge -1\ \text{ and }\ y\le x+1\}.$$

7. The set M can be described as $M=\{(x,y)\in\mathbb{R}^2;\ x^2+y^2\le 4\}$. To find the global extrema we first look at what is happening inside, that is, we check on stationary points of f:

$$\left.\begin{aligned}\tfrac{\partial f}{\partial x}&=0\\ \tfrac{\partial f}{\partial y}&=0\end{aligned}\right\}\implies\left.\begin{aligned}2x-6&=0\\ 2y+6&=0\end{aligned}\right\}\implies x=3,\ y=-3.$$

However, the point $(3,-3)$ is not in M and therefore we disregard it.
Next we have to check on the boundary, that is, we have to find extrema of f given the condition $x^2+y^2=4$. This calls for Lagrange multipliers:

$$\left.\begin{aligned}\tfrac{\partial f}{\partial x}&=\lambda\tfrac{\partial g}{\partial x}\\ \tfrac{\partial f}{\partial y}&=\lambda\tfrac{\partial g}{\partial y}\\ g&=4\end{aligned}\right\}\implies\left.\begin{aligned}2x-6&=\lambda\cdot 2x\\ 2y+6&=\lambda\cdot 2y\\ x^2+y^2&=4\end{aligned}\right\}\implies\begin{aligned}x-3&=\lambda x\\ y+3&=\lambda y\\ x^2+y^2&=4\end{aligned}$$

Now we would like to eliminate λ from the first two equations. Could it happen that $x=0$? Then the first equation would read $-3=0$, not possible; similarly we have $y\ne 0$. Thus we can express λ from the first equation, substitute into the second and eventually obtain $y=-x$. Putting this into the constraint equation we get $x=\pm\sqrt{2}$. Thus there are two candidates, $(\sqrt{2},-\sqrt{2})$ and $(-\sqrt{2},\sqrt{2})$. We put them into f: $f(\sqrt{2},-\sqrt{2})=4-12\sqrt{2}$, $f(-\sqrt{2},\sqrt{2})=4+12\sqrt{2}$, so it seems that the first is the minimum, the second is the maximum.

8. In order to find extrema we need to know $y'(x)$. We can find it relatively easily using implicit differentiation. We differentiate the given equation with respect to x, it is important to remember that also y is a function of x:

$$[y^2+2xy]' = [2x-4x^2]' \implies 2yy'+2y+2xy' = 2-8x \qquad (*) \qquad \implies \qquad y' = \tfrac{2-8x-2y}{2y+2x}.$$

Critical points are characterized by the property $y'(x)=0$:

$$y' = \tfrac{2-8x-2y}{2y+2x} = 0 \implies 2-8x-2y=0 \implies y = 1-4x.$$

The critical points also have to satisfy the given equation (they lie on the curve), so we put this new condition into the curve equation:

$$(1-4x)^2+2x(1-4x) = 2x-4x^2 \qquad \implies \qquad 12x^2-8x+1=0 \qquad \implies \qquad x = \tfrac{1}{2},\ \tfrac{1}{6}.$$

We get candidates $x=\frac{1}{2}$, $y=-1$; and $x=\frac{1}{6}$, $y=\frac{1}{3}$. For both of them we have $y'=0$ (this will come handy later).
To classify them we need the second derivative, so we differentiate the equation $(*)$:

$$2y'y'+2yy''+2y'+2y'+2xy'' = -8 \qquad \implies \qquad y'' = \tfrac{2y'+(y')^2-4}{y+x}.$$

We substitute in the first point, that is, $x=\frac{1}{2}$, $y=-1$ and $y'=0$, to get $y''\big(\frac{1}{2}\big)=8>0$. Thus $y\big(\frac{1}{2}\big)=-1$ is a local minimum.
We plug in the second point, that is, $x=\frac{1}{6}$, $y=\frac{1}{3}$ and $y'=0$, to get $y''\big(\frac{1}{6}\big)=-8<0$. Thus $y\big(\frac{1}{6}\big)=\frac{1}{3}$ is a local maximum.
Alternative solutions are possible, for instance y'' can be obtained by differentiating directly the formula for y'.

Remark for an inquisitive (and advanced) student: We followed the usual procedure for solving this kind of problem, but a careful reader may wonder whether we really found all critical points. Indeed, critical points are defined as points where $y'=0$ or y' does not exist. Are there any such points in our case? The formula for derivative has trouble when $2y+2x=0$, that is, $y=-x$. Again we substitute into the given equation to get $3x^2-2x=0$, that is, $x=0$ or $x=\frac{2}{3}$. These are also critical points, but since there is no derivative at these points, we cannot use the second derivative test to classify them.
What happens there? When the graph approaches the point $(0,0)$ or the point $\big(\frac{2}{3},-\frac{2}{3}\big)$, then the formula for y' tends to infinity. This shows that the real reason why we cannot find derivative there is that the curve has a vertical tangent line there, in particular it cannot have a local extreme there. This is typical, when an implicit function is given by a nice differentiable equation, then the only problems appear at such places of improper derivative. Therefore we do not make a mistake if we focus only on points where $y'=0$.