MA2: Solved problems—Functions of more variables: Integrals

Evaluate the following integrals:

1. $\int 6x^2y^2 + 6x + 6y\,dx$ and $\int\limits_0^1 6x^2y^2 + 6x + 6y\,dx$,

2. $\int \cos(xy + 2z)\,dy$,

3. $\int\limits_0^1 e^{xy+2z}\,dz$.

4. Evaluate $\iint\limits_{\Omega} (x + 1)\,dA$, where Ω is the finite region between the graphs of $y = x^2$ and $y = 5x + 6$.

5. Find the average of the function $f(x, y) = 2e^{x/y}$ over the set

$$\Omega = \{(x, y) \in I\!R^2;\ x \le y \le \sqrt{x}\}.$$

Change the order of differentiation in the following integrals:

6. $\int\limits_2^3 \int\limits_2^{8-2x} f(x, y)\,dy\,dx$,

7. $\int\limits_0^1 \int\limits_0^{e^y} f(x, y)\,dx\,dy$,

8. $\int\limits_0^{\infty} \int\limits_0^{x} f(x, y)\,dy\,dx$.

9. Evaluate $\iint\limits_{\Omega} \frac{1}{y}\frac{1}{x^2 + y^2}\,dA$, where $\Omega = \langle 0, \infty) \times \langle 1, \infty)$. Try it both ways.

Solutions:

1. We pretend that y is a constant, then

$$\int 6x^2y^2 + 6x + 6y\,dx = y^2\int 6x^2\,dx + \int 6x\,dx + y\int 6\,dx = y^2 2x^3 + 3x^2 + y6x + C$$
$$= 2x^3y^2 + 3x^2 + 6xy + C,\ x, y \in I\!R.$$

Now it is easy to evaluate the definite integral, we have to substitute for x. It might be a good idea to remind oneself using a suitable notation as follows.

$$\int\limits_0^1 6x^2y^2 + 6x + 6y\,dx = \Big[2x^3y^2 + 3x^2 + 6xy\Big]_{x=0}^{x=1} = 2y^2 + 3 + 6y.$$

However, experienced integrators often just write $\Big[\ \dots\Big]_0^1$.

2. We pretend that x and z are constants, then

$$\int \cos(xy+2z)\,dy = \left|\begin{aligned} u &= xy+2z \\ du &= \tfrac{\partial}{\partial y}[xy+2z]\,dy = x\,dy \\ dy &= \tfrac{1}{x}\,du \end{aligned}\right| = \int \cos(u)\frac{1}{x}\,du = \frac{1}{x}\int \cos(u)\,du$$

$$= \frac{1}{x}\sin(u) + C = \frac{1}{x}\sin(xy+2z) + C,\ \ y,z \in I\!R,\ \ x \neq 0.$$

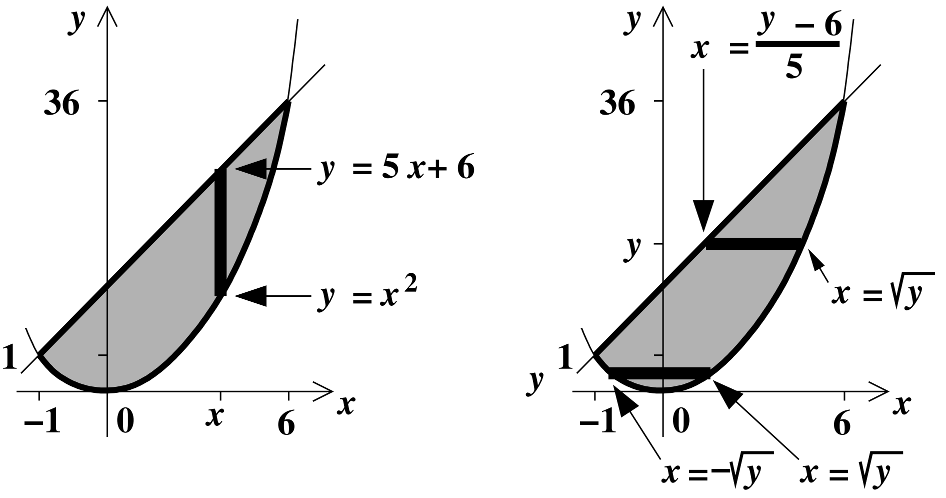
3. Now we pretend that x, y are constants.

$$\int\limits_0^1 e^{xy+2z}\,dz = \left|\begin{aligned} u &= xy+2z \\ du &= \tfrac{\partial}{\partial z}[xy+2z]\,dz = 2\,dz \\ dz &= \tfrac{1}{2}\,du \\ z = 1 &\mapsto u = xy+2 \\ z = 0 &\mapsto u = xy \end{aligned}\right| = \int\limits_{xy}^{xy+2} e^u \frac{1}{2}\,du = \left[\frac{1}{2}e^u\right]_{xy}^{xy+2} = \frac{1}{2}e^{xy+2} - \frac{1}{2}e^{xy}.$$

By the way, the indefinite integral is

$$\int e^{xy+2z}\,dz = \frac{1}{2}e^{xy+2z} + C,\ \ x,y,z \in I\!R.$$

4. How does the region Ω look like? We check whether the curves intersect: $x^2 = 5x+6$ implies $x = -1, 6$. We sketch the region (not to scale):



It is obvious that slicing this region vertically is preferable to horizontal slicing, since horizontal slices would differ in kind depending on their positions.

A vertical slice is determined by a fixed value of x, moving up and down is done by changing y. We will therefore integrate using dy with y ranging between $y = x^2$ and $y = 5x+6$. Thus integrating over this vertical slice leads to $\int\limits_{x^2}^{5x+6} (x+1)\,dy$. Then we put all these slices together by another integration, where we "add" all x than determine our slices:

$$\int\limits_{-1}^{6}\Big(\int\limits_{x^2}^{5x+6} (x+1)\,dy\Big)\,dx = \int\limits_{-1}^{6}\int\limits_{x^2}^{5x+6} (x+1)\,dy\,dx.$$

This double integral is now integrated in the usual way, from inside out.

$$\int_{-1}^{6}\int_{x^2}^{5x+6}(x+1)\,dy\,dx=\int_{-1}^{6}\Big[xy+y\Big]_{y=x^2}^{y=5x+6}dx=\int_{-1}^{6}[x(5x+6)+(5x+6)]-[x\cdot x^2+x^2]\,dx$$

$$=\int_{-1}^{6}4x^2-x^3+11x+6\,dx=\Big[\frac{4}{3}x^3-\frac{1}{4}x^4+\frac{11}{2}x^2+6x\Big]_{-1}^{6}=200+\frac{1}{12}.$$

How would it go if we decided to use horizontal slices? We move left and right by changing x, so integrals over horizontal slices are done with respect to x, each slice is determined by choosing y. If we choose y between 0 and 1, then in the corresponding slice x ranges between $-\sqrt{y}$ and $\sqrt{y}$. If we choose y between 1 and 36, then x ranges between $\frac{1}{5}(y-6)$ and $\sqrt{y}$. Thus we get

$$\int_{0}^{1}\int_{-\sqrt{y}}^{\sqrt{y}}(x+1)\,dx\,dy+\int_{1}^{36}\int_{(y-6)/5}^{\sqrt{y}}(x+1)\,dx\,dy.$$

Now we integrate:

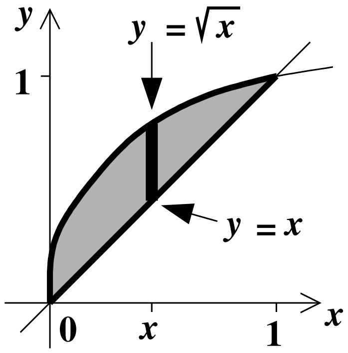
$$\int_{0}^{1}\int_{-\sqrt{y}}^{\sqrt{y}}(x+1)\,dx\,dy+\int_{1}^{36}\int_{(y-6)/5}^{\sqrt{y}}(x+1)\,dx\,dy=\int_{0}^{1}\Big[\frac{1}{2}x^2+x\Big]_{-\sqrt{y}}^{\sqrt{y}}dy+\int_{1}^{36}\Big[\frac{1}{2}x^2+x\Big]_{(y-6)/5}^{\sqrt{y}}dy$$

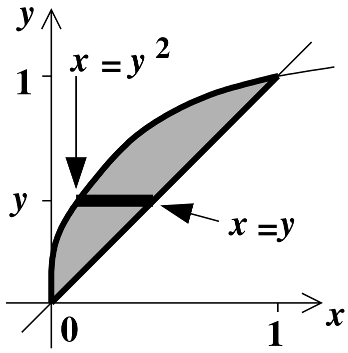
$$=\int_{0}^{1}2\sqrt{y}\,dy+\int_{1}^{36}\frac{1}{2}y+\sqrt{y}-\frac{1}{50}(y-6)^2-\frac{1}{5}(y-6)\,dy$$

$$=\Big[\frac{4}{3}y^{3/2}\Big]_{0}^{1}+\Big[\frac{1}{4}y^2+\frac{2}{3}y^{3/2}-\frac{1}{150}(y-6)^3-\frac{1}{10}(y-6)^2\Big]_{1}^{36}=200+\frac{1}{12}.$$

You probably agree that it was harder this way. A wise choice of slicing can make a big difference.

5. What is the given region?





In order to determine the average we need to know two facts: The area of Ω and the integral of f over Ω. The area can be also obtained using integration over Ω, this time we integrate the function 1. Here both vertical and horizontal slicing seem about the same from geometry point of view (just one integral needed), so we try vertical slices ranging (for a given x) from $y=x$ to $y=\sqrt{x}$.

$$A(\Omega)=\int_{0}^{1}\int_{x}^{\sqrt{x}}1\,dy\,dx=\int_{0}^{1}\Big[y\Big]_{x}^{\sqrt{x}}dx=\int_{0}^{1}\sqrt{x}-x\,dx=\Big[\frac{2}{3}x^{3/2}-\frac{1}{2}x^2\Big]_{0}^{1}=\frac{1}{6}.$$

Now we integrate the given function.

$$\iint\limits_{\Omega} 2e^{x/y}\,dA = \int\limits_0^1 \int\limits_x^{\sqrt{x}} 2e^{x/y}\,dy\,dx.$$

We have a little problem, the integral $\int e^{a/y}\,dy$ is very tough, one of those that can't be expressed using elementary functions. Fortunately there is an alternative, we may try horizontal slicing and hope for better integrals. Given y, values of x on the corresponding horizontal slice range from $x = y^2$ to $x = y$ (nice formulas, perhaps we should have done the area this way as well). We get

$$\iint\limits_{\Omega} 2e^{x/y}\,dA = \int\limits_0^1 \int\limits_{y^2}^{y} 2e^{x/y}\,dx\,dy.$$

This is much easier, we need to find $\int e^{x/a}\,dx$, which is a standard integral that is best solved using substitution. Eventually we will also need integration by parts.

$$\iint\limits_{\Omega} 2e^{x/y}\,dA = \int\limits_0^1 \int\limits_{y^2}^{y} 2e^{x/y}\,dx\,dy = \left|\begin{array}{c} w = \frac{x}{y} \\ dw = \frac{1}{y}\,dx \\ dx = y\,dw \\ x = y \mapsto w = 1 \\ x = y^2 \mapsto w = y \end{array}\right| = \int\limits_0^1 \int\limits_y^1 2e^w y\,dw\,dy$$

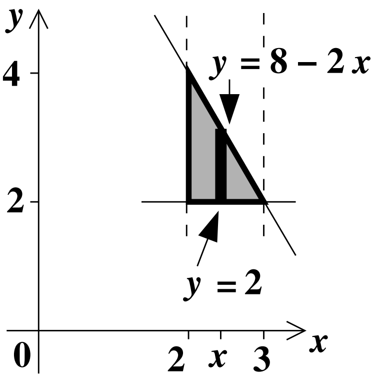
$$= \int\limits_0^1 \Big[2y\,e^w\Big]_{w=y}^{w=1} dy = \int\limits_0^1 2e\,y - 2y\,e^y\,dy = e\int\limits_0^1 2y\,dy - \int\limits_0^1 2y\,e^y\,dy$$

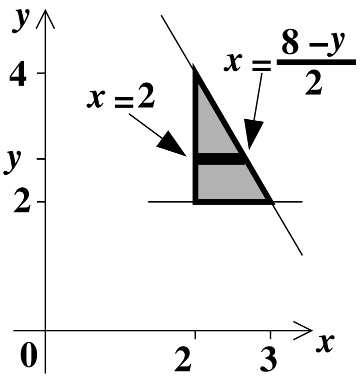
$$= e\Big[y^2\Big]_0^1 - \Big[2y\,e^y\Big]_0^1 + \int\limits_0^1 2\,e^y\,dy = e - 2e + \Big[2\,e^y\Big]_0^1 = e - 2.$$

Thus the average is

$$\frac{1}{A(\Omega)} \iint\limits_{\Omega} 2e^{x/y}\,dA = 6e - 12.$$

6. First we need to determine the region Ω over which we integrate. The inside variable is y that moves us up and down, this means that we follow vertical slices. Positions of these slices are given by x, so the leftmost is on the line $x = 2$ and the rightmost on the line $x = 3$. From the limits of the inside integral we see that given x, the corresponding vertical slice goes from the curve $y = 2$ to the curve $y = 8 - 2x$, so Ω is the region between these two curves. We can draw a picture.

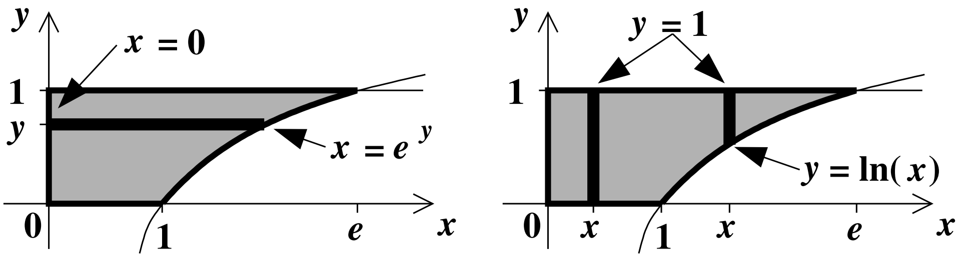




Changing the order of integration means switching to the other direction with slices (see picture on the right). Horizontal slices are given by fixing some y between 2 and 4 (this will be the outer integral), for such y the variable x ranges between $x = 2$ and $x = \frac{1}{2}(8 - y)$ (this we get by solving the relation $y = 8 - 2x$ for x). We get the integral

$$\int_2^4 \int_2^{(8-y)/2} f(x,y)\,dx\,dy.$$

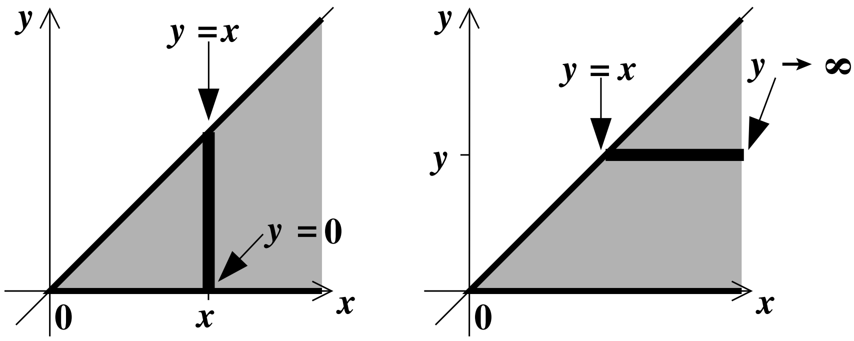
7. We start by determining the region of integration Ω. The inside variable is x that moves us left and right, this means that we follow horizontal slices, the lowest is on the line $y = 0$ and the highest is at $y = 1$. Slices range from the curve $x = 0$ to the curve $x = e^y$, that is, $y = \ln(x)$. We can now draw it.



To change the order of integration we switch to vertical slices, but the picture above clearly shows that we then get two kinds of slices, in other words, we end up with two integrals: For x between 0 and 1, vertical slices go from $y = 0$ to $y = 1$, for x between 1 and e vertival slices go from $y = \ln(x)$ to $y = 1$. We get

$$\int_0^1 \int_0^1 f(x,y)\,dy\,dx + \int_1^e \int_{\ln(x)}^1 f(x,y)\,dy\,dx.$$

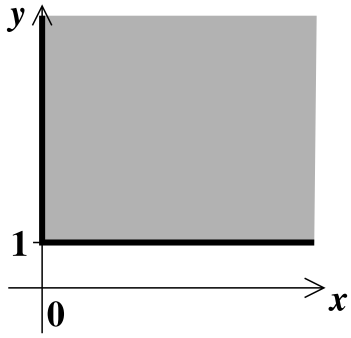
8. We first determine the region of integration Ω. The inner integral with working variable y points to vertical slices, each slice stretches between curves $y = 0$ and $y = x$. We take slices for all $x \geq 0$, the picture is now clear.



Changing the order of differentiation corresponds to taking the other slices, that is, horizontal ones (see the picture on the right). To cover the whole Ω we need to take horizontal slices all the way to infinity, that is, positions are given by y from $\langle 0, \infty)$. For a chosen y then the corresponding slice lets x range between the curve $x = y$ and infinity. Here comes the integral.

$$\int_0^\infty \int_y^\infty f(x,y)\,dx\,dy.$$

9. Since the region is a rectangle,



both integrals will have constant limits of integration and we can order them any way we wish.

1) We start with vertical slicing, which corresponds to the inner integral using y.

$$\iint_\Omega f(x,y)\,dA = \int_0^\infty \int_1^\infty \frac{1}{y}\,\frac{1}{x^2+y^2}\,dy\,dx.$$

We need to figure out $\int \frac{1}{y}\,\frac{1}{a^2+y^2}\,dy$, this calls for partial fractions, x will be a parameter there:

$$\begin{aligned}\int \frac{1}{y(y^2+x^2)}\,dy &= \int \frac{A}{y} + \frac{By+C}{y^2+x^2}\,dy = \int \frac{\frac{1}{x^2}}{y} - \frac{\frac{1}{x^2}y}{y^2+x^2}\,dy\\ &= \frac{1}{x^2}\int \frac{dy}{y} - \frac{1}{x^2}\,\frac{1}{2}\int \frac{2y\,dy}{y^2+x^2} = \left|\begin{matrix} z = y^2+x^2\\ dz = 2y\,dy\end{matrix}\right| = \frac{1}{x^2}\int \frac{dy}{y} - \frac{1}{2x^2}\int \frac{dz}{z}\\ &= \frac{1}{x^2}\ln|y| - \frac{1}{2x^2}\ln|y^2+x^2| + C = \frac{1}{2x^2}\ln|y^2| - \frac{1}{2x^2}\ln|y^2+x^2| + C\\ &= \frac{1}{2x^2}\ln\Big(\frac{y^2}{y^2+x^2}\Big) + C.\end{aligned}$$

Since our integral is improper, it was advisable to express the antiderivative in a compact way. Now we can evaluate

$$\begin{aligned}\int_1^\infty \frac{1}{y}\,\frac{1}{x^2+y^2}\,dy &= \Big[\frac{1}{2x^2}\ln\Big(\frac{y^2}{y^2+x^2}\Big)\Big]_{y=1}^{y=\infty} = \lim_{y\to\infty}\Big(\frac{1}{2x^2}\ln\Big(\frac{1}{1+\frac{x^2}{y^2}}\Big)\Big) - \frac{1}{2x^2}\ln\Big(\frac{1}{1+x^2}\Big)\\ &= \frac{1}{2x^2}\ln(1) + \frac{1}{2x^2}\ln(1+x^2) = \frac{1}{2x^2}\ln(x^2+1).\end{aligned}$$

Back to the given integral:

$$\iint_\Omega f(x,y)\,dA = \int_0^\infty \int_1^\infty \frac{1}{y}\,\frac{1}{x^2+y^2}\,dy\,dx = \int_0^\infty \frac{1}{2x^2}\ln(x^2+1)\,dx$$

This calls for integration by parts.

$$\begin{aligned}\int \frac{1}{2x^2}\ln(x^2+1)\,dx &= \left|\begin{matrix} f = \ln(x^2+1) & g' = \frac{1}{2x^2}\\ f' = \frac{2x}{x^2+1} & g = -\frac{1}{2x}\end{matrix}\right| = -\frac{1}{2x}\ln(x^2+1) - \int -\frac{1}{2x}\,\frac{2x}{x^2+1}\,dx\\ &= -\frac{1}{2x}\ln(x^2+1) + \int \frac{dx}{x^2+1} = -\frac{1}{2x}\ln(x^2+1) + \arctan(x) + C.\end{aligned}$$

Therefore

$$\begin{aligned}\iint_\Omega f(x,y)\,dA &= \Big[\arctan(x) - \frac{\ln(x^2+1)}{2x}\Big]_0^\infty\\ &= \lim_{x\to\infty}\Big(\arctan(x) - \frac{\ln(x^2+1)}{2x}\Big) - \lim_{x\to 0^+}\Big(\arctan(x) - \frac{\ln(x^2+1)}{2x}\Big)\end{aligned}$$

$$= \frac{\pi}{2} - \lim_{x\to\infty}\Big(\frac{\ln(x^2+1)}{2x}\Big) - 0 + \lim_{x\to 0^+}\Big(\frac{\ln(x^2+1)}{2x}\Big)$$
$$\underset{\text{l'H}}{=\!=} \frac{\pi}{2} - \lim_{x\to\infty}\Big(\frac{2x}{2(x^2+1)}\Big) + \lim_{x\to 0^+}\Big(\frac{2x}{2(x^2+1)}\Big) = \frac{\pi}{2} - 0 + 0 = \frac{\pi}{2}.$$

Wow, that was some integral. A nice review of basic integrating techniques.

2) Now we try horizontal slicing.

$$\iint\limits_{\Omega} f(x,y)\,dA = \int_1^\infty\int_0^\infty \frac{1}{y}\,\frac{1}{x^2+y^2}\,dx\,dy = \int_1^\infty \frac{1}{y}\int_0^\infty \frac{1}{x^2+y^2}\,dx\,dy.$$

The inside integral $\int \frac{dx}{x^2+a^2}$ is standard, some people even remember it. Those who do not remember may use the recommended indirect substitution.

$$\int_0^\infty \frac{dx}{x^2+y^2} = \left|\begin{array}{c} x = yt \\ dx = y\,dt \\ x=0 \mapsto t=0 \\ x=\infty \mapsto t=\infty \end{array}\right| = \int_0^\infty \frac{y\,dt}{y^2t^2+y^2} = \frac{1}{y}\int_0^\infty \frac{dt}{t^2+1}$$
$$= \Big[\frac{1}{y}\arctan(t)\Big]_{t=0}^{t=\infty} = \lim_{t\to\infty}\Big(\frac{1}{y}\arctan(t)\Big) - 0 = \frac{\pi}{2}\,\frac{1}{y}.$$

Therefore

$$\iint\limits_{\Omega} f(x,y)\,dA = \int_1^\infty \frac{1}{y}\int_0^\infty \frac{1}{x^2+y^2}\,dx\,dy = \int_1^\infty \frac{1}{y}\cdot\frac{\pi}{2}\,\frac{1}{y}\,dy = \frac{\pi}{2}\int_1^\infty y^{-2}\,dy$$
$$= \frac{\pi}{2}\Big[-\frac{1}{y}\Big]_1^\infty = \lim_{y\to\infty}\Big(-\frac{\pi}{2}\,\frac{1}{y}\Big) - \frac{\pi}{2}(-1) = 0 + \frac{\pi}{2} = \frac{\pi}{2}.$$

This way it seems a bit easier.