MA2: Cvičné příklady—Funkce více proměnných: Integrály
Stručná řešení

1. $\int_0^1\int_0^2 6xy+2x+2y\,dy\,dx=\int_0^1\Big[3xy^2+2xy+y^2\Big]_0^2dx=\int_0^1 16x+4\,dx\,dx=\Big[8x^2+4x\Big]_0^1=12.$

Alternativa: $\int_0^2\int_0^1 6xy+2x+2y\,dx\,dy=\int_0^2\Big[3x^2y+x^2+2xy\Big]_0^1dy=\int_0^2 5y+1\,dx=\Big[\frac{5}{2}y^2+y\Big]_0^2=12.$

2. $\int_0^\pi\int_0^{2\pi} 3\sin(3x+y)\,dy\,dx=\left|\begin{array}{l} z=3x+y\\ dz=\frac{\partial}{\partial y}[3x+y]\,dy\\ dz=dy\end{array}\right|=\int_0^\pi\Big[-3\cos(3x+y)\Big]_0^{2\pi}dx$

$=\int_0^\pi -3\cos(3x+2\pi)+3\cos(3x)\,dx=\Big[-\sin(3x+2\pi)+\sin(3x)\Big]_0^\pi=0.$

Poznámka: Vlastně $\cos(3x+2\pi)-\cos(3x)=\cos(3x)-\cos(3x)=0$.

Alternativa: $\int_0^{2\pi}\int_0^\pi 3\sin(3x+y)\,dx\,dy=\left|\begin{array}{l} z=3x+y\\ dz=\frac{\partial}{\partial x}[3x+y]\,dx\\ dz=3\,dx\end{array}\right|=\int_0^{2\pi}\Big[-\cos(3x+y)\Big]_0^\pi dy$

$=\int_0^{2\pi} -\cos(3\pi+y)+\cos(y)\,dy=\Big[-\sin(3\pi+y)+\sin(y)\Big]_0^{2\pi}=0.$

3. $\int_0^1\int_0^{2\pi} x\sin(xy)\,dy\,dx=\left|\begin{array}{l} z=xy\\ dz=\frac{\partial}{\partial y}[xy]\,dy\\ dz=x\,dy\end{array}\right|=\int_0^1\Big[-\cos(xy)\Big]_0^{2\pi}dx=\int_0^1 -\cos(2\pi x)+\cos(0)\,dx$

$=\int_0^1 1-\cos(2\pi x)\,dx=\Big[x-\frac{1}{2\pi}\sin(2\pi x)\Big]_0^1=1.$

Alternativa: $\int_0^{2\pi}\int_0^1 x\sin(xy)\,dx\,dy=\left|\begin{array}{ll} f=x & g'=\sin(xy)\\ f'=1 & g=-\frac{1}{y}\cos(xy)\end{array}\right|=\int_0^{2\pi}\Big[-\frac{1}{y}x\cos(xy)+\frac{1}{y^2}\sin(xy)\Big]_0^1dy$

$=\int_0^{2\pi}\frac{1}{y^2}\sin(y)-\frac{1}{y}\cos(y)\,dy,$

a máme problém, $\int\frac{1}{y}\cos(y)\,dy$ je jeden z těch integrálů, které nelze vyjádřit pomocí vzorce s elementárními funkcemi.

4. $\int_0^1\int_0^2 x^2y\,e^{xy}dy\,dx=\left|\begin{array}{ll} f=xy & g'=x\,e^{xy}\\ f'=x & g=e^{xy}\end{array}\right|\left|\begin{array}{l} z=xy\\ dz=\frac{\partial}{\partial y}[xy]\,dy\\ dz=x\,dy\end{array}\right|=\int_0^1\Big[xy\,e^{xy}-e^{xy}\Big]_0^2dx$

$=\int_0^1 2x\,e^{2x}-e^{2x}+1\,dx=\int_0^1 2x\,e^{2x}\,dx+\int_0^1 1-e^{2x}\,dx=\Big[x\,e^{2x}\Big]_0^1-\int_0^1 2e^{2x}\,dx+\Big[x-\frac{1}{2}e^{2x}\Big]_0^1$

$=e^2-\Big[\frac{1}{2}e^{2x}\Big]_0^1+1-\frac{1}{2}e^2+\frac{1}{2}=2.$

Alternativa: $\int_0^2\int_0^1 x^2y\,e^{xy}dx\,dy=\left|\begin{array}{ll} f=x^2 & g'=y\,e^{xy}\\ f'=2x & g=e^{xy}\end{array}\right|=\int_0^2\Big(\Big[x^2\,e^{xy}\Big]_0^1-\int_0^1 2x\,e^{xy}dx\Big)dy$

$=\left|\begin{array}{ll} f=2x & g'=e^{xy}\\ f'=2 & g=\frac{1}{y}e^{xy}\end{array}\right|=\int_0^2 e^y-\Big[2x\frac{1}{y}\,e^{xy}-\frac{2}{y^2}\,e^{xy}\Big]_0^1dy=\int_0^2 e^y-\frac{2}{y}\,e^y+\frac{2}{y^2}\,e^y-\frac{2}{y^2}dy.$

Toto není pěkný integrál, první způsob je lepší.

5. $\int_0^1\int_0^1\frac{y}{(1+x^2+y^2)^2}\,dy\,dx=\left|\begin{array}{l} z=y^2+x^2+1\\ dz=\frac{\partial}{\partial y}[y^2+x^2+1]\,dy\\ dz=2y\,dy\end{array}\right|=\int_0^1\Big[-\frac{1}{2}\frac{1}{1+x^2+y^2}\Big]_0^1dx$

$$= \int_0^1 \frac{1}{2}\frac{1}{1+x^2} - \frac{1}{2}\frac{1}{2+x^2}\,dx = \Big[\tfrac{1}{2}\operatorname{arctg}(x) - \tfrac{1}{2\sqrt{2}}\operatorname{arctg}\big(\tfrac{x}{\sqrt{2}}\big)\Big]_0^1 = \tfrac{\pi}{8} - \tfrac{\sqrt{2}}{4}\operatorname{arctg}\big(\tfrac{1}{\sqrt{2}}\big).$$

Alternativa: $\int\limits_0^1\int\limits_0^1 \frac{y}{(1+x^2+y^2)^2}\,dx\,dy$

Integrál $\int \frac{A}{(x^2+B)^2}\,dx$ je velmi nepříjemný, první přístup je lepší.

6. $$\int\limits_0^1\int\limits_0^\infty \frac{1}{1+x^2+y^2+x^2y^2}\,dy\,dx = \int\limits_0^1\int\limits_0^\infty \frac{1}{(1+x^2)y^2+(1+x^2)}\,dy\,dx = \int\limits_0^1 \frac{1}{x^2+1}\int\limits_0^\infty \frac{1}{y^2+1}\,dy\,dx$$
$$= \int\limits_0^1 \frac{1}{x^2+1}\Big[\operatorname{arctg}(y)\Big]_0^\infty dx = \int\limits_0^1 \frac{1}{x^2+1}\frac{\pi}{2}\,dx = \Big[\tfrac{\pi}{2}\operatorname{arctg}(x)\Big]_0^1 = \tfrac{\pi^2}{8}.$$

7. $$\int\limits_0^\infty\int\limits_0^\infty e^{-x-y}\,dy\,dx = \int\limits_0^\infty e^{-x}\int\limits_0^\infty e^{-y}\,dy\,dx = \int\limits_0^\infty e^{-x}\Big[-e^{-y}\Big]_0^\infty dx = \int\limits_0^\infty e^{-x}\cdot 1\,dx = \Big[-e^{-x}\Big]_0^\infty = 1.$$

Bonus: Co kdybychom měli $\int\limits_0^\infty\int\limits_0^\infty e^{-x^2-y^2}\,dy\,dx$? Pak máme průšvih, protože víme, že primitivní funkci k e^{-y^2} nelze vyjádřit pomocí vzorce s elementárními funkcemi. Je tu ale zajímavý trik. Je známo, že $\int_{-\infty}^{\infty} e^{-z^2/2}dz = \sqrt{2\pi}$. Proto $\int_{-\infty}^{\infty} e^{-y^2}\,dy = \left|\begin{matrix} z = \sqrt{2}y \\ dz = \sqrt{2}\,dy\end{matrix}\right| = \frac{1}{\sqrt{2}}\int_{-\infty}^{\infty} e^{-z^2/2}\,dz = \sqrt{\pi}$. Ze symetrie $\int_0^\infty e^{-y^2}\,dy = \frac{\sqrt{\pi}}{2}$. Proto
$$\int\limits_0^\infty\int\limits_0^\infty e^{-x^2-y^2}\,dy\,dx = \int\limits_0^\infty e^{-x^2}\int\limits_0^\infty e^{-y^2}\,dy\,dx = \int\limits_0^\infty \tfrac{\sqrt{\pi}}{2}e^{-x^2}\,dx = \tfrac{\sqrt{\pi}}{2}\int\limits_0^\infty e^{-x^2}\,dx = \tfrac{\pi}{4}.$$

Pro obrázky Ω z 8–15 viz níže.

8. Svislé řezy vypadají přirozeně.
$$\int\limits_{-1}^1\int\limits_{x^2}^1 3x^3e^{xy}\,dy\,dx = \int\limits_{-1}^1\Big[3x^2e^{xy}\Big]_{x^2}^1 dx = \int\limits_{-1}^1 3x^2e^x - 3x^2e^{x^3}\,dx = \Big[3x^2e^x - 6x\,e^x + 6\,e^x - e^{x^3}\Big]_{-1}^1 = 2e - 14e^{-1}.$$

9. Svislé řezy se zdají lepší.
$$\int\limits_1^2\int\limits_{\pi x}^{2\pi x} x^2\sin(xy)\,dy\,dx = \int\limits_1^2\Big[-x\cos(xy)\Big]_{\pi x}^{2\pi x} dx = \int\limits_1^2 -x\cos(2\pi x^2) + x\cos(\pi x^2)\,dx$$
$$= \Big[-\tfrac{1}{4\pi}\sin(2\pi x^2) + \tfrac{1}{2\pi}\sin(\pi x^2)\Big]_1^2 = 0.$$

10. Průsečíky: $(1,-2)$, $(2,1)$. Vodorovné řezy jsou nejlepší.
$$\int\limits_{-2}^1\int\limits_{(y+5)/3}^{\sqrt{5-y^2}} 9y\,dx\,dy = \int\limits_{-2}^1\Big[9xy\Big]_{(y+5)/3}^{\sqrt{5-y^2}} dy = \int\limits_{-2}^1 9y\sqrt{5-y^2} - 3y^2 - 15y\,dx = \Big[-3(5-y^2)^{3/2} - y^3 - \tfrac{15}{2}y^2\Big]_{-2}^1 = -\tfrac{15}{2}.$$

11. Průsečíky: $(1,1)$, $(3,3)$, $(4,2)$. Je to trojúhelník.
$$\int\limits_1^3\int\limits_{(x+2)/3}^{x} 9x\,dy\,dx + \int\limits_3^4\int\limits_{(x+2)/3}^{6-x} 9x\,dy\,dx = \int\limits_1^3\Big[9xy\Big]_{(x+2)/3}^{x} dx + \int\limits_3^4\Big[9xy\Big]_{(x+2)/3}^{6-x} dx$$
$$= \int\limits_1^3 9x^2 - 3x(x+2)\,dx + \int\limits_3^4 9x(6-x) - 3x(x+2)\,dx = \int\limits_1^3 6x^2 - 6x\,dx + \int\limits_3^4 48x - 12x^2\,dx$$
$$= \Big[2x^3 - 3x^2\Big]_1^3 + \Big[24x^2 - 4x^3\Big]_3^4 = 48.$$

Alternativa:

$$\int_1^2\int_y^{3y-2} 9x\,dx\,dy+\int_2^3\int_y^{6-y} 9x\,dx\,dy=\int_1^2\Big[\tfrac{9}{2}x^2\Big]_y^{3y-2}dy+\int_2^3\Big[\tfrac{9}{2}x^2\Big]_y^{6-y}dy$$

$$=\int_1^2 36y^2-54y+18\,dy+\int_2^3 162-54y\,dy=\Big[12y^3-27y^2+18y\Big]_1^2+\Big[162y-27y^2\Big]_2^3=48.$$

12. Trojúhelník s vrcholy $(0,0)$, $(0,2)$, $(1,2)$.

Svislé řezy: $\int\limits_0^1\int\limits_{2x}^2\big(4e^{y^2}+8xy\big)\,dy\,dx.$

My ale víme, že primitivní funkci k e^{y^2} nelze vyjádřit vzorcem s elementárními funkcemi, takže to není dobrý nápad.

Vodorovné řezy:

$$\int_0^2\int_0^{y/2} 4e^{y^2}+8xy\,dx\,dy=\int_0^2\Big[4e^{y^2}x+4x^2y\Big]_{x=0}^{x=y/2}dy=\int_0^2 2y\,e^{y^2}+y^3\,dy=\Big[e^{y^2}+\tfrac{1}{4}y^4\Big]_0^2=e^4+3.$$

13. Trojúhelník, vrcholy $(0,0)$, $(3,0)$, $(0,3)$.

Svislé řezy:

$$\int_0^3\int_0^{3-x}\frac{y}{x}\,dy\,dx=\int_0^3\Big[\frac{y^2}{2x}\Big]_0^{3-x}dx=\int_0^3\frac{(3-x)^2}{2x}\,dx=\int_0^3\tfrac{9}{2}\tfrac{1}{x}x-3+\tfrac{1}{2}x\,dx=\Big[\tfrac{9}{2}\ln|x|-3x+\tfrac{1}{4}x^2\Big]_0^3=\infty.$$

Vodorovné řezy:

$$\int_0^3\int_0^{3-y}\frac{y}{x}\,dx\,dy=\int_0^3\Big[y\ln|x|\Big]_0^{3-y}dy=\int_0^3\infty\,dy.$$ Takto to nepůjde.

14. Svislé řezy:

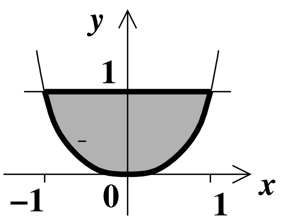
$$\int_0^\infty\int_x^\infty e^{-x-y}\,dy\,dx=\int_0^\infty\Big[-e^{-x-y}\Big]_x^\infty dx=\int_0^\infty -0+e^{-2x}\,dx=\Big[-\tfrac{1}{2}e^{-2x}\Big]_0^\infty=\tfrac{1}{2}-0=\tfrac{1}{2}.$$

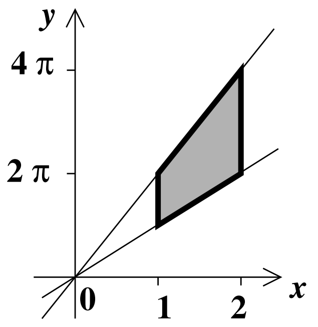
Vodorovné řezy:

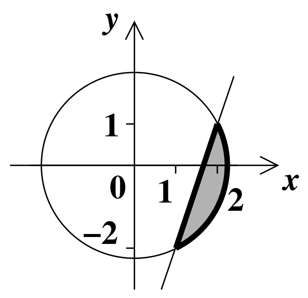
$$\int_0^\infty\int_0^y e^{-x-y}\,dx\,dy=\int_0^\infty\Big[-e^{-x-y}\Big]_0^y dy=\int_0^\infty -e^{-2y}+e^{-y}\,dy=\Big[\tfrac{1}{2}e^{-2y}-e^{-y}\Big]_0^\infty=0-\tfrac{1}{2}+1=\tfrac{1}{2}.$$

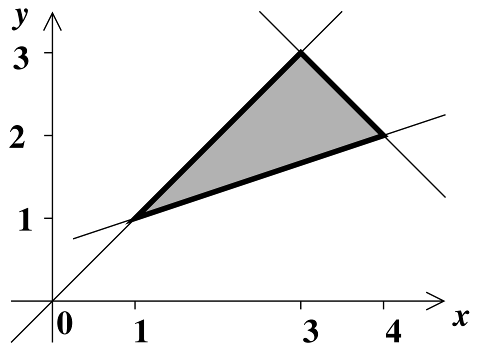
15. Svislé řezy:

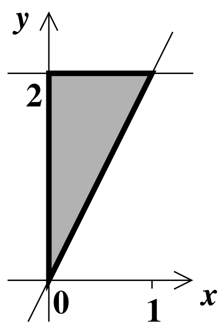
$$\int_1^\infty\int_{-1/\sqrt{x}}^{1/\sqrt{x}}\frac{1}{x}\,dy\,dx=\int_1^\infty\Big[\frac{y}{x}\Big]_{-1/\sqrt{x}}^{1/\sqrt{x}}dx=\int_1^\infty\frac{2}{x^{3/2}}\,dx=\Big[-\tfrac{4}{\sqrt{x}}\Big]_1^\infty=-0+4=4.$$

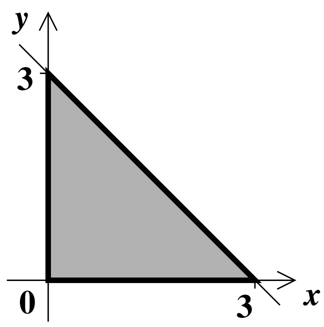


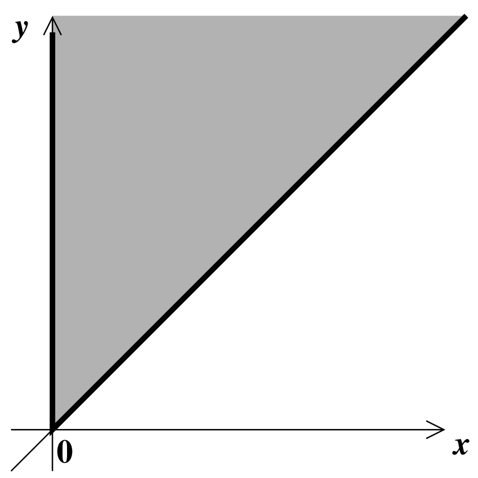


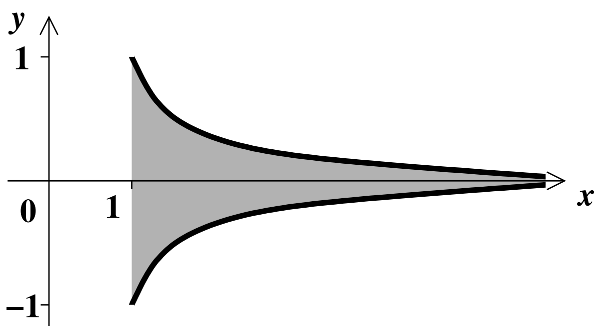












Pro obrázky Ω z 16–21 viz níže.

16. $\displaystyle\int_0^1\int_{x^3}^{x^2} xy\,dy\,dx = \int_0^1 \Big[\tfrac{1}{2}xy^2\Big]_{x^3}^{x^2} dx = \int_0^1 \tfrac{1}{2}x^5 - \tfrac{1}{2}x^7\,dx = \Big[\tfrac{1}{12}x^6 - \tfrac{1}{16}x^8\Big]_0^1 = \tfrac{1}{12} - \tfrac{1}{16} = \tfrac{1}{48}.$

$$\int_0^1\int_{\sqrt{y}}^{y^{1/3}} xy\,dx\,dy = \int_0^1 \Big[\tfrac{1}{2}x^2y\Big]_{\sqrt{y}}^{y^{1/3}} dy = \int_0^1 \tfrac{1}{2}y^{5/3} - \tfrac{1}{2}y^2\,dy = \Big[\tfrac{3}{16}y^{8/3} - \tfrac{1}{6}y^3\Big]_0^1 = \tfrac{1}{48}.$$

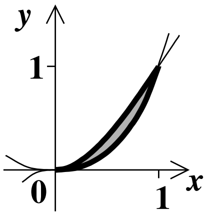
17. $\displaystyle\int_0^{\pi/2}\int_{2x/\pi}^{\sin(x)} f(x,y)\,dy\,dx.$

18. $\displaystyle\int_0^1\int_1^{1/y} f(x,y)\,dx\,dy.$

19. $\displaystyle\int_0^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)\,dx\,dy.$

20. $\displaystyle\int_0^1\int_y^{2-y} f(x,y)\,dx\,dy.$

21. $\displaystyle\int_0^3\int_{\sqrt{y/3}}^{y+1} f(x,y)\,dx\,dy.$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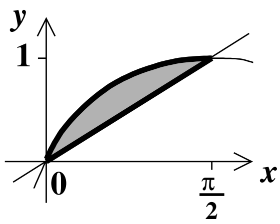
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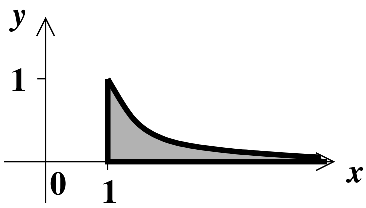
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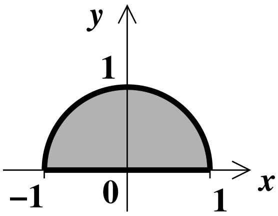
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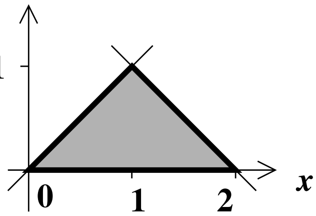
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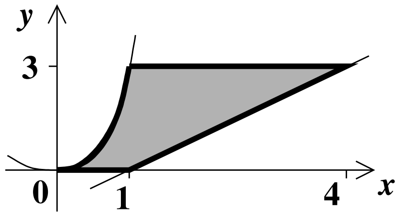
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