ODE: Practice problems—Separable equations

1. For the equation $\dot{x} = \dfrac{x^2 - x}{t}$ solve the following Cauchy problems:

a) $x(1) = 2$; b) $x(4) = \frac{1}{2}$; c) $x(-1) = \frac{2}{3}$; d) $x(1) = \frac{3}{4}$;
e) $x(-2) = 1$; f) $x(3) = 0$; g) $x(1) = -1$; h) $x(0) = 3$.

For the following problems, first find a general solution and discuss conditions on existence (perhaps depending on C). If solutions exist on a neighborhood of infinity, discuss their asymptotic behaviour as $x \to \infty$. Finally, solve the given initial value problem.

2. $\dfrac{y'}{y+1} = -4x^3, \quad y(0) = 0;$

3. $3y' = \dfrac{1}{y^2}, \quad y(2) = 1;$

4. $y' = e^{x-y}, \quad y(0) = \ln(4);$

5. $\dot{x} = \dfrac{x^2}{t^2}, \quad x(-1) = -\frac{1}{2};$

6. $y' = \dfrac{2xy}{x^2 - 4}, \quad y(1) = -6;$

7. $yy' = -x, \quad y(4) = -3;$

8. $\dfrac{2y'}{1 - y^2} = \dfrac{2}{x}, \quad y(1) = -\frac{5}{3};$

9. $\dfrac{y'}{2\sqrt{y}} = e^x, \quad y(2) = e^4 - 4e^2 + 4;$

10. $x' = \dfrac{(x+2)\cos(t)}{\sin(t) + 2}, \quad x(0) = 0;$

11. $y' = -2x^3y^3, \quad y(5) = \frac{1}{\sqrt{621}};$

12. $\dfrac{e^y y'}{e^y - 1} = \dfrac{4}{x}, \quad y(1) = \ln(2);$

13. $y' = \cos(x)y^2, \quad y\left(\frac{\pi}{2}\right) = 1;$

14. $y' - y^2 = 1, \quad y(\pi) = 0;$

15. $\dfrac{y'}{y} = -\dfrac{1}{x}, \quad y(-2) = -3;$

16. $y' = 4t\sqrt{y}, \quad y(0) = 1;$

17. $2y' + 1 = y^2, \quad y(1) = \frac{1+2e}{1-2e};$

18. $\dfrac{y'}{y-1} = \dfrac{3}{x}, \quad y(1) = 2;$

19. $\dfrac{y'}{y-1} = -\dfrac{y}{x}, \quad y(2) = -1;$

20. $2\sqrt{x}y' = y^2, \quad y(9) = -1;$

21. $\dfrac{y'}{y+1} = \dfrac{\cos(x)}{\sin(x)}, \quad y\left(\frac{\pi}{2}\right) = 1;$

22. $\dot{x} = \dfrac{e^{-x}}{t}, \quad x(1) = 0;$

23. $y' = \dfrac{1 - y^2}{1 - x^2}, \quad y(0) = 0.$

Solutions

1. Conditions from the equation: $t \neq 0$.
Stationary solutions: $x(t) = 0$ on $(-\infty, 0)$ and on $(0, \infty)$, also $x(t) = 1$ on $(-\infty, 0)$ and on $(0, \infty)$. We separate and integrate: $\int \frac{dx}{x^2-x} = \int \frac{dt}{t}$, partial fractions

$$\int \frac{dx}{x(x-1)} = \int \frac{1}{x-1} - \frac{1}{x}\, dx = \ln|x-1| - \ln|x|.$$

Equation $\ln\left|\frac{x-1}{x}\right| = \ln|t| + c$, $\frac{x-1}{x} = \pm e^c t$, the usual trick with $C = \pm e^c \neq 0$, hence $1 - \frac{1}{x} = Ct$, general solution $x(t) = \frac{1}{1-Ct}$, $t \neq 0$, $t \neq \frac{1}{C}$, satisfies $x \neq 0$ and $x \neq 1$ due to $C \neq 0$ and $t \neq 0$.
Remark: Choice $C = 0$ yields stationary solution $x(t) = 1$, but $x(t) = 0$ cannot be obtained this way.
Initial conditions:
a) $C = \frac{1}{2}$, hence $x_a(t) = \frac{1}{1-t/2}$, $t \in (0, 2)$;
b) $C = -\frac{1}{4}$, hence $x_b(t) = \frac{1}{1+t/4}$, $t \in (0, \infty)$;
c) $C = \frac{1}{2}$, hence $x_c(t) = \frac{1}{1-t/2}$, $t \in (-\infty, 0)$;
d) $C = -\frac{1}{3}$, hence $x_d(t) = \frac{1}{1+t/3}$, $t \in (0, \infty)$;
e) $C = 0$, hence stacionrn een $x_e(t) = 1$, $t \in (-\infty, 0)$;
f) C not found, stationary solution $x_f(t) = 0$, $t \in (0, \infty)$;
g) $C = 2$, hence $x_g(t) = \frac{1}{1-2t}$, $t \in \left(\frac{1}{2}, \infty\right)$;
h) $x_h(t)$ does not exist.

2. Conditions from the equation: $y \neq -1$.
Separation: $\int \frac{dy}{y+1} = -\int 4x^3 dx$. Stat. sol.: candidate $y(x) = -1$ ruled out by the condition.
Integration: $\ln|y+1| = -x^4 + c$, the usual trick with $C = \pm e^c \neq 0$, general solution: $y(x) = C\, e^{-x^4} - 1$.
Existence: Due to $C \neq 0$ we have $y \neq -1$, hence $x \in I\!R$.
For $x \sim \infty$ we have $y(x) \to -1$.
Init. cond.: $C = 1$, solution $y(x) = e^{-x^4} - 1$, $x \in I\!R$.

3. Conditions from the equation: $y \neq 0$
Separation: $\int 3y^2 dy = \int 1 dx$. Stat. sol.: DNE.
Integration: $y^3 = x + C$, general solution: $y(x) = (x + C)^{1/3}$.
Existence: $y \neq 0 \implies x \neq -C$, two intervals.
For $x \sim \infty$ we have $y(x) \sim x^{1/3}$.
Init. cond.: $C = -1$, solution $y(x) = (x-1)^{1/3}$, $x \in (1, \infty)$.

4. Conditions from the equation: none.
Separation: $\int e^y dy = \int e^x dx$. Stat. sol.: DNE.
Integration: $e^y = e^x + C$, general solution: $y(x) = \ln(e^x + C)$.
Existence: $x \in I\!R$.
For $x \sim \infty$ we have $y(x) \sim \ln(e^x) = x$.
Init. cond.: $C = 3$, solution $y(x) = \ln(e^x + 3)$, $x \in I\!R$.

5. Conditions from the equation: $t \neq 0$.
Separation: $\int \frac{dx}{x^2} = \int \frac{dt}{t^2}$. Stat. sol.: $x(t) = 0$, $t \neq 0$.
Integration: $-\frac{1}{x} = -\frac{1}{t} - C$, general solution: $x(t) = 0$ or $x(t) = \frac{-1}{-1/t - C} = \frac{t}{Ct+1}$, this satisfies $x \neq 0$ due to $t \neq 0$.
Existence: $t \neq 0$, $Ct + 1 \neq 0$, that is, $t \neq -\frac{1}{C}$ for $C \neq 0$.
For $t \sim \infty$ we have $x(t) \sim \frac{1}{C}$ for $C \neq 0$, for $C = 0$ we get $x(t) = t$, for stat. sol. we have $x(t) = 0$.
Init. cond.: $C = -1$, solution $x(t) = \frac{t}{1-t}$, $t \in (-\infty, 0)$.

6. Conditions from the equation: $x \neq \pm 2$.
Separation: $\int \frac{dy}{y} = \int \frac{2x\, dx}{x^2-4}$. Stat. sol.: $y(x) = 0$, $x \neq \pm 2$.
Integration: substitution $w = x^2 - 4$, $\ln|y| = \ln|x^2 - 4| + c$, the usual trick with $C = \pm e^c \neq 0$, general solution: $y(x) = 0$ or $y(x) = C(x^2 - 4)$, it satisfies $y \neq 0$ due to $C \neq 0$ and $x \neq \pm 2$. Choice $C = 0$ includes the stationary sol.
Existence: $x \neq \pm 2$.
For $x \sim \infty$ we have $y(x) \sim Cx^2$.
Init. cond.: $C = 2$, solution $y(x) = 2(x^2 - 4)$, $x \in (-2, 2)$.

7. Conditions from the equation: none.
Separation: $\int y dy = -\int x\, dx$. Stat. sol.: DNE.
Integration: $\frac{1}{2}y^2 = -\frac{1}{2}x^2 + c$, we choose $C = 2c$, general solution: $y(x) = \pm\sqrt{C - x^2}$.

Existence: $C - x^2 \geq 0$, for $C < 0$ it means $|x| \leq C$, otherwise this does not make sense.
Init. cond.: $C = 25$, solution $y(x) = -\sqrt{25 - x^2}$, $x \in [-5, 5]$.

8. Conditions from the equation: $x \neq 0$, $y \neq \pm 1$.
Separation: $\int \frac{2\,dy}{1-y^2} = \int \frac{2}{x}\,dx$. Stat. sol.: Candidates $y(x) = \pm 1$ ruled out by the condition.
Integration: partial fractions, $\ln\left|\frac{y+1}{y-1}\right| = 2\ln|x| + c = \ln|x^2| + c = \ln(x^2) + c$, the usual trick with $C = \pm e^c \neq 0$, general solution: $y(x) = \frac{Cx^2+1}{Cx^2-1}$, due to $C \neq 0$ we have $y \neq \pm 1$.
Existence: $x \neq 0$, $Cx^2 - 1 \neq 0$, this is restrictive only for $C > 0$, then $x \neq \pm\frac{1}{\sqrt{C}}$.
For $x \sim \infty$ we have $y(x) \to 1$.
Init. cond.: $C = \frac{1}{4}$, solution $y(x) = \frac{x^2+4}{x^2-4}$, $x \in (0, 2)$.

9. Conditions from the equation: $y > 0$.
Separation: $\int \frac{dy}{2\sqrt{y}} = \int e^x dx$. Stat. sol.: candidate $y(x) = 0$ ruled out by the condition.
Integration: $\sqrt{y} = e^x + C$, beware it demands condition $e^x + C \geq 0$, general solution: $y(x) = (e^x + C)^2$.
Existence: $e^x + C > 0$, for $C < 0$ it means $x > \ln(-C)$, for $C \geq 0$ it means $x \in I\!R$.
For $x \sim \infty$ we have $y(x) \sim e^{2x}$.
Init. cond.: $y(2) = e^4 - 4e^2 + 4$, solution $C = -2$, $y(x) = (e^x - 2)^2$, $x \in (\ln(2), \infty)$.

10. Conditions from the equation: none.
Separation: $\int \frac{dx}{x+2} = \int \frac{\cos(t)}{\sin(t)+2}\,dt$. Stat. sol.: $x(t) = -2$.
Integration: substitution $w = \sin(t)+2$, $\ln|x+2| = \ln|\sin(t)+2| + c$, the usual trick with $C = \pm e^c \neq 0$, general solution: $x(t) = -2$ or $x(t) = C(\sin(t)+2) - 2$, where $x \neq -2$ due to $\sin(t) + 2 \neq 0$ and $C \neq 0$. Choice $C = 0$ includes the stationary solution.
Existence: $t \in I\!R$.
For $x \sim \infty$ we cannot simplify the solution. It is bounded.
Init. cond.: $C = 1$, solution $x(t) = \sin(t)$, $t \in I\!R$.

11. Conditions from the equation: none.
Separation: $\int \frac{dy}{y^3} = -\int 2x^3 dx$. Stat. sol.: $y(x) = 0$.
Integration: $\frac{-1}{2y^2} = -\frac{1}{2}x^4 + c$, trik $C = 2c$, general solution: $y(x) = 0$ or $y(x) = \pm\frac{1}{\sqrt{x^4 - C}}$, it satisfies $y \neq 0$.
Existence: $x^4 - C > 0$, for $C \geq 0$ it means $x \in I\!R$, for $C < 0$ it means $|x| > C^{1/4}$.
For $x \sim \infty$ we have $y(x) \sim \pm\frac{1}{x^2}$.
Init. cond.: $C = 4$, solution $y(x) = \frac{1}{\sqrt{x^4-4}}$, $x \in (\sqrt{2}, \infty)$.

12. Conditions from the equation: $x \neq 0$, $y \neq 0$.
Separation: $\int \frac{e^y dy}{e^y - 1} = \int \frac{4}{x} dx$. Stat. sol.: candidate $y(x) = 0$ ruled out by the condition.
Integration: substitution $w = e^y - 1$, $\ln|e^y - 1| = 4\ln|x| + c = \ln(x^4) + c$, the usual trick with $C = \pm e^c \neq 0$, general solution: $y(x) = \ln(Cx^4 + 1)$.
Existence: $Cx^4 + 1 > 0$, for $C < 0$ it means $|x| < |C|^{1/4}$, for $C > 0$ it means $x \in I\!R$; we also have $x \neq 0$; case $y = 0$ can't happen due to $C \neq 0$ and $x \neq 0$.
For $C > 0$ we can do $x \sim \infty$, then $y(x) \sim 4\ln(x)$.
Init. cond.: $C = 1$, solution $y(x) = \ln(x^4 + 1)$, $x \in (0, \infty)$.

13. Conditions from the equation: none.
Separation: $\int \frac{dy}{y^2} = \int \cos(x) dx$. Stat. sol.: $y(x) = 0$.
Integration: $-\frac{1}{y} = \sin(x) + C$, general solution: $y(x) = \frac{-1}{\sin(x)+C}$.
Existence: $\sin(x) + C \neq 0$, meaning depends on C.
Not clear whether we can go $x \to \infty$. If it were possible, $y(x)$ can't be simplified.
Init. cond.: $C = -2$, solution $y(x) = \frac{1}{2-\sin(x)}$, $x \in I\!R$.

14. Conditions from the equation: none.
Separation: $y' = 1 + y^2 \implies \int \frac{dy}{y^2+1} = \int 1 dx$. Stat. sol.: DNE.
Integration: $\arctan(y) = x + C$, general solution: $y(x) = \tan(x + C)$.
Existence: $x \neq \frac{\pi}{2} - C + k\pi$.
Function does not exist on any neighborhood of infinity, so it does not make sense to ask about $x \sim \infty$.
Init. cond.: $C = 0$, solution $y(x) = \tan(x)$, $x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$.

15. Conditions from the equation: $x, y \neq 0$.
Separation: $\int \frac{dy}{y} = -\int \frac{1}{x} dx$. Stat. sol.: candidate $y(x) = 0$ ruled out by the condition.
Integration: $\ln|y| = -\ln|x| + c = \ln\left|\frac{1}{x}\right| + c$, the usual trick with $C = \pm e^c \neq 0$, general solution: $y(x) = \frac{C}{x}$.
Existence: $x \neq 0$, due to $C \neq 0$ we cannot have $y = 0$.
For $x \sim \infty$ we have $y(x) \to 0$.
Init. cond.: $C = 6$, solution $y(x) = \frac{6}{x}$, $x \in (-\infty, 0)$.

16. Conditions from the equation: $y \geq 0$.
Separation: $\int \frac{dy}{\sqrt{y}} = \int 4t\, dt$. Stat. sol.: $y(x) = 0$.
Integration: $2\sqrt{y} = 2t^2 + c$, trik $C = \frac{1}{2}c$, hence $\sqrt{y} = t^2 + C$, thus $t^2 + C \geq 0$, general solution: $y(x) = 0$ or $y(t) = (t^2 + C)^2$.
Existence: $t^2 + C \geq 0$, for $C \geq 0$ it means $t \in I\!R$, for $C < 0$ it means $|t| \geq \sqrt{|C|}$.
Then $(t^2 + C)^2 = 0$ can happen, hence general solutions can be glued with the stationary one.
For $x \sim \infty$ we have $y(t) \sim t^4$.
Init. cond.: $C = 1$, solution $y(t) = (t^2 + 1)^2$, $t \in I\!R$. Then $y(t) = 0$ will not happen, hence uniqueness of this solution.
Remark: For instance, the initial condition $y(2) = 1$ yields $y(t) = (t^2 - 1)^2$, then for $t = 1$ we get $y(t) = 0$ and we have to investigate the possibility of connecting with the stationary solution. The function given by

$$f(t) = \begin{cases} 0, & t \leq 1; \\ (t^2 - 1)^2, & t \geq 1 \end{cases}$$

is continuous and satisfies the euqation on $(-\infty, 1)$ and $(1, \infty)$. At $t = 1$ both formulas have derivative equal to zero, hence the function f is differentiable on $I\!R$ and satisfies the equation there. Similarly we show that we can connect to $y(t) = 0$ at arbitrary point $t = -\sqrt{D}$ with the function $(t^2 - D)^2$, so there are infinitely many solutions satisfying the condition $y(2) = 1$, for any $D \geq 0$ it is

$$f(t) = \begin{cases} (t^2 - D)^2, & t \leq -\sqrt{D}; \\ 0, & -\sqrt{D} \leq t \leq 1; \\ (t^2 - 1)^2, & t \geq 1. \end{cases}$$

17. Conditions from the equation: none.
Separation: $2dy = y^2 - 1 \implies \int \frac{2dy}{y^2 - 1} = \int 1\, dx$. Stat. sol.: $y(x) = \pm 1$.
Integration: $\ln\left|\frac{y-1}{y+1}\right| = x + c$, the usual trick with $C = \pm e^c \neq 0$, general solution: $y(x) = -1$ or $y(x) = 1$ or $y(x) = \frac{1 + C\, e^x}{1 - C\, e^x}$, which due to $C \neq 0$ satisfies $y \neq \pm 1$. Choice $C = 0$ includes $y(x) = 1$.
Existence: $1 - C\, e^x \neq 0$, for $C < 0$ it means $x \in I\!R$, for $C > 0$ it means $x \neq \ln(1/C)$, that is, $x \neq -\ln(C)$.
For $x \sim \infty$ we have $y(x) \to -1$.
Init. cond.: $C = 2$, solution $y(x) = \frac{1 + 2e^x}{1 - 2e^x}$, $x \in (-\ln(2), \infty)$.

18. Conditions from the equation: $y \neq 1$, $x \neq 0$.
Separation: $\int \frac{dy}{y - 1} = \int \frac{3}{x}\, dx$. Stat. sol.: candidate $y(x) = 1$ ruled out by the condition.
Integration: $\ln|y - 1| = 3\ln|x| + c = \ln|x^3| + c$, the usual trick with $C = \pm e^c \neq 0$, general solution: $y(x) = 1 + Cx^3$, the case $y = 1$ won't happen due to $C \neq 0$ and $x \neq 0$.
For $x \sim \infty$ we have $y(x) \sim Cx^3$.
Init. cond.: $C = 1$, solution $y(x) = 1 + x^3$, $x \in (0, \infty)$.

19. Conditions from the equation: $x \neq 0$, $y \neq 1$.
Separation: $\int \frac{dy}{y^2 - y} = -\int \frac{1}{x}\, dx$. Stat. sol.: $y(x) = 0$, candidate $y(x) = 1$ ruled out by the condition.
Integration: partial fractions, $\ln\left|\frac{y-1}{y}\right| = -\ln|x| + c = \ln\left|\frac{1}{x}\right| + c$, the usual trick with $C = \pm e^c \neq 0$, general solution: $y(x) = 0$ or $y(x) = \frac{1}{1 - C/x} = \frac{x}{x - C}$, the case $y = 0$ won't happen due to $x \neq 0$.
Existence: $x \neq 0$, $x - C \neq 0$; due to $C \neq 0$ we won't get $y = 1$.
For $x \sim \infty$ we have $y(x) \to 1$.
Init. cond.: $C = -4$, solution $y(x) = \frac{x}{x - 4}$, $x \in (0, 4)$.

20. Conditions from the equation: $x \geq 0$.
Separation: $\int \frac{dy}{y^2} = \int \frac{dx}{2\sqrt{x}}$. Stat. sol.: $y(x) = 0$.
Integration: $-\frac{1}{y} = \sqrt{x} + C$, general solution: $y(x) = 0$ or $y(x) = \frac{-1}{\sqrt{x}+C}$, where $y \neq 0$. Stationary solution cannot be included by a choice of C.
Existence: $x \geq 0$, $\sqrt{x} + C > 0$ which for $C > 0$ means $x \geq 0$, for $C \leq 0$ it means $x \neq C^2$.
For $x \sim \infty$ we have $y(x) \sim \frac{-1}{\sqrt{x}}$ and $y(x) \to 0$.
Init. cond.: $C = -2$, solution $y(x) = \frac{1}{2-\sqrt{x}}$, $x \in (4, \infty)$.

21. Conditions from the equation: $y \neq -1$, $x \neq k\pi$.
Separation: $\int \frac{dy}{y+1} = \int \frac{\cos(x)}{\sin(x)} dx$. Stat. sol.: candidate $y(x) = -1$ ruled out by the condition.
Integration: substitution $w = \sin(x)$, $\ln|y+1| = \ln|\sin(x)| + c$, the usual trick with $C = \pm e^c \neq 0$, general solution: $y(x) = C\sin(x) - 1$, satisfies $y \neq -1$ due to $C \neq 0$ and $k \neq k\pi$.
Existence: $x \neq k\pi$.
The solution does not exist on any neighborhood of infinity, so it does not make sense to ask about $x \sim \infty$.
Init. cond.: $C = 2$, solution $y(x) = 2\sin(x) - 1$, $x \in (0, \pi)$.

22. Conditions from the equation: $t \neq 0$.
Separation: $\int e^x dx = \int \frac{dt}{t}$. Stat. sol.: DNE.
Integration: $e^x = \ln|t| + C$, general solution: $x(t) = \ln(\ln|t| + C)$.
Existence: $t \neq 0$, $\ln|t| + C > 0$, that is, $|t| > e^{-C}$. Since $e^{-C} > 0$, it also includes $t \neq 0$. Possible intervals $(-\infty, -e^{-C})$ and (e^{-C}, ∞).
For $t \sim \infty$ we have $x(t) \sim \ln(\ln(t))$.
Init. cond.: $C = 1$, solution $x(t) = \ln(\ln(t) + 1)$, $x \in \left(\frac{1}{e}, \infty\right)$.

23. Conditions from the equation: $x \neq \pm 1$.
Separation: $\int \frac{dy}{1-y^2} = \int \frac{dx}{1-x^2}$. Stat. sol.: $y(x) = \pm 1$.
Integration: partial fractions, $\ln\left|\frac{y-1}{y+1}\right| = \ln\left|\frac{x-1}{x+1}\right| + c$, the usual trick with $C = \pm e^c \neq 0$, general solution: $y(x) = -1$ or $y(x) = 1$ or $y(x) = \frac{C(x+1)+(x-1)}{C(x+1)-(x-1)}$, where $y \neq \pm 1$ due to $C \neq 0$ and $x \neq \pm 1$.
Choice $C = 0$ includes $y(x) = -1$, the second stationary solution cannot be included.
Existence: $x \neq \pm 1$, also $C(x+1) - (x-1) \neq 0$, that is, $x \neq \frac{C+1}{C-1}$ for $C \neq 1$, otherwise no restriction.
For $x \sim \infty$ we have $y(x) \sim \frac{C+1}{C-1}$ if $C \neq 1$, otherwise $y(x) = x$.
Init. cond.: $C = 1$, solution $y(x) = x$, $x \in (-1, 1)$.