

### ODE: Practice problems—Homogeneous linear differential equations

For the following equations, find their their general solutions and discuss typical asymptotic behaviour near infinity.

If initial conditions are given, find the corresponding particular solution.

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|------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------|
| <p>1. <math>y'' + y' - 6y = 0</math>,<br/><math>y(1) = e^2, y'(1) = 2e^2</math>;</p>                                               | <p>9. <math>y'' - A^2y = 0, \quad A &gt; 0</math>,<br/><math>y(0) = 1, y'(0) = 0</math>;</p>               |
| <p>2. <math>y'' - y' = 0</math>,<br/><math>y(-1) = 13, y'(-1) = 0</math>;</p>                                                      | <p>10. <math>x''' - x'' - 2x' = 0</math>,<br/><math>x(0) = 0, x'(0) = 1, x''(0) = -1</math>;</p>           |
| <p>3. <math>x'' + 9x = 0</math>,<br/><math>x(\pi) = 0, x'(\pi) = 3</math>;</p>                                                     | <p>11. <math>y''' - 2y'' + 10y' = 0</math>;</p>                                                            |
| <p>4. <math>y'' - 3y' + 2y = 0</math>,<br/><math>y(0) = 3, y'(0) = 4</math>;</p>                                                   | <p>12. <math>y''' - 3y'' + 3y' - y = 0</math>;</p>                                                         |
| <p>5. <math>x'' - 6x' + 9x = 0</math>,<br/><math>x(0) = 0, x'(0) = 3</math>;</p>                                                   | <p>13. <math>x''' - A^2x' = 0, A &gt; 0</math>,<br/><math>x(0) = 1, x'(0) = 2A, x''(0) = 0</math>;</p>     |
| <p>6. <math>y'' - 2y = 0</math>,<br/><math>y(0) = 2, y'(0) = 0</math>;</p>                                                         | <p>14. <math>y'''' - 2y''' + y'' = 0</math>;</p>                                                           |
| <p>7. <math>y'' - 6y' + 13y = 0</math>,<br/><math>y(0) = 0, y'(0) = 2</math>;</p>                                                  | <p>15. <math>x^{(4)} - x'' = 0</math>,<br/><math>x(1) = 14, x'(1) = 1, x''(0) = 0, x'''(0) = 0</math>;</p> |
| <p>8. <math>\ddot{x} + \omega^2x = 0, \quad \omega \in \mathbb{N}</math>,<br/><math>x(2\pi) = 1, \dot{x}(2\pi) = \omega</math></p> | <p>16. <math>y^{(4)} - y = 0</math>.</p>                                                                   |

#### Bonus:

Bonus questions for problem 5.:

a) Find some initial conditions at  $t_0 = 0$  so that the corresponding solution satisfies  $x(2) = 2e^6$ ,  $x'(2) = 7e^6$ .

b) Find some initial conditions at  $t_0 = 0$  so that the corresponding solution grows at infinity at the rate  $O(e^{3t})$ .

17. Consider the equation  $y'' + 4y' + py = 0$ , where  $p \in \mathbb{R}$  is a parameter.

Determine values of  $p$  for which all solutions will converge to zero at infinity.

18. Consider the equation  $y'' - 2y' + \frac{1}{2}py = 0$ , where  $p \in \mathbb{R}$  is a parameter.

Determine values of  $p$  for which the natural fundamental system consists of monotone functions.

19. Consider the equation  $y'' + py' + qy = 0$ , where  $p, q \in \mathbb{R}$  are parameters.

Determine values of  $p, q$  for which all solutions are bounded on  $\mathbb{R}$ .

20. Consider the equation  $y'' + py' + qy = 0$ , where  $p, q \in \mathbb{R}$  are parameters.

Determine values of  $p, q$  for which all solutions are both concave up and concave down.

## Solutions

**1.** Char. pol.  $p(\lambda) = \lambda^2 + \lambda - 6 = (\lambda + 3)(\lambda - 2)$ , char. numbers  $\lambda = -3, 2$ ;  
 fund. system:  $\{e^{-3x}, e^{2x}\}$ , general solution:  $y(x) = a e^{-3x} + b e^{2x}$ ,  $x \in \mathbb{R}$ .  
 For  $x \sim \infty$  we have  $y(x) \sim b e^{2x}$ .  
 Init. cond.:  $y(x) = e^{2x}$ ,  $x \in \mathbb{R}$ .

**2.** Char. pol.  $p(\lambda) = \lambda^2 - \lambda = \lambda(\lambda - 1)$ , char. numbers  $\lambda = 0, 1$ ;  
 fund. system:  $\{e^0 = 1, e^x\}$ ; general solution:  $y(x) = a + b e^x$ ,  $x \in \mathbb{R}$ .  
 For  $x \sim \infty$  we have  $y(x) \sim b e^x$ .  
 Init. cond.:  $y(x) = 13$ ,  $x \in \mathbb{R}$ .

**3.** Char. pol.  $p(\lambda) = \lambda^2 + 9$ , char. numbers  $\lambda = \pm 3i$ ;  
 fund. system:  $\{\sin(3t), \cos(3t)\}$ ; general solution:  $x(t) = a \sin(3t) + b \cos(3t)$ ,  $t \in \mathbb{R}$ .  
 For  $t \sim \infty$  we cannot simplify the function, the solution is bounded.  
 Init. cond.:  $x(t) = -\sin(3t)$ ,  $t \in \mathbb{R}$ .

**4.** Char. pol.  $p(\lambda) = \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2)$ , char. numbers  $\lambda = 1, 2$ ;  
 fund. system:  $\{e^x, e^{2x}\}$ ; general solution:  $y(x) = a e^x + b e^{2x}$ ,  $x \in \mathbb{R}$ .  
 For  $x \sim \infty$  we have  $y(x) \sim b e^{2x}$ .  
 Init. cond.:  $y(x) = 2e^x + e^{2x}$ ,  $x \in \mathbb{R}$ .

**5.** Char. pol.  $p(\lambda) = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2$ , char. numbers  $\lambda = 3$  ( $2\times$ );  
 fund. system:  $\{e^{3t}, t e^{3t}\}$ ; general solution:  $x(t) = a e^{3t} + b t e^{3t}$ ,  $t \in \mathbb{R}$ .  
 For  $t \sim \infty$  we have  $x(t) \sim b t e^{3t}$ .  
 Init. cond.:  $x(t) = 3t e^{3t}$ ,  $t \in \mathbb{R}$ .

Bonus: a) From the given conditions, the desired solution is  $x(t) = t e^{3t}$ . At  $t_0 = 0$  it satisfies  $x(0) = 0$ ,  $x'(0) = 1$ , so these are the desired initial conditions.

b) In order for the general solution  $x(t) = a e^{3t} + b t e^{3t}$  to grow at the rate  $e^{3t}$ , we need to prevent the dominant term  $t e^{3t}$  from appearing, so we want  $b = 0$ .

For instance  $x(t) = e^{3t}$  will do, and it satisfies the initial conditions  $x(0) = 1$ ,  $x'(0) = 3$ .

Any answer of the form  $x(0) = a$ ,  $x'(0) = 3a$  for some  $a \in \mathbb{R}$  will do. In fact, the correct answers are recognizable by the test  $x'(0) = 3x(0)$ .

Really? How about  $x(0) = x'(0) = 0$ ? The corresponding solution  $x(t) = 0$  grows at the rate  $O(1) = O(t^0)$ , which actually also is  $O(e^{3t})$ , the latter is dominant. So it is fine.

**6.** Char. pol.  $p(\lambda) = \lambda^2 - 2$ , char. numbers  $\lambda = \pm\sqrt{2}$ ;  
 fund. system:  $\{e^{-\sqrt{2}x}, e^{\sqrt{2}x}\}$ ; general solution:  $y(x) = a e^{-\sqrt{2}x} + b e^{\sqrt{2}x}$ ,  $x \in \mathbb{R}$ .  
 For  $x \sim \infty$  we have  $y(x) \sim b e^{\sqrt{2}x}$ .  
 Init. cond.:  $y(x) = e^{-\sqrt{2}x} + e^{\sqrt{2}x}$ ,  $x \in \mathbb{R}$ .

**7.** Char. pol.  $p(\lambda) = \lambda^2 - 6\lambda + 13$ , char. numbers  $\lambda = 3 \pm 2i$ ;  
 fund. system:  $\{e^{3x} \sin(2x), e^{3x} \cos(2x)\}$ ; general solution:  $y(x) = a e^{3x} \sin(2x) + b e^{3x} \cos(2x)$ ,  $x \in \mathbb{R}$ .  
 For  $x \sim \infty$  the function cannot be asymptotically simplified.  
 Init. cond.:  $y(x) = e^{3x} \sin(2x)$ ,  $x \in \mathbb{R}$ .

**8.** Char. pol.  $p(\lambda) = \lambda^2 + \omega^2$ , char. numbers  $\lambda = \pm\omega i$ ;  
 fund. system:  $\{\sin(\omega t), \cos(\omega t)\}$ ; general solution:  $x(t) = a \sin(\omega t) + b \cos(\omega t)$ ,  $t \in \mathbb{R}$ .  
 For  $t \sim \infty$  we cannot simplify the function, the solution is bounded.  
 Init. cond.:  $x(t) = \sin(\omega t) + \cos(\omega t)$ ,  $t \in \mathbb{R}$ .

**9.** Char. pol.  $p(\lambda) = \lambda^2 - A^2$ , char. numbers  $\lambda = \pm A$ ;  
 fund. system:  $\{e^{-Ax}, e^{Ax}\}$ ; general solution:  $y(x) = a e^{-Ax} + b e^{Ax}$ ,  $x \in \mathbb{R}$ .  
 For  $x \sim \infty$  we have  $y(x) \sim b e^{Ax}$ .  
 Init. cond.:  $y(x) = \frac{1}{2} e^{-Ax} + \frac{1}{2} e^{Ax} = \cosh(Ax)$ ,  $x \in \mathbb{R}$ .

**10.** Char. pol.  $p(\lambda) = \lambda^3 - \lambda^2 - 2\lambda = \lambda(\lambda + 1)(\lambda - 2)$ , char. numbers  $\lambda = 0, -1, 2$ ;  
 fund. system:  $\{1, e^{-t}, e^{2t}\}$ ; general solution:  $x(t) = a + b e^{-t} + c e^{2t}$ ,  $t \in \mathbb{R}$ .  
 For  $t \sim \infty$  we have  $x(t) \sim c e^{2t}$ .  
 Init. cond.:  $x(t) = 1 - e^{-t}$ ,  $t \in \mathbb{R}$ .

**11.** Char. pol.  $p(\lambda) = \lambda^3 - 2\lambda^2 + 10\lambda = \lambda(\lambda^2 - 2\lambda + 10)$ , char. numbers  $\lambda = 0, 1 \pm 3i$ ;  
fund. system:  $\{1, e^x \sin(3x), e^x \cos(3x)\}$ ; general solution:  $y(x) = a + b e^x \sin(3x) + c e^x \cos(3x)$ ,  $x \in \mathbb{R}$ .  
For  $x \sim \infty$  the function cannot be asymptotically simplified.

**12.** Char. pol.  $p(\lambda) = \lambda^3 - 3\lambda^2 + 3\lambda - 1 = (\lambda - 1)^3$ , char. numbers  $\lambda = 1$  ( $3\times$ );  
fund. system:  $\{e^x, x e^x, x^2 e^x\}$ ; general solution:  $y(x) = a e^x + b x e^x + c x^2 e^x = (a + b x + c x^2) e^x$ ,  $x \in \mathbb{R}$ .  
For  $x \sim \infty$  we have  $y(x) \sim c x^2 e^x$ .

**13.** Char. pol.  $p(\lambda) = \lambda^3 - A^2 \lambda = \lambda(\lambda + A)(\lambda - A)$ , char. numbers  $\lambda = 0, \pm A$ ;  
fund. system:  $\{1, e^{-At}, e^{At}\}$ ; general solution:  $x(t) = a + b e^{-At} + c e^{At}$ ,  $t \in \mathbb{R}$ .  
For  $t \sim \infty$  we have  $x(t) \sim c e^{At}$ .  
Init. cond.:  $x(t) = 1 - e^{-At} + e^{At} = 1 - 2 \sinh(At)$ ,  $t \in \mathbb{R}$ .

**14.** Char. pol.  $p(\lambda) = \lambda^4 - 2\lambda^3 + \lambda^2 = \lambda^2(\lambda - 1)^2$ , char. numbers  $\lambda = 0$  ( $2\times$ ),  $1$  ( $2\times$ );  
fund. system:  $\{1, x, e^x, x e^x\}$ ; general solution:  $y(x) = a + b x + c e^x + d x e^x$ ,  $x \in \mathbb{R}$ .  
For  $x \sim \infty$  we have  $y(x) \sim d x e^x$ .

**15.** Char. pol.  $p(\lambda) = \lambda^4 - \lambda^2 = \lambda^2(\lambda - 1)(\lambda + 1)$ , char. numbers  $\lambda = 0$  ( $2\times$ ),  $\pm 1$ ;  
fund. system:  $\{1, t, e^{-t}, e^t\}$ ; general solution:  $x(t) = a + b t + c e^{-t} + d e^t$ ,  $t \in \mathbb{R}$ .  
For  $t \sim \infty$  we have  $x(t) \sim d e^t$ .  
Init. cond.:  $x(t) = 13 + t$ ,  $t \in \mathbb{R}$ .

**16.** Char. pol.  $p(\lambda) = \lambda^4 - 1 = (\lambda^2 + 1)(\lambda^2 - 1) = (\lambda^2 + 1)(\lambda + 1)(\lambda - 1)$ , char. numbers  $\lambda = \pm 1, \pm i$ ;  
fund. system:  $\{e^{-x}, e^x, \sin(x), \cos(x)\}$ ; general solution:  $y(x) = a e^{-x} + b e^x + c \sin(x) + d \cos(x)$ ,  $x \in \mathbb{R}$ .  
For  $x \sim \infty$  we have  $y(x) \sim b e^x$ .

**17.**  $\lambda = \frac{-4 \pm \sqrt{16 - 4p}}{2} = -2 \pm \sqrt{4 - p}$ .

Cases:  $p > 4$ : complex  $\lambda$ , solutions of the type  $a e^{-2x} \cos(\omega x) + b e^{-2x} \sin(\omega x)$ , these go to zero. So this  $p$  is fine.

$p = 4$ : Double  $\lambda$ , solution  $a e^{-2x} + b x e^{-2x}$ , this also goes to zero (we know  $\frac{x}{e^{2x}} \rightarrow 0$ ), so this is OK.

$p < 4$ : Two real solutions, we cannot allow one of them to be zero or positive. We thus want  $-2 + \sqrt{4 - p} < 0$ , hence  $p > 0$ . So  $p$  from  $(0, 4)$  are fine.

Conclusion: We can take  $p > 0$ .

**18.**  $\lambda = \frac{2 \pm \sqrt{4 - 2p}}{2} = 1 \pm \frac{1}{2} \sqrt{4 - 2p}$ .

Cases:  $p > 2$ : the fundamental system is  $\{e^x \cos(\omega x), e^x \sin(\omega x)\}$ . Functions are not monotone, we do not want that.

$p = 2$ : Double  $\lambda$ , fund. system  $\{e^x, x e^x\}$ . The first function is increasing, how about  $x e^x$ ? We have  $[x e^x]' = (1 + x)e^x$ , this could be both positive and negative. Thus  $x e^x$  is not monotone, we do not want that.

$p < 2$ : Two real solutions, so two exponentials, or perhaps for  $\lambda = 0$  a constant. The constant is monotone, exponentials too, we like this.

Conclusion: We can take  $p < 2$ .

**19.**  $\lambda = \frac{-p \pm \sqrt{p^2 - 4q}}{2} = -\frac{1}{2} p \pm \frac{1}{2} \sqrt{p^2 - 4q}$ .

Cases:  $p \neq 0$ : Then at least one of lambdas has a non-zero real part, the corresponding solution then features an exponential and exponentials are not bounded on  $\mathbb{R}$ . We do not want this, so  $p = 0$  is a must.

$p = 0$ : Then  $\lambda = \pm \frac{1}{2} \sqrt{-4q} = \pm \sqrt{-q}$ .

If  $q < 0$ , we get two real  $\lambda$  and we get exponentials again. We do not want that.

If  $q = 0$ , we get a double  $\lambda = 0$ , with general solution  $a + b x$ . This need not be bounded, we do not want this.

If  $q > 0$ , then the solution is  $a \cos(\sqrt{-q} x) + b \sin(\sqrt{-q} x)$ , this is bounded, we want this.

Conclusion: We can take  $p = 0$  and  $q > 0$ .

**20.**  $\lambda = \frac{-p \pm \sqrt{p^2 - 4q}}{2} = -\frac{1}{2} p \pm \frac{1}{2} \sqrt{p^2 - 4q}$ .

Analysis: The only functions that are simultaneously concave up and concave down are linear functions, that is, of the form  $a + b x$ . To get them as a solution, we need to have the repeated root  $\lambda = 0$ . This means that  $\sqrt{p^2 - 4q} = 0$ , that is,  $q = \frac{1}{4} p^2$ , then we get  $\lambda = -\frac{1}{2} p$  and the situation is clear.

Conclusion: We can only take  $p = 0$  and  $q = 0$ .