ODE: Practice problems—Non-homogeneous systems of equations

Solve the following systems of equations. Use the matrix approach, but if you want, you can also try elimination for fun.

1. $$\begin{aligned} y_1' &= y_1 - y_2 + 2e^x \\ y_2' &= -y_1 + y_2 + e^x \end{aligned} \qquad y_1(0) = 0,\ y_2(0) = 3;$$

2. $$\begin{aligned} x_1' &= x_1 - x_2 + 9 \\ x_2' &= 10x_1 - x_2 + 10e^t \end{aligned} \qquad x_1(0) = -1,\ x_2(0) = 6.$$

Find a general solution for the following system of equations.

3. $$\begin{aligned} y_1' &= y_1 + y_3 + 6e^x \\ y_2' &= y_1 - y_2 - 6 \\ y_3' &= y_1 + y_3 \end{aligned}$$

Solutions

1. 1) General solution. **Eigenvalues:** $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, $|A - \lambda E| = \begin{vmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda = 0$ gives $\lambda = 0, 2$.

$\lambda = 0$: $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $v_1 - v_2 = 0$, choose $v_2 = 1$, then $v_1 = 1$, $\vec{y}_a(x) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

$\lambda = 2$: $\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $-v_1 - v_2 = 0$, choose $v_2 = -1$, then $v_1 = 1$, $\vec{y}_b(x) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2x}$.

General solution of the associated homogeneous equation:

$\vec{y}_h(x) = a\vec{y}_a + b\vec{y}_b = a\begin{pmatrix} e^x \\ -e^x \end{pmatrix} + b\begin{pmatrix} e^{2x} \\ -e^{2x} \end{pmatrix} = \begin{pmatrix} a + be^{2x} \\ a - be^{2x} \end{pmatrix}$.

Rewrite: $y_{1h}(x) = a + be^{2x}$, $y_{2h}(x) = a - be^{2x}$, $x \in I\!R$.

Remark: Fundamental matrix $Y(x) = \begin{pmatrix} 1 & e^{2x} \\ 1 & -e^{2x} \end{pmatrix}$.

Non-zero right hand-side: Method of undetermined coefficients (guessing). One factor e^x with $\lambda = 1$, no correction. Guess $y_{1p} = A\,e^x$, $y_{2p} = B\,e^x$. Substituting into the equations we get $A\,e^x = A\,e^x - B\,e^x + 2e^x$, $B\,e^x = -A\,e^x + B\,e^x + e^x$, this yields $A = 1$, $B = 2$. Solution $y_{1p}(x) = e^x$, $y_{2p}(x) = 2e^x$.

General solution: $y_1(x) = e^x + a + be^{2x}$, $y_2(x) = 2e^x + a - be^{2x}$, $x \in I\!R$.

Alternative: variation of parameter. General solution of associated homogeneous $\vec{y}_h = Y(x)\vec{c}$, variation $\vec{y}(x) = Y(x)\vec{c}(x)$, equation $Y(x)\vec{c}\,'(x) = \vec{b}(x)$. We need Y^{-1}:

$\begin{pmatrix} 1 & e^{2x} & 1 & 0 \\ 1 & -e^{2x} & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{2}e^{-2x} & -\frac{1}{2}e^{-2x} \end{pmatrix}$, hence $Y^{-1}(x) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ e^{-2x} & -e^{-2x} \end{pmatrix}$.

Then $\vec{c}\,' = Y^{-1}\vec{b} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ e^{-2x} & -e^{-2x} \end{pmatrix}\begin{pmatrix} 2e^x \\ e^x \end{pmatrix} = \begin{pmatrix} \frac{3}{2}e^x \\ \frac{1}{2}e^{-x} \end{pmatrix}$, hence $\vec{c}(x) = \begin{pmatrix} \frac{3}{2}e^x \\ -\frac{1}{2}e^{-x} \end{pmatrix}$ and

$\vec{y}_p(x) = Y(x)\vec{c}(x) = \begin{pmatrix} e^x \\ 2e^x \end{pmatrix}$. Thus $\vec{y}(x) = y_p(x) + y_h(x) = \begin{pmatrix} e^x + a + be^{2x} \\ 2e^x + a - be^{2x} \end{pmatrix}$, $x \in I\!R$.

Alternative: Row variation. First homogeneous solution (by elimination or via eigenvalues):

$\left\{\begin{matrix} y_{1h}(x) = a + be^{2x} \\ y_{2h}(x) = a - be^{2x} \end{matrix}\right\}$, then variation $\left\{\begin{matrix} y_1(x) = a(x) + b(x)e^{2x} \\ y_2(x) = a(x) - b(x)e^{2x} \end{matrix}\right\}$. Equations $\left\{\begin{matrix} a'(x) + b'(x)e^{2x} = 2e^x \\ a'(x) - b'(x)e^{2x} = e^x \end{matrix}\right\}$ yield $\left\{\begin{matrix} a'(x) = \frac{3}{2}e^x \\ b'(x) = \frac{1}{2}e^{-x} \end{matrix}\right\}$, hence $\left\{\begin{matrix} a(x) = \frac{3}{2}e^x \\ b(x) = -\frac{1}{2}e^{-x} \end{matrix}\right\}$ and $\left\{\begin{matrix} y_{1p}(x) = e^x \\ y_{2p}(x) = 2e^x \end{matrix}\right\}$.

Elimination: From (#1) $y_2 = y_1 - y_1' + 2e^x$ $(*)$, into (#2) gives $y_1'' - 2y_1' = -e^x$, char. n. $\lambda = 0, 2$, solution of homogeneous equation $y_h(x) = a + be^{2x}$. Non-zero RHS is handled best by guessing, $y_p = Ae^x$, substituting into $y_1'' - 2y_1' = -e^x$ gives $A = 1$, hence $y_1(x) = y_p + y_h = e^x + a + be^{2x}$, from $(*)$ we get $y_2(x) = 2e^x + a - be^{2x}$.

2) Init. conditions yield $\vec{y}(x) = \begin{pmatrix} e^x - e^{2x} \\ 2e^x + e^{2x} \end{pmatrix}$, $x \in I\!R$.

2. 1) General solution. **Eigenvalues:** $A = \begin{pmatrix} 1 & -1 \\ 10 & -1 \end{pmatrix}$, $|A - \lambda E| = \begin{vmatrix} 1-\lambda & -1 \\ 10 & -1-\lambda \end{vmatrix} = \lambda^2 + 9 = 0$ gives $\lambda = \pm 3j$.

$\lambda = 3j$: $\begin{pmatrix} 1-3j & -1 \\ 10 & -1-3j \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $(1-3j)v_1 - v_2 = 0$, choose $v_1 = 1$, then $v_2 = 1 - 3j$, so

$\vec{x}_C(t) = \begin{pmatrix} 1 \\ 1-3j \end{pmatrix} e^{3jt} = \begin{pmatrix} 1 \\ 1-3j \end{pmatrix}[\cos(3t) + j\sin(3t)] = \begin{pmatrix} \cos(3t) + j\sin(3t) \\ \cos(3t) + 3\sin(3t) + j[\sin(3t) - 3\cos(3t)] \end{pmatrix}$.

We take $\vec{x}_a(t) = \mathrm{Im}(\vec{x}_C) = \begin{pmatrix} \sin(3t) \\ \sin(3t) - 3\cos(3t) \end{pmatrix}$, $\vec{x}_b(t) = \mathrm{Re}(\vec{x}_C) = \begin{pmatrix} \cos(3t) \\ 3\sin(3t) + \cos(3t) \end{pmatrix}$.

General solution of associated homogeneous

$$\vec{x}_h(t) = a\vec{x}_a + b\vec{x}_b = a\begin{pmatrix} \sin(3t) \\ \sin(3t) - 3\cos(3t) \end{pmatrix} + b\begin{pmatrix} \cos(3t) \\ 3\sin(3t) + \cos(3t) \end{pmatrix}$$
$$= \begin{pmatrix} a\sin(3t) + b\cos(3t) \\ a[\sin(3t) - 3\cos(3t)] + b[3\sin(3t) + \cos(3t)] \end{pmatrix},\ t \in I\!R.$$

Rewrite: $x_{1h}(t) = a\sin(3t) + b\cos(3t)$, $x_{2h}(t) = a[\sin(3t) - 3\cos(3t)] + b[3\sin(3t) + \cos(3t)]$, $t \in I\!R$.

Remark: Fundamental matrix sol. $X(t) = \begin{pmatrix} \sin(3t) & \cos(3t) \\ \sin(3t) - 3\cos(3t) & 3\sin(3t) + \cos(3t) \end{pmatrix}$.

Non-zero right hand-side: Method of undetermined coefficients (guessing). Factor 9 with $\lambda = 0$, no correction. Factor $10e^t$ with $\lambda = 1$, no correction. Guess $x_{1p} = A + B\,e^t$, $x_{2p} = C + D\,e^t$.
Substituting into the equations we get $B\,e^t = A+B\,e^t-C-D\,e^t+9$, $D\,e^t = 10A+10B\,e^t-C-D\,e^t+10e^t$, this yields $A = 1$, $B = -1$, $C = 10$, $D = 0$. Solution $x_{1p} = 1 - e^t$, $x_{2p} = 10$.
General solution:
$x_1(t) = 1 - e^t + a\sin(3t) + b\cos(3t)$, $x_2(t) = 10 + a[\sin(3t) - 3\cos(3t)] + b[3\sin(3t) + \cos(3t)]$, $t \in I\!R$.
Alternative: variation of parameter. General solution of associated homogeneous $\vec{x}_h = X(t)\vec{c}$, variation $\vec{x}(t) = X(t)\vec{c}(t)$, equation $X(t)\vec{c}\,'(t) = \vec{b}(t)$. We need X^{-1}:

$$\begin{pmatrix} \sin(3t) & \cos(3t) & 1 & 0 \\ \sin(3t) - 3\cos(3t) & 3\sin(3t) + \cos(3t) & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & \sin(3t) + \frac{1}{3}\cos(3t) & -\frac{1}{3}\cos(3t) \\ 0 & 1 & \cos(3t) - \frac{1}{3}\sin(3t) & \frac{1}{3}\sin(3t) \end{pmatrix},$$

hence $X^{-1}(t) = \begin{pmatrix} \sin(3t) + \frac{1}{3}\cos(3t) & -\frac{1}{3}\cos(3t) \\ \cos(3t) - \frac{1}{3}\sin(3t) & \frac{1}{3}\sin(3t) \end{pmatrix}$.

Then $\vec{c}\,' = X^{-1}\vec{b} \begin{pmatrix} \sin(3t) + \frac{1}{3}\cos(3t) & -\frac{1}{3}\cos(3t) \\ \cos(3t) - \frac{1}{3}\sin(3t) & \frac{1}{3}\sin(3t) \end{pmatrix} \begin{pmatrix} 9 \\ 10e^t \end{pmatrix} = \begin{pmatrix} 9\sin(3t) + 3\cos(3t) - \frac{10}{3}e^t\cos(3t) \\ 9\cos(3t) - 3\sin(3t) + \frac{10}{3}e^t\sin(3t) \end{pmatrix}$,

so $\vec{c}(x) = \begin{pmatrix} -3\cos(3t) + \sin(3t) - \frac{1}{3}e^t\cos(3t) - e^t\sin(3t) + a \\ 3\sin(3t) + \cos(3t) + \frac{1}{3}e^t\sin(3t) - e^t\cos(3t) + b \end{pmatrix}$ and $\vec{x}_p(t) = X(t)\vec{c}(t) = \begin{pmatrix} 1 - e^t \\ 10 \end{pmatrix}$.

Thus $\vec{x}(t) = \begin{pmatrix} 1 - e^t + a\sin(3t) + b\cos(3t) \\ 10 + a[\sin(3t) - 3\cos(3t)] + b[3\sin(3t) + \cos(3t)] \end{pmatrix}$.

Alternative: Row variation. First homogeneous solution (by elimination or via eigenvalues):

$$\left\{ \begin{array}{l} x_{1h}(t) = a\sin(3t) + b\cos(3t) \\ x_{2h}(t) = a[\sin(3t) - 3\cos(3t)] + b[3\sin(3t) + \cos(3t)] \end{array} \right\},$$

then variation $\left\{ \begin{array}{l} x_1(t) = a(t)\sin(3t) + b(t)\cos(3t) \\ x_2(t) = a(t)[\sin(3t) - 3\cos(3t)] + b(t)[3\sin(3t) + \cos(3t)] \end{array} \right\}$.

Equations $\left\{ \begin{array}{l} a'(t)\sin(3t) + b'(t)\cos(3t) = 9 \\ a'(t)[\sin(3t) - 3\cos(3t)] + b'(t)[3\sin(3t) + \cos(3t)] = 10e^t \end{array} \right\}$ yield

$\left\{ \begin{array}{l} a'(t) = 9\sin(3t) + 3\cos(3t) - \frac{10}{3}e^t\cos(3t) \\ b'(t) = 9\cos(3t) - 3\sin(3t) + \frac{10}{3}e^t\sin(3t) \end{array} \right\}$, hence

$\left\{ \begin{array}{l} a(t) = -3\cos(3t) + \sin(3t) - \frac{1}{3}e^t\cos(3t) - e^t\sin(3t) \\ b(t) = 3\sin(3t) + \cos(3t) + \frac{1}{3}e^t\sin(3t) - e^t\cos(3t) \end{array} \right\}$ and $\left\{ \begin{array}{l} x_{1p}(t) = 1 - e^t \\ x_{2p}(t) = 10 \end{array} \right\}$.

Elimination: From (#1) $x_2 = x_1 - x_1' + 9$ $(*)$, into (#2) gives $x_1'' + 9x_1 = 9 - 10e^t$, char. n. $\lambda = \pm 3j$, solution of homogeneous equation $x_h(t) = a\sin(3t) + b\cos(3t)$. Non-zero RHS is done best by guessing, $x_p = A + Be^t$, substitution into $x_1'' + 9x_1 = 9 - 10e^t$ gives $A = 1$, $B = -1$, therefore $x_1(t) = x_p + x_h = 1 - e^t + a\sin(3t) + b\cos(3t)$, from $(*)$ get $x_2(t) = 10 + a[\sin(3t) - 3\cos(3t)] + b[3\sin(3t) + \cos(3t)]$.

2) Init. conditions yield $\vec{x}(t) = \begin{pmatrix} 1 - e^t + \sin(3t) - \cos(3t) \\ 10 - 2\sin(3t) - 4\cos(3t) \end{pmatrix}$, $x \in I\!R$.

3. Eigenvalues: $\lambda = 0, -1, 2$, $\vec{y}(x) = \begin{pmatrix} a + 3ce^{2x} \\ a + be^{-x} + ce^{2x} \\ -a + 3ce^{2x} \end{pmatrix}$, $x \in I\!R$, see Homogeneous systems.

Remark: Fundamental matrix $Y(x) = \begin{pmatrix} 1 & 0 & 3e^{2x} \\ 1 & e^{-x} & e^{2x} \\ -1 & 0 & 3e^{2x} \end{pmatrix}$.

Non-zero right hand-side: Method of undetermined coefficients (guessing). Factor $6e^x$ with $\lambda = 1$, no correction. Factor -6 with $\lambda = 0$, correction $m = 1$. One needs to include terms with and also without correction.
Guess $y_{1p} = A\,e^x + B + Cx$, $y_{2p} = D\,e^x + E + Fx$, $y_{3p} = G\,e^x + H + Ix$.
Substituting into the equations we get $A\,e^x + C = A\,e^x + B + Cx + G\,e^x + H + Ix + 6e^x$,
$D\,e^x + F = A\,e^x + B + Cx - D\,e^x - E - Fx - 6$, $G\,e^x + I = A\,e^x + B + Cx + G\,e^x + H + Ix$.
Rewrite: $-G\,e^x + (C - B - H) - (C + I)x = 6e^x$, $(2D - A)\,e^x + (F + E - B) + (F - C)x = -6$,
$-A\,e^x - (B + H) - (C + I)x = 0$, this yields $A = 0$, $C = 0$, $D = 0$, $F = 0$, $G = -6$, $I = 0$. Also two independent euations for three variables $B + H = 0$, $E - B = -6$, choice $B = 0$ leads to $H = 0$, $E = -6$. Solution $y_{1p} = 0$, $y_{2p} = -6$, $y_{3p} = -6e^x$.
General solution: $y_1(x) = a + 3ce^{2x}$, $y_2(x) = -6 + a + be^{-x} + ce^{2x}$, $y_3(x) = -6e^x - a + 3ce^{2x}$.
Alternative: variation of parameter. General solution of associated homogeneous $\vec{y}_h = Y(x)\vec{c}$, variation $\vec{y}(x) = Y(x)\vec{c}(x)$, equation $Y(x)\vec{c}\,'(x) = \vec{b}(x)$. We need Y^{-1}:

$$\begin{pmatrix} 1 & 0 & 3e^{2x} & 1 & 0 & 0 \\ 1 & e^{-x} & e^{2x} & 0 & 1 & 0 \\ -1 & 0 & 3e^{2x} & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & -\frac{2}{3}e^{x} & e^{x} & \frac{1}{3}e^{x} \\ 0 & 0 & 1 & \frac{1}{6}e^{-2x} & 0 & \frac{1}{6}e^{-2x} \end{pmatrix},$$

hence $Y^{-1}(x) = \frac{1}{6}\begin{pmatrix} 3 & 0 & -3 \\ -4e^{x} & 6e^{x} & 2e^{x} \\ e^{-2x} & 0 & e^{-2x} \end{pmatrix}$.

So $\vec{c}\,' = Y^{-1}\vec{b} = \frac{1}{6}\begin{pmatrix} 3 & 0 & -3 \\ -4e^{x} & 6e^{x} & 2e^{x} \\ e^{-2x} & 0 & e^{-2x} \end{pmatrix}\begin{pmatrix} 6e^{x} \\ -6 \\ 0 \end{pmatrix} = \begin{pmatrix} 3e^{x} \\ -4e^{2x} - 6e^{x} \\ e^{-x} \end{pmatrix}$, hence $\vec{c}(x) = \begin{pmatrix} 3e^{x} \\ -2e^{2x} - 6e^{x} \\ -e^{-x} \end{pmatrix}$

and $\vec{y}_p(x) = Y(x)\vec{c}(x) = \begin{pmatrix} 0 \\ -6 \\ -6e^{x} \end{pmatrix}$.

Thus $\vec{y}(x) = \begin{pmatrix} a + 3ce^{2x} \\ -6 + a + be^{-x} + ce^{2x} \\ -6e^{x} - a + 3ce^{2x} \end{pmatrix}$.

Alternative: Row variation. First homogeneous solution (by elimination or via eigenvalues):

$\left\{\begin{array}{l} y_{1h}(x) = a + 3ce^{2x} \\ y_{2h}(x) = a + be^{-x} + ce^{2x} \\ y_{3h}(x) = -a + 3ce^{2x} \end{array}\right\}$, then variation $\left\{\begin{array}{l} y_1(x) = a(x) + 3c(x)e^{2x} \\ y_2(x) = a(x) + b(x)e^{-x} + c(x)e^{2x} \\ y_3(x) = -a(x) + 3c(x)e^{2x} \end{array}\right\}$.

Equations $\left\{\begin{array}{l} a'(x) + 3c'(x)e^{2x} = 6e^{x} \\ a'(x) + b'(x)e^{-x} + c'(x)e^{2x} = -6 \\ -a'(x) + 3c'(x)e^{2x} = 0 \end{array}\right\}$ yield $\left\{\begin{array}{l} a'(x) = 3e^{x} \\ b'(x) = -6e^{x} - 4e^{2x} \\ c'(x) = e^{-x} \end{array}\right\}$, hence

$\left\{\begin{array}{l} a(x) = 3e^{x} \\ b(x) = -6e^{x} - 2e^{2x} \\ c(x) = -e^{x} \end{array}\right\}$ and $\left\{\begin{array}{l} y_{1p}(x) = 0 \\ y_{2p}(x) = -6 \\ y_{3p}(x) = -6e^{x} \end{array}\right\}$.

Elimination: From (#2) $y_1 = y_2' + y_2 + 6$ $(*)$, into (#1) and (#3) gives $\left\{\begin{array}{l} (1^*)\ y_2'' = y_2 + y_3 + 6 + 6e^{x} \\ (2^*)\ y_3' = y_2' + y_2 + y_3 + 6 \end{array}\right\}$, from (#1*) $y_3 = y_2'' - y_2 - 6 - 6e^{x}$ $(\star)$, into (#2*) gives $y_2''' - y_2'' - 2y_2' = -6$. Char. n. $\lambda = 0, -1, 2$, solution of associated homogeneous $y_h(x) = a + be^{-x} + ce^{2x}$.

Non-zero right hand-side is done best by guessing, $k = 1$, therefore $y_p = x^1 A = Ax$, substitution into $y_2''' - y_2'' - 2y_2' = -6$ gives $A = 3$, hence $y_2(x) = y_p + y_h = 3 + a + be^{-x} + ce^{2x}$, from $(\star)$ we get $y_3(x) = -9 - 6e^{x} - a + 3ce^{2x}$, from $(*)$ we get $y_1(x) = 9 + a + 3ce^{2x}$.