ODE: Solved problems—Analyzing solutions

1. For the equation $y' = \dfrac{x(1-y)}{2x-y^2}$ sketch its slope field and determine stationary solutions (if there are any).

2. For the autonomous equation $y' = \dfrac{y(y^2-4)}{1-y}$ sketch its slope field, determine stationary solutions (if there are any), and determine stability of equilibria (if there are any).

3. For the equation $y' = \dfrac{e^{13y}}{x-23}$ discuss existence and uniqueness using the Picard theorem.

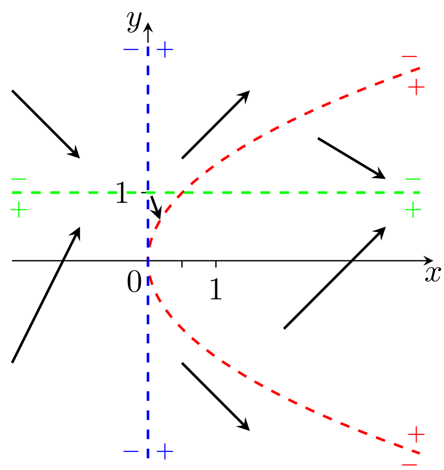
Solutions

1. The sign of the right-hand side $f = \dfrac{x(1-y)}{2x-y^2}$ can change if:

- it is zero, which happens when $x=0$ and $y=1$;
- it does not exist, which happens when $2x-y^2=0$, that is, $x=\frac{1}{2}y^2$.

We have three dividing curves: the vertical line $x=0$ a.k.a. the y-axis, the horizontal line $y=1$ and the curve $x=\frac{1}{2}y^2$, which is a parabola with vertex $(0,0)$ open to the right (flipped axes). It intersect the line $y=1$ at the point $\left(\frac{1}{2},1\right)$.

This determines several regions in the plane and we have to identify the signs of f in them. The simplest (and most reliable) method is to pick a point from each region and substitute into $f(x,y) = \dfrac{x(1-y)}{2x-y^2}$, determining the sign. For instance, for the larger region inside parabola we can choose the point $(2,0)$ and according to $f(2,0)=\frac{1}{2}>0$ the sign is positive there, so we draw an arrow suggesting that solutions grow there. Similarly we handle other regions, not forgetting the tiny one.
Another possibility to identify signs is to use the fact that f consists of three multiplied and divided factors, so we can determine the sign contribution of each and put them together in the usual way.



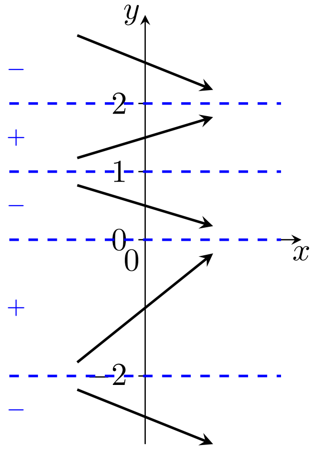
The factor x in the numerator has the y-axis as its dividing line for sign, on the right it imparts positive, on the left negative sign. For the factor $1-y$ we get the line $y=1$ as its dividing curve, it imparts negative sign in the upper half-plane and positive in the lower half-plane. For the factor $2x-y^2$ we have the parabola $x=\frac{1}{2}y^2$, the positive sign contribution inside can be determined for instance by substitution into $2x-y^2$, outside it is negative.

Now for each region we evaluate influences. For instance, consider the sort-of-triangle above the line $y=1$ going to the right. We apply the negative sign from $1-y$ and positive contributions from x and $2x-y^2$.

Stationary solutions appear when for a certain value y_0, the derivative $y' = f(x,y)$ becames zero regardless of x. This happens here for $y_0=1$, so we get a stationary solution $y(x)=1$. Since the equation is not autonomous (we see x in the right-hand side), we do not discuss stability.

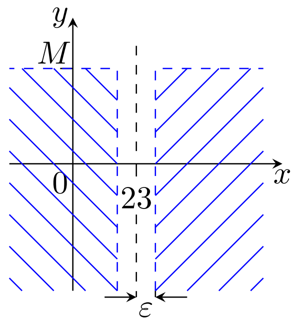
2. When dealing with an autonomous equation, it is sufficient to investigate influence of y on the sign. We do it in the usual way known from calculus.
We have factors y, $y^2 - 4$ and $1 - y$, yielding four dividing points.

$$\begin{array}{ccccccccc} - & -2 & + & 0 & - & 1 & + & 2 & - \end{array}$$



When we carry this information over to the plane, we obtain the slope field.
There are three stationary solutions, namely $y(x) = 0$ and $y(x) = \pm 2$. The value $y = 1$ divides the sign, but it does not create a stationary value.
We have three equilibria, from the slope field we deduce that $y = 2$ and $y = 0$ are stable, while $y = -2$ is unstable.

3. We use the approach with derivative. We find $\frac{\partial f}{\partial y} = 13\frac{e^{13y}}{x-23}$ and we ask where its boundedness is endangered. We see that we are in trouble when $y \to \infty$ and $x \to 23$. Thus we need to cut off these parts of the plane. In order to keep y away from infinity, we restrict it to an interval of the form $(-\infty, M)$ for some $M > 0$. For x we have to make sure that $|x - 23| > \varepsilon$ for some $\varepsilon > 0$, which suggests two possible intervals.



Thus we consider rectangles in $\mathbb{R}^2$ of the type $(-\infty, 23 - \varepsilon) \times (-\infty, M)$ and of the type $(23 + \varepsilon, \infty) \times (-\infty, M)$ for $M, \varepsilon > 0$. We claim that $\frac{\partial f}{\partial y} = 13\frac{e^{13y}}{x-23}$ is bounded on them. Indeed, consider some $M, \varepsilon > 0$. If (x, y) lies in one of the two corresponding rectangles, then $y \leq M$ and $|x - 23| > \varepsilon$.
We also see that $e^y > 0$ and it is an increasing function, hence

$$\left|\frac{\partial f}{\partial y}\right| = 13e^{13y}\frac{1}{|x-23|} \leq 13e^M\frac{1}{\varepsilon}.$$

Thus the derivative is bounded, and according to the Picard theorem the given differential equation has unique solutions through all points of sets $(-\infty, 23-\varepsilon) \times (-\infty, M)$ and $(23+\varepsilon, \infty) \times (-\infty, M)$ for $M, \varepsilon > 0$.
Now we start to enlarge these regions by letting $M \to \infty$ and $\varepsilon \to 0^+$. It is obvious that all points in the plane $\mathbb{R}^2$ with exception of those on the line $x = 23$ belong to some of the growing rectangles and thus the observation on existence and uniqueness applies to them.
Conclusion: A solution passes through every point $(x, y) \in \mathbb{R}^2$ satisfying $x \neq 23$ and is determined uniquely.