ODE: Solved problems–Separable equations

1. Find the solution of the equation $y' = \dfrac{2y}{x-1}$ that satisfies the initial condition $y(0) = 13$.

2. Find a general solution of the equation $y' = 3x^2(y-1)^2$.
Then find the solution that satisfies the initial condition $y(1) = \frac{8}{9}$.

3. For the equation $y' = \dfrac{y^2-1}{2x}$ solve the following Cauchy problems:

a) $y(1) = 0$; b) $y(5) = -4$; c) $y(1) = 2$; d) $y(0) = 3$; e) $y(1) = -1$.

4. For the equation $y' = \dfrac{y}{2x} + \dfrac{1}{2xy}$ solve the following Cauchy problems:

a) $y(2) = -1$; b) $y(0) = 1$; c) $y(-2) = 1$; d) $y(-1) = 0$.

Solutions

1. As usual we first find a **general solution**. We start by observing that we have the condition $x \neq 1$, we will have to keep this in mind.
The equation is not linear, so we hope that it sill be separable. We try it:

$$\frac{dy}{dx} = \frac{2y}{x-1} \implies \frac{dy}{y} = \frac{2\,dx}{x-1}.$$

It worked, so we quickly add integral signs so that it makes sense. We also notice that this step works only for $y \neq 0$. Theory says that the constant function $y(x) = 0$ is therefore a stationary solution.
Let's move on, we integrate:

$$\int \frac{dy}{y} = \int \frac{2\,dx}{x-1} \implies \ln|y| = 2\ln|x-1| + C.$$

We want to solve the resulting equation for y. In order to get rid of the logarithm on the left we apply the exponential to both sides (treated as whole units, this in particular applies to the right-hand side):

$$|y| = e^{2\ln|x-1|+C} = e^C \cdot e^{2\ln|x-1|}.$$

Unfortunately the two prevents the exponential from cancelling againts the logarithm. Such situations are handled algebraically, either by rewriting the expression as $\left(e^{\ln|x-1|}\right)^2$, or by hiding the two in the logarithm (which can be done immediately after integration): $2\ln|x-1| = \ln(|x-1|^2)$. Both approaches lead to

$$|y| = e^C|x-1|^2.$$

There is a trick to handle the absolute value that is typically used in this situation, we replace it with plus or minus. However, we do not have to apply it on the right, as the square always yields a non-negative number. Either way, we obtain

$$y = \pm e^C(x-1)^2.$$

Which brings us to the second trick that is often used in this context. We can choose the constant C as we wish, so we can also make the number e^C into anything positive. But the sign is also arbitrary, so $\pm e^C$ is an arbitrary non-zero number, we will denote it $D = \pm e^C$. We obtain

$$y(x) = D(x-1)^2.$$

This solution was derived under the assumption that $y \neq 0$, which is true for this formula due to the condition $x \neq 1$ (see the equation) and $D \neq 0$. Everything is fine.

There are two choices for a general solution, $y(x) = D(x-1)^2$ with constant $D \neq 0$ and the stationary one $y(x) = 0$.
Sometimes these can be merged. We try $D = 0$ in the general fomrula and we obtain $y(x) = 0$ indeed. The last step is to inquire about conditions. No restriction from the formula, there is one from the equation. Conclusion: The general solutions is

$$y(x) = D(x-1)^2,\ x \neq 1.$$

(Here we have no restriction on D, just like we usually do for parameters in a general solution.)

Initial condition: We want $y(0) = 13$, that is, $D(0-1)^2 = 13$. From this we easily get $D = 13$. it remains to determine the interval. The condition $x \neq 1$ tells us to choose from $(-\infty, 1)$ and $(1, \infty)$. We want an interval that allows us to substitute the initial value $x_0 = 0$ (see the condition), this determines the interval.
Conclusion: The solution is $y(x) = 13(x-1)^2$, $x \in (-\infty, 1)$.

2. There are no restrictions on x, y from the equation. Can we separate?

$$\frac{dy}{dx} = 3x^2(y-1)^2 \implies \int (y-1)^{-2} dy = \int 3x^2\, dx.$$

Separation was successful. However, we should check whether it was correct (division by zero), and we see the restriction $y \neq 1$. We have stationary solution $y(x) = 1$ on $I\!R$.
For $y \neq 1$ we integrate, $-(y-1)^{-1} = x^3 + C$. Solve for y:

$$\frac{1}{y-1} = -C - x^3 \implies y(x) = 1 - \frac{1}{x^3 + C}.$$

This came under the assumption that $y \neq 1$, which is true for this formula, so it is in order.
Can we include the stationary solution by chooosing some value for C? No, there is no C that would make $1 - \frac{1}{x^3+C} = 1$ always. Condition on existence: Nothing from the equation, the solution wants $x^3 + C \neq 0$.
Conclusion: the general solution is given by two formulas, $y(x) = 1 - \frac{1}{x^3+C}$ for $x^3 \neq -C$ and $y(x) = 1$ for $x \in I\!R$.
Remark: If you want, you can replace C with the constant $-C$, then you get an alternative formula $y(x) = 1 - \frac{1}{x^3-C}$ for $x^3 \neq C$ that some may find nicer.

Initial condition: We want $y(1) = \frac{8}{9}$, that is, $1 - \frac{1}{1^3+C} = \frac{8}{9}$. We get $C = 8$, $y(x) = 1 - \frac{1}{x^3+8}$. There is a condition on existence $x^3 \neq -8$, that is, $x \neq -2$. This creates two intervals, we choose the one that includes $x_0 = 1$.
The solution is $y(x) = 1 - \frac{1}{x^3+8}$, $x \in (-2, \infty)$.

3. We can separate the equation, $y' = \frac{1}{2x} \cdot (y^2 - 1)$, so we use the appropriate method. We see the condition $x \neq 0$.
First we check on stationary solutions: $y^2 - 1 = 0$ for values $1, -1$, so we get solution $y(x) = -1$ on $(-\infty, 0)$ and on $(0, \infty)$ and solution $y(x) = 1$ on $(-\infty, 0)$ and on $(0, \infty)$.
For $y \neq \pm 1$ we separate x and y and then integrate: $\displaystyle\int \frac{2\,dy}{y^2-1} = \int \frac{dx}{x}$. The integral on the right is elementary, the left one is done easily using partial fractions decomposition:

$$\int \frac{2\,dy}{(y-1)(y+1)} = \int \frac{1}{y-1} - \frac{1}{y+1}\, dy = \ln|y-1| - \ln|y+1| = \ln\Big|\frac{y-1}{y+1}\Big|.$$

We obtain $\ln\big|\frac{y-1}{y+1}\big| = \ln|x| + c$, passing to exponential we get $\big|\frac{y-1}{y+1}\big| = e^{\ln|x|+c} = e^c \cdot e^{\ln|x|} = e^c|x|$.

Trick with absolute value yields $\frac{y-1}{y+1} = \pm e^c x$, we hide the signs into C: $\frac{y-1}{y+1} = Cx$, where $C \neq 0$. From this $y(x) = \frac{1+Cx}{1-Cx}$, $x \neq 0, \frac{1}{C}$. We have $C \neq 0$; but $C = 0$ gives one of the stationary solutions, so we include it; the second one can't be included (in some sense we get it by choice $C = \infty$). A general solution thus is $y(x) = -1$ for $x \neq 0$ (two possible intervals of solution) and $y(x) = \frac{1+Cx}{1-Cx}$ for $x \neq 0$ and $x \neq \frac{1}{C}$ when $C \neq 0$ (three possible intervals of solution).
Init. conditions: a) Substitute: $0 = \frac{1+C}{1-C}$, hence $C = -1$. Conditions are now $x \neq 0$, $x \neq -1$, they determine three intervals and we want the one that includes $x_0 = 1$. Solution is $y_a(x) = \frac{1-x}{1+x}$, $x \in (0, \infty)$.
b) Substitute: $-4 = \frac{1+5C}{1-5C}$, hence $C = \frac{1}{3}$. Conditions are now $x \neq 0$, $x \neq 3$, they determine three intervals and we want the one that includes $x_0 = 5$. Solution is $y_b(x) = \frac{1+\frac{x}{3}}{1-\frac{x}{3}} = \frac{3+x}{3-x}$, $x \in (3, \infty)$.
c) Substitute: $2 = \frac{1+C}{1-C}$, hence $C = \frac{1}{3}$. Conditions are now $x \neq 0$, $x \neq 3$, they determine three intervals and we want the one that includes $x_0 = 1$. Solution is $y_c(x) = \frac{1+\frac{x}{3}}{1-\frac{x}{3}} = \frac{3+x}{3-x}$, $x \in (0, 3)$.
Remark: The solution from b) and c) are given by the same formula, but they are different because of the interval.
d) Since $x_0 = 0$ cannot be used in the given equation, no solution $y_d(x)$ exists.
e) Substitute: $-1 = \frac{1+C}{1-C}$ has no solution. Now we use stationary solutions from the beginning, we choose the right one: Solution is $y_e(x) = -1$, $x \in (0, \infty)$.

4. The given equation is definitely not linear (y in the denominator), so the only chance is to try to separate, which fortunately works after we use common denominator:

$$y' = \frac{y^2+1}{2yx} = \frac{1}{x} \cdot \frac{y^2+1}{2y}.$$

We see conditions $x \neq 0$, $y \neq 0$, and since $y^2 + 1$ can't be made equal to zero by any choice of y, there will be no stationary solution. For $x, y \neq 0$ we separate and integrate: $\displaystyle\int \frac{2y\,dy}{y^2+1} = \int \frac{dx}{x}$.
The first integral is easy by subtitution $z = y^2 + 1$, we get $\ln|y^2 + 1| = \ln|x| + c$, $|y^2 + 1| = e^c|x|$, trick gives $y^2 + 1 = \pm e^c x = Cx$, hence $y^2 = Cx - 1$ and $y(x) = \pm\sqrt{Cx - 1}$, $C \neq 0$ (note that for $C = 0$ the formula makes no sense). These solutions exist wherever the condition $Cx - 1 > 0$ is true, since the root requires that $Cx - 1 \geq 0$ and we also have condition $y \neq 0$. Solution of this inequality depends on the sign of C, we get domains $\left(-\infty, \frac{1}{C}\right)$ for $C < 0$ and $\left(\frac{1}{C}, \infty\right)$ for $C > 0$.
Init. conditions: a) Here we will choose minus the root, since $y_0 < 0$.
Substitute: $-1 = -\sqrt{2C - 1}$, hence $C = 1$. Solution: $y_a(x) = -\sqrt{x - 1}$, $x \in (1, \infty)$.
b) y_b does not exist, since $x_0 = 0$ is not possible.
c) Here we choose plus the root, since $y_0 > 0$. Substitute: $1 = \sqrt{-2C - 1}$, hence $C = -1$.
Solution: $y_c(x) = \sqrt{-x - 1}$, $x \in (-\infty, -1)$.
d) y_d does not exist because of the condition $y \neq 0$.