ODE: Solved problems—Linear equations

1. Find a general solution of the equation $y''''' - 2y'''' + 5y''' - 8y'' + 4y' = 0$.
Determine the asymptotic growth of a typical solution at infinity.

2. Find a general solution of the equation $y'' - y' - 2y = -2x\,e^x + 3e^{-x}$.
Determine the asymptotic growth of a typical solution at infinity.

3. For the equation $y'' + y' = 4x + 2x\,e^x$ solve the Cauchy problem $y(0) = \frac{1}{2}$, $y'(0) = -\frac{1}{2}$.

4. Find a general solution of the equation $\ddot{x} - 2\dot{x} + x = 2\sin(t) - 25\cos(2t) + 4e^t$.

Solutions

1. It is a homogeneous linear equation with constant coefficients, therefore we can find a general solution directly using characteristic numbers. We are supposed to solve the equation

$$\lambda^5 - 2\lambda^4 + 5\lambda^3 - 8\lambda^2 + 4\lambda = 0.$$

An obvious step is to factor out λ:

$$\lambda(\lambda^4 - 2\lambda^3 + 5\lambda^2 - 8\lambda + 4) = 0.$$

This is about as far as standard procedures can get us, now it's time to guess. Problems at school typically have nice roots, we try to substitute $\lambda = 1$ and it works. We pull out the corresponding factor:

$$\lambda(\lambda - 1)(\lambda^3 - \lambda^2 + 4\lambda - 4) = 0.$$

Now we try $\lambda = 1$ in the cubic polynomial and it works again. We pull out another factor:

$$\lambda(\lambda - 1)(\lambda - 1)(\lambda^2 + 4) = 0.$$

Finally we have a quadratic polynomial, we can find its roots.
So the characteristic numbers are $\lambda = 0$, $\lambda = 1$ (double), and $\lambda = \pm 2i = 0 \pm 2i$. Thus the fundamental system is

$$\{e^{0\cdot x} = 1, e^{1\cdot x}, x \cdot e^{1\cdot x}, e^{0\cdot x}\sin(2\cdot x), e^{0\cdot x}\cos(2\cdot x)\}.$$

We actually do not need this, but it gave us time to thing about the functions that will be used to compose our general solution:

$$y(x) = a + b\,e^x + cx\,e^x + d\sin(2x) + f\cos(2x), \qquad x \in I\!R.$$

What is the asymptotic behaviour at infinity? In a typical solution all constants are non-zero, so we just find the dominant term. We notice that there are terms going to infinity, so we can ignore the part $a + d\sin(2x) + f\cos(2x)$ that is bounded. We are choosing between e^x and $x\,e^x$, the latter is obviously dominant.
Conclusion: For $x \to \infty$ we have $y(x) \sim cx\,e^x$.

2. It is a non-homogeneous linear ODE of order 2, there's no problem in it, so we have solutions on $I\!R$. We start with the associated homogeneous equation. It has constant coefficients, so we go to characteristic things: Characteristic polynomial is $p(\lambda) = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1)$, equation $p(\lambda) = 0$ gives characteristic numbers $\lambda = -1, 2$. Fundamental system of solution is thus $\{e^{-x}, e^{2x}\}$ and general solution of homogeneous equation is $y_h(x) = ae^{-x} + be^{2x}$, $x \in I\!R$.
Now we pass to non-homogeneous equation. Right hand-side is special, so we do not need to do variation. In fact we have a sum of two distinct special terms, since $2x\,e^x$ and $3e^{-x}$ cannot be written using one common $e^{\alpha x}$.

We start with the first term. To get e^x on the output, we have to put e^x as the input, we do not have to create any sines or cosines, but we do have to worry about the polynomial in $2x\,e^x$. It should be generalized into a polynomial of the same degree: $(Ax+B)e^x$. This is the basic form. Is correction needed?
There are no sines or cosines in $e^{1\cdot x}$, so it is coded with $\lambda = 1 + \pm 0i = 1$. This number is not among characteristic numbers (see the homogeneous version above), so no match/overlap and thus no correction (formally, $m = 0$). The first part in the right-hand side will therefore be created using $(Ax+B)e^x$.
Remark: If you are curious how this fits with the general theorem, you would first need to write it as

$$2x\,e^x = e^{1\cdot x}(0 + (2x)\cdot 1) = e^{1\cdot x}(0\sin(0\cdot x) + (2x)\cos(0\cdot x)),$$

then the parameters are $\alpha = 1$, $\beta = 0$, multiplicity of $\alpha + \beta i = 1$ as a characteristic number is $m = 0$ (so no correction), the highest degree of polynomial is $d = 1$ ($2x$ is a polynomial of degree 1). According to the theorem there is a solution of the form

$$y(x) = x^0 e^{1\cdot x}((C + Dx)\sin(0) + (Ax+B)\cos(0)) = (Ax+B)e^x.$$

End of remark.
Similarly we guess that the basic form corresponding to the second part $3e^{-x}$ is $A\,e^{-x}$. It is described by the number $\lambda = -1 \pm 0i = -1$, and we see this number among characteristic numbers, namely once. Thus we need to apply correction x^1 (formally, $m = 1$). Thus the proper form for the second part is $y(x) = x^1 C e^{-x} = Cx\,e^{-x}$ (we cannot use A here, it is already taken). We get a particular solution of the given equation (by principle of superposition) by adding the two, so we are looking for a solution of the form

$$y(x) = (Ax+B)e^x + Cx\,e^{-x}.$$

Substituting into the left-hand side of the equation we get the output

$$\begin{aligned}&[(Ax+2A+B)e^x + (Cx-2C)e^{-x}] - [(Ax+A+B)e^x + (-Cx+C)e^{-x}]\\ &\quad - 2[(Ax+B)e^x + Cx\,e^{-x}] = [-2Ax + (A-2B)]e^x + (-3C)e^{-x}.\end{aligned}$$

We want this to be equal to $-2x\,e^x + 3e^{-x}$, and since functions e^x and e^{-x} are independent, they cannot influence one another. In other words, we treat them separately and demand that

$$[-2Ax + (A-2B)]e^x = -2x\,e^x, \qquad (-3C)e^{-x} = 3e^{-x}.$$

This yields

$$\begin{aligned}&-2Ax + (A-2B) = -2x + 0\\ &-3C = 3\end{aligned} \implies \begin{aligned}&-2Ax = -2x\\ &A - 2B = 0\\ &C = -1\end{aligned} \implies A = 1,\ B = \tfrac{1}{2},\ C = -1.$$

We obtain the particular solution $y_p(x) = \left(x + \frac{1}{2}\right)e^x - x\,e^{-x}$ and general solution given by $y_p + y_h$:

$$y(x) = \left(x + \tfrac{1}{2}\right)e^x - x\,e^{-x} + ae^{-x} + be^{2x}, \qquad x \in I\!R.$$

Asymptotic growth at infinity? In a typical solution all constants are non-zero, so we just find the dominant term. We notice that there are terms going to infinity, so we can ignore the part $ae^{-x} \to 0$ as $x \to \infty$.
Another interesting term is $xe^{-x} = \frac{x}{e^x}$. Since exponentials dominate powers around infinity, we have $\frac{x}{e^x} \to 0$ (if you do not see this, try l'Hospital's rule), so this term is out as well. For large x

we also have $x+\frac{1}{2}\sim x$, so we are deciding between candidates $x\,e^x$ and e^{2x}. Mutual comparison $\frac{e^{2x}}{x\,e^x}=\frac{e^x}{x}\to\infty$ favours e^{2x}, so here is the conclusion: For $x\to\infty$ we get $y(x)\sim b\,e^{2x}$.
Alternative: We could use variation of parameters, from $y_h(x)$ we get $y(x)=a(x)e^{-x}+b(x)e^{2x}$, thus we get equations

$$\begin{aligned}a'(x)e^{-x}+b'(x)e^{2x}&=0\\ -a'(x)e^{-x}+2b'(x)e^{2x}&=-2x\,e^x+3e^{-x}\end{aligned}\implies\begin{aligned}a'(x)&=\frac{2}{3}x\,e^{2x}-1\\ b'(x)&=-\frac{2}{3}x\,e^{-x}+e^{-3x}\end{aligned}$$

From this using integration by parts $a(x)=\frac{1}{3}x\,e^{2x}-\frac{1}{6}e^{2x}-x$, $b(x)=\frac{2}{3}x\,e^{-x}+\frac{2}{3}e^{-x}-\frac{1}{3}e^{-3x}$, substituting back we get y_p and then $y=y_p+y_h$.
Also the other problems below can be solved using variation, but usually it is longer that way.

3. It is a non-homogeneous linear ODE of order 2, no problem with it, so all solutions will be on $I\!R$. We start with associated homogeneous equation. It has constant coefficients, so we go to characteristic things: Characteristic polynomial $p(\lambda)=\lambda^2+\lambda=\lambda(\lambda+1)$, equation $p(\lambda)=0$ gives characteristic numbers $\lambda=0,-1$. Fundamental system of solution is then $\{e^{0\cdot x},e^{-x}\}=\{1,e^{-x}\}$ and general solution of homogeneous equation is $y_h(x)=a+be^{-x}$, $x\in I\!R$.
Now we turn to non-homogeneous equation. Right hand-side is special, so we do not have to do variation of parameters. It is actually a sum of two distinct special RHS's, since $4x$ and $2x\,e^x$ can't be written using one $e^{\alpha x}$.
The term $4x$ has no exponential nor (co)sines, so we need not include them in the input, we just try to hit the polynomial, which we do with $Ax+B$. Correction: A polynomial without exponentials and (co)sines is coded using $\lambda=0\pm 0i=0$. We find this number once among the characteristic numbers, which implies a correction x^1 (here $m=1$). Thus there must be a solution of the form $y(x)=x^1(Ax+B)=Ax^2+Bx$.
Remark: This fits the general theorem as follows:

$$4x=e^0(0+(4x)\cdot 1)=e^{0\cdot x}(0\sin(0\cdot x)+(4x)\cos(0\cdot x))$$

has parameters $\alpha=0$, $\beta=0$, multiplicity of $\alpha+\beta i=0$ as characteristic number is $m=1$, highest degree of polynomial is $d=1$ ($4x$ is polynomial of degree 1). Thus there is a solution of the form

$$y(x)=x^1e^{0\cdot x}((Cx+D)\sin(0)+(Ax+B)\cos(0))=x(Ax+B)=Ax^2+Bx.$$

The term $2x\,e^x$ has basic form of guess $(Cx+D)e^x$. Its number is $\lambda=1$, no match with characteristic numbers, so no correction, the original guess is the right one.
We add the two guesses, our estimate for the form of a solution is

$$y(x)=Ax^2+Bx+(Cc+D)e^x.$$

We substitute into the given equation and get

$$[2A+(Cx+2C+D)e^x]+[(2Ax+B+(Cx+C+D)e^x]=[2Ax+(2A+B)]+[2Cx+(3C+2D)]e^x.$$

We want this to be $4x+2x\,e^x$. Since the functions e^x and 1 are independent, the equality happens only if the two parts match. Thus we get

$$\begin{aligned}2Ax+(2A+B)&=4x+0\\ 2Cx+(3C+2D)&=2x+0\end{aligned}\implies\begin{aligned}2Ax&=4x\\ 2A+B&=0\\ 2Cx&=2x\\ 3C+2D&=0\end{aligned}\implies A=2,B=-4,\ C=1,\ D=-\tfrac{3}{2}.$$

So we have a particular solution $y_p(x) = 2x^2 - 4x + \big(x - \frac{3}{2}\big)e^x$ and a general solution given by $y_p + y_h$:

$$y(x) = 2x^2 - 4x + \big(x - \tfrac{3}{2}\big)e^x + a + be^{-x}, \qquad x \in I\!R.$$

Init. conditions: We have $y(x) = 2x^2 - 4x + \big(x - \frac{3}{2}\big)e^x + a + be^{-x}$, $y'(x) = 4x - 4 + \big(x - \frac{1}{2}\big)e^x - be^{-x}$; we put in the conditions and we get

$$\begin{array}{l} \frac{1}{2} = -\frac{3}{2} + a + b \\ -\frac{1}{2} = -4 - \frac{1}{2} - b \end{array} \implies \begin{array}{l} a + b = 2 \\ b = -4 \end{array} \implies \quad a = 6,\ b = -4.$$

Solution is $y(x) = 2x^2 - 4x + 6 + \big(x - \frac{3}{2}\big)e^x - 4e^{-x}, \quad x \in I\!R.$

4. It is a non-homogeneous linear ODE of order 2, no problem, so all solutions will be on $I\!R$ (variable is t!). We start with associated homogeneous equation. It has constant coefficients, so we go to characteristic things: Characteristic polynomial $p(\lambda) = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$, equation $p(\lambda) = 0$ gives characteristic numbers $\lambda = 1$ ($2\times$). Fundamental system of solution is then $\{e^t, t\,e^t\}$ and general solution of homogeneous equation is $x_h(t) = ae^t + bt\,e^t$, $t \in I\!R$.
Now we turn to non-homogeneous equation. Right hand-side is special, it is actually a sum of two distinct special RHS's, since $\sin(t)$ and $\cos(2t)$ can't be expressed as $\sin(\beta t)$, $\cos(\beta t)$ for one β.
In order to obtain $2\sin(t)$ we need not put an exponential in our input, but we do have to put in $\sin(t)$ and also another term with $\cos(t)$ (they always come together when guessing). Then we need to generalize the polynomial 2 of degree 0, so the basic form of our guess is $A\sin(t) + B\cos(t)$.
Correction? $2\sin(1 \cdot t)$ is coded with $\lambda = 0 \pm 1i = \pm i$. We do not find this among characteristic numbers, so no overlap, hence no correction ($m = 0$). The basic form is therefore accepted as is.
Remark: To fit the general theorem we could write $2\sin(t) = e^0(2\sin(1 \cdot t) + 0\cos(1 \cdot t))$. THis expression has parameters $\alpha = 0$, $\beta = 1$, multiplicity of $\alpha + \beta i = i$ as characteristic number is $m = 0$, maximal degree of polynomial is $d = 0$ (2 is polynomial of degree 0). Thus there is a solution of the form $x(t) = t^0 e^{0 \cdot t}(A\sin(1 \cdot t) + B\cos(1 \cdot t)) = A\sin(t) + B\cos(t)$. End of remark.
The term $-25\cos(2t)$ is handled similarly: The basic form is $C\sin(2t) + D\cos(2t)$, and since $\lambda = \pm 2i$ is not among characteristic numbers, there is no correction.
For the third term e^t we guess the basic form of solution as $E\,e^t$, since there is a polynomial 1 of degree 0 hidden in front of $e^t = 1 \cdot e^t$ and it has to be generalized. Correction: e^t is coded by $\lambda = 1$ which is also among characteristic numbers, it is actually twice there (double characteristic number). Thus we have to apply correction with $m = 2$ and the guess for our solution is $t^2 E\,e^t = Et^2e^t$.
We add all components, so we look for a solution of the form

$$x(t) = A\sin(t) + B\cos(t) + C\sin(2t) + D\cos(2t) + Et^2e^t.$$

We substitute into the left-hand side of the given equation and get

$$\begin{aligned} &[-A\sin(t) - B\cos(t) - 4C\sin(2t) - 4D\cos(2t) + 2Ee^t + 4Ete^t + Et^2e^t] \\ &\qquad - 2[A\cos(t) - B\sin(t) + 2C\cos(2t) - 2D\sin(2t) + 2Ete^t + Et^2e^t] \\ &\qquad + [A\sin(t) + B\cos(t) + C\sin(2t) + D\cos(2t) + Et^2e^t] \\ &= 2B\sin(t) - 2A\cos(t) + [4D - 3C]\sin(2t) + [-4C - 3D]\cos(2t) + 2Ee^t. \end{aligned}$$

We want this to be $2\sin(t) - 25\cos(t) + 4e^t$. Due to independence we compare term by term and obtain equations $2B = 2$, $-2A = 0$, $-3C + 4D = 0$, $-4C - 3D = -25$, $2E = 4$; from this $A = 0$, $B = 1$, $C = 4$, $D = 3$, $E = 2$.
Thus we have a particular solution $x_p(t) = \cos(t) + 4\sin(2t) + 3\cos(2t) + 2t^2e^t$ a general solution given by the formula $x_p + x_h$:

$$x(t) = \cos(t) + 4\sin(2t) + 3\cos(2t) + 2t^2e^t + ae^t + bt\,e^t, \qquad t \in I\!R.$$