## ODE: Solved problems—Method of variation

**1.** For the equation $y' + \frac{x^2}{x^3-1}y = 4\sqrt[3]{(x^3-1)^2}$ solve the following Cauchy problems:

a) $y(0) = -1$; b) $y(1) = 3$; c) $y(2) = \frac{8}{\sqrt[3]{7}}$.

**2.** Find a general solution of the equation $y'' - 2y' + y = \frac{e^x}{\sqrt{1-x^2}}$.

### Solutions

**1.** It is a non-homogeneous linear ODE, so we start with the associated homogeneous equation:

$$y' + \frac{x^2}{x^3-1}y = 0.$$

It does not have constant coefficients, so we cannot use characteristic numbers $\lambda$. Fortunately, separation is possible.

We have condition $x \neq 1$. We separate and integrate: $\int \frac{dy}{y} = -\int \frac{x^2}{x^3-1}\,dx$.

The second integral is done via substitution $z = x^3 - 1$, we get $\ln|y| = -\frac{1}{3}\ln|x^3-1| + c = \ln\left|\frac{1}{\sqrt[3]{x^3-1}}\right| + c$, $|y| = e^c\left|\frac{1}{\sqrt[3]{x^3-1}}\right|$, trick $y = \pm e^c\frac{1}{\sqrt[3]{x^3-1}}$, hence a general solution of the homogeneous equation is $y_h(x) = \frac{C}{\sqrt[3]{x^3-1}}$, $x \neq 1$. Here $C \neq 0$, but $C = 0$ gives stationary solution $y(x) = 0$, $x \neq 1$.

Now we do variation of parameter: $y(x) = \frac{C(x)}{\sqrt[3]{x^3-1}}$. We can substitute into the given equation and cancel:

$$\left[\frac{C(x)}{\sqrt[3]{x^3-1}}\right]' + \frac{x^2}{x^3-1}\cdot\frac{C(x)}{\sqrt[3]{x^3-1}} = 4\sqrt[3]{(x^3-1)^2}$$

$$\implies C'(x)\frac{1}{\sqrt[3]{x^3-1}} + C(x)\frac{-x^2}{\sqrt[3]{(x^3-1)^4}} + \frac{x^2}{x^3-1}\cdot\frac{C(x)}{\sqrt[3]{x^3-1}} = 4\sqrt[3]{(x^3-1)^2}$$

$$\implies C'(x)\frac{1}{\sqrt[3]{x^3-1}} - C(x)\frac{x^2}{\sqrt[3]{(x^3-1)^4}} + \frac{x^2}{\sqrt[3]{(x^3-1)^4}}\cdot C(x) = 4\sqrt[3]{(x^3-1)^2}$$

$$\implies C'(x)\frac{1}{\sqrt[3]{x^3-1}} = 4\sqrt[3]{(x^3-1)^2}.$$

Or we simply rememeber that the same expression as $y(x)$ but with differentiated $C(x)$ is equal to the right-hand side:

$$\frac{C'(x)}{\sqrt[3]{x^3-1}} = 4\sqrt[3]{(x^3-1)^2}$$

Either way, we have an equation to solve.

$$C'(x) = 4(x^3-1) \implies C(x) = x^4 - 4x.$$

Thus we get $y_p(x) = \frac{x^4-4x}{\sqrt[3]{x^3-1}}$. Using $y = y_h + y_p$ we get a general solution

$$y(x) = \frac{x^4 - 4x + C}{\sqrt[3]{x^3-1}},\ x \neq 1.$$

Alternative: $C(x) = x^4 - 4x + C$, then after substituting we immediately get the general solution above.
**Init. conditions**: a) Substitute: $-1 = \frac{C}{-1}$, hence $C = 1$, and $x_0 = 0$ must be in an interval given by condition $x \neq 1$. Solution: $y_a(x) = \dfrac{x^4 - 4x + 1}{\sqrt[3]{x^3 - 1}}$, $x \in (-\infty, 1)$.
b) $y_b$ does not exist, we can't have $x_0 = 1$.
c) Substitute: $\frac{8}{\sqrt[3]{7}} = \frac{8+C}{\sqrt[3]{7}}$, hence $C = 0$. Solution: $y_c(x) = \dfrac{x^4 - 4x}{\sqrt[3]{x^3 - 1}}$, $x \in (1, \infty)$.

**2.** It is a non-homogeneous linear ODE of order 2, existence of solution is determined by continuity of coefficients and its right-hand side, so here we have solutions on $(-1, 1)$. We start with the associated homogeneous equation. It has constant coefficients, so we go to characteristic things: From the equation $\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2 = 0$ we get the characteristic number $\lambda = 1$ $(2\times)$. The fundamental system of solutions is thus $\{e^x, x\,e^x\}$ and a general solution of homogeneous equation is $y_h(x) = ae^x + bx\,e^x$, $x \in I\!R$.
Now we pass to non-homogeneous equation. The right-hand side is not special, so we have to use variation of parameters. We see condition $|x| < 1$. We have $y(x) = a(x)e^x + b(x)x\,e^x$, hence equations

$$\begin{aligned} a'(x)e^x + b'(x)x\,e^x &= 0 \\ a'(x)e^x + b'(x)(x+1)e^x &= \frac{e^x}{\sqrt{1-x^2}} \end{aligned} \quad \Longrightarrow \quad \begin{aligned} a'(x) + b'(x)x &= 0 \\ a'(x) + b'(x)(x+1) &= \frac{1}{\sqrt{1-x^2}} \end{aligned}.$$

Subtracting (#2)$-$(#1) we get $b'(x) = \frac{1}{\sqrt{1-x^2}}$, hence $b(x) = \arcsin(x)$. Substituting back into the first equation we have $a'(x) = -\frac{x}{\sqrt{1-x^2}}$, this time we integrate using substitution $z = 1 - x^2$, hence $a(x) = \sqrt{1 - x^2}$.
We can also solve the system using determinants (Cramer rule):

$$D = \begin{vmatrix} 1 & x \\ 1 & x+1 \end{vmatrix} = 1, \quad D_{a'} = \begin{vmatrix} 0 & x \\ \frac{1}{\sqrt{1-x^2}} & x+1 \end{vmatrix} = -\frac{x}{\sqrt{1-x^2}}, \quad D_{b'} = \begin{vmatrix} 1 & 0 \\ 1 & \frac{1}{\sqrt{1-x^2}} \end{vmatrix} = \frac{1}{\sqrt{1-x^2}}.$$

Then $a'(x) = \frac{D_{a'}}{D}$ and $b'(x) = \frac{D_{b'}}{D}$ and we get exactly the same formulas as above.
Anyway, we get particular solution $y_p(x) = \sqrt{1-x^2}\,e^x + \arcsin(x)\,x\,e^x$ and general solution by adding particular and homogeneous ones:

$$y(x) = \sqrt{1-x^2}\,e^x + \arcsin(x)\,x\,e^x + ae^x + bx\,e^x, \qquad x \in (-1, 1).$$

We could have obtained this by putting constants right away when integrating $a(x) = \sqrt{1-x^2} + a$, $b(x) = \arcsin(x) + b$.