ODE: Solved problems—Systems of equations

For the following problems, the approach via matrices is our primary solution method. If you want, you can also try elimination for fun.

1. For a system of equations $\begin{aligned} y_1' &= -2y_1 + y_2 \\ y_2' &= 2y_1 - y_2 \end{aligned}$ a) find a fundamental matrix solution;

b) solve the Cauchy problem $y_1(0) = 3$, $y_2(0) = 0$;

b) determine stability of the trivial stationary solution $y_1(x) = y_2(x) = 0$.

2. Find the solutions of the system $\begin{aligned} y_1' &= 4y_1 - y_2 \\ y_2' &= 4y_1 \end{aligned}$ that satisfies $y_1(1) = 3e^2$, $y_2(1) = 4e^2$.

Determine stability of the trivial stationary solution $y_1(x) = y_2(x) = 0$.

3. Find a general solution of the system of equations $\begin{aligned} y_1' &= 2y_1 - y_2 + 2 \\ y_2' &= -6y_1 + y_2 - 5e^{-x} \end{aligned}$.

4. (tougher) Find a general solution of the system of equations $\begin{aligned} \dot{x}_1 &= x_1 - x_2 \\ \dot{x}_2 &= 5x_1 - x_2 - 6\sin^2(t) \end{aligned}$.

Solutions

1. Eigenvalue method: The system is homogeneous, so the eigenvalue approach gives directly a general solution. Matrix of the system is $A = \begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix}$. To get eigenvalues we solve the equation

$$0 = \begin{vmatrix} -2-\lambda & 1 \\ 2 & -1-\lambda \end{vmatrix} = (-2-\lambda)(-1-\lambda) - 2 = (\lambda+2)(\lambda+1) - 2 = \lambda^2 + 3\lambda = \lambda(\lambda+3),$$

and we see that eigenvalues are $\lambda = 0, -3$. We find eigenvectors:

$\underline{\lambda = 0}$: $\begin{pmatrix} -2-0 & 1 \\ 2 & -1-0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, this reads $\begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Both equations are the same, so it is enough to consider the first equation $-2v_1 + v_2 = 0$, we want $\vec{v} \neq \vec{0}$, hence we choose something not zero, say $v_1 = 1$, then $v_2 = 2$. We get a solution $\vec{y}_a(x) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{0x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

$\underline{\lambda = -3}$: $\begin{pmatrix} -2-(-3) & 1 \\ 2 & -1-(-3) \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, this reads $\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Again it is enough to consider the first equation $v_1 + v_2 = 0$, we want $\vec{v} \neq \vec{0}$, so we choose something not zero, say $v_1 = 1$, then $v_2 = -1$. We have a solution $\vec{y}_b(x) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3x} = \begin{pmatrix} e^{-3x} \\ -e^{-3x} \end{pmatrix}$.

Thus the fundamental matrix is $Y(x) = \begin{pmatrix} 1 & e^{-3x} \\ 2 & -e^{-3x} \end{pmatrix}$.

General solution is

$$\vec{y}(x) = a\begin{pmatrix} 1 \\ 2 \end{pmatrix} + b\begin{pmatrix} e^{-3x} \\ -e^{-3x} \end{pmatrix} = \begin{pmatrix} a + be^{-3x} \\ 2a - be^{-3x} \end{pmatrix}, \quad x \in I\!R.$$

Since the problem was not stated in matrix form, we prefer this form for the answer:

$$y_1(x) = a + be^{-3x},\ y_2(x) = 2a - be^{-3x},\ x \in I\!R.$$

Obviously, a typical solution $\vec{y}$ does not go to $\vec{0}$ at infinity, thus we infer that $y_1(x) = y_2(x) = 0$ is an unstable stationary solution, that is, that $(0,0)$ is an unstable equilibrium. The stability theorem confirms it, it is not true that all eigenvalues would be negative (have negative real parts).

Bonus: Since one of the (real) eigenvalues is zero and the other positive, this is not one of the six basic types.
Init. conditions: We rewrite them using the general solution and get

$$\begin{aligned} 3 &= a + b \\ 0 &= 2a - b \end{aligned} \implies a = 1,\ b = 2.$$

The solution is $y_1(x) = 1 + 2e^{-3x}$, $y_2(x) = 2 - 2e^{-3x}$ for $x \in I\!R$.
If you want, you can write $\vec{y}(x) = \begin{pmatrix} 1 + 2e^{-3x} \\ 2 - 2e^{-3x} \end{pmatrix}$, $x \in I\!R$.

Alternative: **Elimination method**: From the first equation we express $y_2 = y_1' + 2y_1 \quad (*)$ and substitute into the second: $y_1'' + 3y_1' = 0$. This homogeneous equation with constant coefficients is solved through characteristic things. Characteristic polynomial: $p(\lambda) = \lambda^2 + 3\lambda$, char. numbers $\lambda = 0, -3$, general solution is $y_1(x) = a + be^{-3x}$.
Substituting into $(*)$ we get $y_2(x) = 2a - be^{-3x}$ and we have our general solution, then we use initial conditions.
We can rewrite our solution as

$$\vec{y}(x) = \begin{pmatrix} a + be^{-3x} \\ 2a - be^{-3x} \end{pmatrix} = a \begin{pmatrix} 1 \\ 2 \end{pmatrix} + b \begin{pmatrix} e^{-3x} \\ -e^{-3x} \end{pmatrix}, \quad x \in I\!R.$$

We see that the fundamental matrix is $Y(x) = \begin{pmatrix} 1 & e^{-3x} \\ 2 & -e^{-3x} \end{pmatrix}$.

2. Eigenvalue method: The system is homogeneous, its matrix is $A = \begin{pmatrix} 4 & -1 \\ 4 & 0 \end{pmatrix}$. Eigenvalues: we solve the equation

$$0 = \begin{vmatrix} 4 - \lambda & -1 \\ 4 & -\lambda \end{vmatrix} = (4 - \lambda)(-\lambda) + 4 = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2, \qquad \text{eigenvalue is} \quad \lambda = 2\ (2\times).$$

<u>$\lambda = 2$</u>: $\begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, need to solve $2v_1 - v_2 = 0$, choose $v_1 = 1$, hence $v_2 = 2$. We get solution $\vec{y}_a(x) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2x} = \begin{pmatrix} e^{2x} \\ 2e^{2x} \end{pmatrix}$.
For a double eigenvalue there is a special prodecure to get a second solution: We solve the equation $\begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, need to solve the first equation $2v_1 - v_2 = 1$, now for $\vec{v} \neq \vec{0}$ we can choose $v_1 = 0$, then $v_2 = -1$. We get solution

$$\vec{y}_b(x) = \left[\begin{pmatrix} 1 \\ 2 \end{pmatrix} x + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right] e^{2x} = \begin{pmatrix} x\, e^{2x} \\ (2x - 1)e^{2x} \end{pmatrix}.$$

The general solution is

$$\vec{y}(x) = a \begin{pmatrix} e^{2x} \\ 2e^{2x} \end{pmatrix} + b \begin{pmatrix} x\, e^{2x} \\ (2x - 1)e^{2x} \end{pmatrix} = \begin{pmatrix} ae^{2x} + bx\, e^{2x} \\ 2ae^{2x} + b(2x - 1)e^{2x} \end{pmatrix}, \quad x \in I\!R,$$

that is, $y_1(x) = ae^{2x} + bx\, e^{2x}$, $y_2(x) = 2ae^{2x} + b(2x - 1)e^{2x}$ for $x \in I\!R$.
Init. conditions: We rewrite them using the general solution and get

$$\begin{aligned} 3e^2 &= ae^2 + be^2 \\ 4e^2 &= 2ae^2 + be^2 \end{aligned} \implies a = 1,\ b = 2,$$

so the solution is $y_1(x) = (2x+1)e^{2x}$, $y_2(x) = 4x\,e^{2x}$ for $x \in I\!R$.
Because it is not true that all eigenvalues would be negative (have negative real parts), we conclude that $y_1(x) = y_2(x) = 0$ is an unstable stationary solution or that $(0,0)$ is an unstable equilibrium. After all, We see that a typical solution definitely does not go to the origin.
Bonus: This is an unstable node or knot.

Alternative: **Elimination method**: From the first equation we express $y_2 = 4y_1 - y_1' \quad (*)$ and put into the other: $y_1'' - 4y_1' + 4y_1 = 0$. Homogeneous equation with constant coefficients, so characteristic polynomial $p(\lambda) = \lambda^2 - 4\lambda + 4 = (\lambda-2)^2$, char. number $\lambda = 2$ $(2\times)$ and thus a general solution $y_1(x) = ae^{2x} + bx\,e^{2x}$.
Substituting into $(*)$ we get $y_2(x) = 2ae^{2x} + b(2x-1)e^{2x}$, then we handle initial conditions.
Remark: This solution is given by the same basic vectors as in the eigenvalue approach:

$$\vec{y}(x) = \begin{pmatrix} ae^{2x} + bx\,e^{2x} \\ 2ae^{2x} + b(2x-1)e^{2x} \end{pmatrix} = a\begin{pmatrix} e^{2x} \\ 2e^{2x} \end{pmatrix} + b\begin{pmatrix} x\,e^{2x} \\ (2x-1)e^{2x} \end{pmatrix}.$$

This is just a coincidence. If we choose when constructing the second basic vector, say, $v_1 = 1$, we would get $v_2 = 1$ and thus

$$\vec{y}(x) = \left[\begin{pmatrix} 1 \\ 2 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right] e^{2x} = \begin{pmatrix} (x+1)\,e^{2x} \\ (2x+1)e^{2x} \end{pmatrix}.$$

However, this vector is a combination of the two vecors from elimination (namely their sum), conversely the second vector from elimination is a combination of $\vec{y} = \begin{pmatrix} e^{2x} \\ 2e^{2x} \end{pmatrix}$ and $\begin{pmatrix} (x+1)\,e^{2x} \\ (2x+1)e^{2x} \end{pmatrix}$.
Thus this new couple defines the same space and everything is fine. Conclusion: If elimination and eigenvectors yield different vectors, then this does not automatically mean that we made a mistake somewhere; we have to check that the vectors from one basis can be obtained from vectors of the other basis and vice versa.

3. Eigenvalue method: First we solve homogeneous version of the equation. Matrix of the system is $A = \begin{pmatrix} 2 & -1 \\ -6 & 1 \end{pmatrix}$, we solve equation

$$0 = \begin{vmatrix} 2-\lambda & -1 \\ -6 & 1-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda) - 6 = \lambda^2 - 3\lambda - 4 = (\lambda-4)(\lambda+1), \text{ eigenvalues are } \lambda = -1, 4.$$

$\underline{\lambda = -1}$: $\begin{pmatrix} 3 & -1 \\ -6 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, need to solve $3v_1 - v_2 = 0$, choose $v_1 = 1$, hence $v_2 = 3$, we have solution $\vec{y}_a(x) = \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-x} = \begin{pmatrix} e^{-x} \\ 3e^{-x} \end{pmatrix}$.
$\underline{\lambda = 4}$: $\begin{pmatrix} -2 & -1 \\ -6 & -3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, need to solve $-2v_1 - v_2 = 0$, choose $v_1 = 1$, hence $v_2 = -2$, we have solution $\vec{y}_b(x) = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{4x} = \begin{pmatrix} e^{4x} \\ -2e^{4x} \end{pmatrix}$.
We get the general homogeneous solution

$$\vec{y}_h(x) = \begin{pmatrix} ae^{-x} + be^{4x} \\ 3ae^{-x} - 2be^{4x} \end{pmatrix}, \quad x \in I\!R.$$

By the way, because it is not true that all eigenvalues would be negative (have negative real parts), we conclude that $y_1(x) = y_2(x) = 0$ is an unstable stationary solution or that $(0,0)$ is an unstable equilibrium of this homogeneous system.
Bonus: This is a saddle.

Right-hand side: It looks like **special** righ hand-side, so we try guessing. We see the term 2 with $\lambda = 0$ (no correction) and the term $5e^{-x}$ with $\lambda = -1$, here we have an overlap. Special rule for systems of equations says that both functions in our guess for solution form must include all types from all right-hand sides and with corrections going from the right one to none. Thus we guess the form of a particular solution as $y_1(x) = A + Bx\,e^{-x} + C\,e^{-x}$ and $y_2(x) = D + Ex\,e^{-x} + Fe^{-x}$. We substitute into the given equation and get

$$\begin{aligned}
[A + Bx\,e^{-x} + Ce^{-x}]' &= 2(A + Bx\,e^{-x} + Ce^{-x}) - (D + Ex\,e^{-x} + Fe^{-x}) + 2\\
[D + Ex\,e^{-x} + Fe^{-x}]' &= -6(A + Bx\,e^{-x} + Ce^{-x}) + (D + Ex\,e^{-x} + Fe^{-x}) - 5e^{-x}\\
\Longrightarrow\quad & (-2A + D) + (-3B + E)x\,e^{-x} + (B - 3C + F)e^{-x} = 2\\
& (6A - D) + (6B - 2E)x\,e^{-x} + (6C + E - 2F)e^{-x} = -5e^{-x}
\end{aligned}$$

Therefore

$$\begin{aligned}
-2A + D &= 2\\
-3B + E &= 0\\
B - 3C + F &= 0\\
6A - D &= 0\\
6B - 2E &= 0\\
6C + E - 2F &= -5
\end{aligned}
\quad\Longrightarrow\quad
\begin{aligned}
A &= \frac{1}{2}\\
D &= 3\\
E &= 3B\\
B - 3C + F &= 0\\
3B + 6C - 2F &= -5
\end{aligned}$$

This system of equations is interesting, because the two equations for B, E are in fact identical, so we have five equations for six unknowns. However, B or E happen to be exactly those unknowns that we are not free to choose. For instance, if we tried the natural choice $B = E = 0$ then the last two equations would not be solvable. In fact, those last two equations tell us that $B = -1$ and $E = -3$, and it turnes out that we are free to choose one of the constants C, F. The choice $C = -1$ yields $F = -2$ and we have $\vec{y}_p(x) = \begin{pmatrix} \frac{1}{2} - x\,e^{-x} - e^{-x} \\ 3 - 3x\,e^{-x} - 2e^{-x} \end{pmatrix}$. The formula $\vec{y} = \vec{y}_p + \vec{y}_h$ then provides us with a general solution

$$\vec{y}(x) = \begin{pmatrix} \frac{1}{2} - x\,e^{-x} - e^{-x} + ae^{-x} + be^{4x} \\ 3 - 3x\,e^{-x} - 2e^{-x} + a3e^{-x} - 2be^{4x} \end{pmatrix}, \quad x \in I\!R.$$

As usual we prefer notation corresponding to the way the problem was given, so we write $y_1(x) = \frac{1}{2} - e^{-x} - x\,e^{-x} + ae^{-x} + be^{4x}$, $y_2(x) = 3 - 2e^{-x} - 3x\,e^{-x} + a3e^{-x} - 2be^{4x}$ pro $x \in I\!R$.

Alternative: It is possible to use the method of variation, in its matrix form or in its row form.

Matrix form: The homogeneous solution is $\vec{y}_h(x) = \begin{pmatrix} e^{-x} & e^{4x} \\ 3e^{-x} & -2e^{4x} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$.

Variation: $\vec{y}_h(x) = \begin{pmatrix} e^{-x} & e^{4x} \\ 3e^{-x} & -2e^{4x} \end{pmatrix} \begin{pmatrix} a(x) \\ b(x) \end{pmatrix}$. Substituting into the system we obtain the matrix equation

$$\begin{pmatrix} e^{-x} & e^{4x} \\ 3e^{-x} & -2e^{4x} \end{pmatrix} \begin{pmatrix} a'(x) \\ b'(x) \end{pmatrix} = \begin{pmatrix} 2 \\ -5e^{-x} \end{pmatrix}.$$

We need to find the inverse matrix of the fundamental matrix $Y(x)$:

$$\begin{pmatrix} e^{-x} & e^{4x} & 1 & 0 \\ 3e^{-x} & -2e^{4x} & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & \frac{2}{5}e^{x} & \frac{1}{5}e^{x} \\ 0 & 1 & \frac{3}{5}e^{-4x} & -\frac{1}{5}e^{-4x} \end{pmatrix}$$

We have $\begin{pmatrix} a'(x) \\ b'(x) \end{pmatrix} = Y(x)^{-1} \begin{pmatrix} 2 \\ -5e^{-x} \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 2e^{x} & e^{x} \\ 3e^{-4x} & -e^{-4x} \end{pmatrix} \begin{pmatrix} 2 \\ -5e^{-x} \end{pmatrix} = \begin{pmatrix} \frac{4}{5}e^{x} - 1 \\ \frac{6}{5}e^{-4x} + e^{-5x} \end{pmatrix}$, thus

$$\begin{pmatrix} a(x) \\ b(x) \end{pmatrix} = \begin{pmatrix} \int \frac{4}{5}e^{x} - 1\,dx \\ \int \frac{6}{5}e^{-5x} + e^{-5x}dx \end{pmatrix} = \begin{pmatrix} \frac{4}{5}e^{x} - x \\ -\frac{3}{10}e^{-4x} - \frac{1}{5}e^{-5x} \end{pmatrix}.$$

Substituting into the right place we obtain

$$\vec{y}_p = Y(x)\vec{c}(x) = \begin{pmatrix} e^{-x} & e^{4x} \\ 3e^{-x} & -2e^{4x} \end{pmatrix} \begin{pmatrix} \frac{4}{5}e^{x} - x \\ -\frac{3}{10}e^{-4x} - \frac{1}{5}e^{-5x} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} - x\,e^{-x} - \frac{1}{5}e^{-x} \\ 3 - 3x\,e^{-x} + \frac{2}{5}e^{-x} \end{pmatrix}.$$

We have $\vec{y}_p$ and $\vec{y}_h$, so $\vec{y} = \vec{y}_p + \vec{y}_h$ as before.
Note that the particular solution from variation is different from the one obtained by guessing. If they are both correct, they should differ by a vector from the space of homogeneous solutions. Indeed,

$$\begin{pmatrix} \frac{1}{2} - x\,e^{-x} - \frac{1}{5}e^{-x} \\ 3 - 3x\,e^{-x} + \frac{2}{5}e^{-x} \end{pmatrix} - \begin{pmatrix} \frac{1}{2} - x\,e^{-x} - e^{-x} \\ 3 - 3x\,e^{-x} - 2e^{-x} \end{pmatrix} = \begin{pmatrix} \frac{4}{5}e^{-x} \\ tfrac125e^{-x} \end{pmatrix} = \frac{4}{5}\begin{pmatrix} e^{-x} \\ 3e^{-x} \end{pmatrix}.$$

Row form of variation: From the homogeneous solution we guess the form of a particular solution: $y_1(x) = a(x)e^{-x} + b(x)e^{4x}$, $y_2(x) = 3a(x)e^{-x} - 2b(x)e^{4x}$. We substitute into the system:

$$\begin{aligned} &[a(x)e^{-x} + b(x)e^{4x}]' = 2(a(x)e^{-x} + b(x)e^{4x}) - (3a(x)e^{-x} - 2b(x)e^{4x}) + 2 \\ &[3a(x)e^{-x} - 2b(x)e^{4x}]' = -6(a(x)e^{-x} + b(x)e^{4x}) + (3a(x)e^{-x} - 2b(x)e^{4x}) - 5e^{-x} \\ &\qquad\Longrightarrow \quad \begin{aligned} a'(x)e^{-x} + b'(x)e^{4x} &= 2 \\ 3a'(x)e^{-x} - 2b'(x)e^{4x} &= -5e^{-x} \end{aligned} \end{aligned}$$

This we solve easily by row operations or perhaps the Cramer rule and obtain
$a'(x) = \frac{4}{5}e^{x} - 1$ a $b'(x) = \frac{6}{5}e^{-4x} + e^{-5x}$.
We integrate, the resulting functions $a(x) = \frac{4}{5}e^{x} - x$, $b(x) = -\frac{3}{10}e^{-4x} - \frac{1}{5}e^{-5x}$ are substituted into y_1 and y_2 and we get the particular solution $y_{1p}(x) = \frac{1}{2} - x\,e^{-x} - \frac{1}{5}e^{-x}$, $y_{2p}(x) = 3 - 3x\,e^{-x} + \frac{2}{5}e^{-x}$ just like with the matrix variation.
Alternative: **Elimination method**: From the first equation we express $y_2 = 2y_1 - y_1' + 2 \quad (*)$ and put into the other: $y_1'' - 3y_1' - 4y_1 = 5e^{-x} - 2$. This is a non-homogeneous linear equation, first we solve homogeneous, it has constant coefficients, so $\lambda^2 - 3\lambda - 4 = (\lambda - 4)(\lambda + 1) = 0$ yields char. numbers $\lambda = -1, 4$.
We have solution $y_h(x) = ae^{-x} + be^{4x}$ of homogeneous equation, now we need to solve non-homogeneous. It looks like **special** righ hand-side, more precisely a combination of two types, one is $5e^{-x}$ with solution $x\,A\,e^{-x}$ (parameter $\lambda = -1$ has multiplicity $m = 1$) and the second is -2 with solution B (parameter $\lambda = 0$ has multiplicity $m = 0$), thus we look for a solution of the form $y(x) = Ax\,e^{-x} + B$. We substitute into the given equation and get

$$[(Ax - 2A)e^{-x}] - 3[(-Ax + A)e^{-x}] - 4[Ax\,e^{-x} + B] = 5e^{-x} - 2 \implies -5Ae^{-x} - 4B = 5e^{-x} - 2.$$

We have $A = -1$, $B = \frac{1}{2}$, hence $y_p(x) = -x\,e^{-x} + \frac{1}{2}$, $y_1(x) = y_p(x) + y_h(x) = \frac{1}{2} - x\,e^{-x} + ae^{-x} + be^{4x}$.
Substituting into $(*)$ we get $y_2 = 3 + (1 - 3x)e^{-x} + 3ae^{-x} - 2be^{4x}$. Conclusion:
A general solution is
$y_1(x) = \frac{1}{2} - x\,e^{-x} + ae^{-x} + be^{4x}$, $y_2 = 3 + (1 - 3x)e^{-x} + 3ae^{-x} - 2be^{4x}$, $x \in I\!R$.
Again, we can compare this with solutions from variation or guessing method, the particular solution component differs by a vector from the homogeneous solution.

4. Eigenvalue method: First we solve the homogeneous version of the equation. Matrix of the system is $A = \begin{pmatrix} 1 & -1 \\ 5 & -1 \end{pmatrix}$, we solve equation

$$0 = \begin{vmatrix} 1 - \lambda & -1 \\ 5 & -1 - \lambda \end{vmatrix} = (1 - \lambda)(-1 - \lambda) + 5 = \lambda^2 + 4, \qquad \text{eigenvalues are} \quad \lambda = \pm 2j.$$

$\underline{\lambda = 2i}$: $\begin{pmatrix} 1-2i & -1 \\ 5 & -1-2i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, need to solve $(1-2i)v_1 - v_2 = 0$, choose $v_1 = 1$, hence $v_2 = 1-2i$, we have solution

$$\begin{aligned}\vec{y}(x) &= \begin{pmatrix} 1 \\ 1-2i \end{pmatrix} e^{2it} = \begin{pmatrix} \cos(2t) + i\sin(2t) \\ (1-2i)[\cos(2t) + i\sin(2t)] \end{pmatrix} \\ &= \begin{pmatrix} \cos(2t) + i\sin(2t) \\ [\cos(2t) + 2\sin(2t)] + i[\sin(2t) - 2\cos(2t)] \end{pmatrix}.\end{aligned}$$

We find two solutions for our basis by taking the real and the imaginary part of this, so

$$\vec{x}_a(t) = \begin{pmatrix} \cos(2t) \\ \cos(2t) + 2\sin(2t) \end{pmatrix}, \qquad \vec{x}_b(t) = \begin{pmatrix} \sin(2t) \\ \sin(2t) - 2\cos(2t) \end{pmatrix}.$$

The general (homogeneous) solution is therefore

$$\vec{x}_h(t) = \begin{pmatrix} a\cos(2t) + b\sin(2t) \\ a[\cos(2t) + 2\sin(2t)] + b[\sin(2t) - 2\cos(2t)] \end{pmatrix}$$

that is, $x_{1h}(t) = a\cos(2t) + b\sin(2t)$, $x_{2h}(t) = a[\cos(2t) + 2\sin(2t)] + b[\sin(2t) - 2\cos(2t)]$, $t \in I\!R$. Particular solution: The right-hand side includes the term $\sin^2(t)$ for which we cannot guess, which leaves us with variation. We will try the row version:

$$\begin{aligned} x_{1p}(t) &= a(t)\cos(2t) + b(t)\sin(2t), \\ x_{2p}(t) &= a(t)[\cos(2t) + 2\sin(2t)] + b(t)[\sin(2t) - 2\cos(2t)]. \end{aligned}$$

We substitute into the system and obtain equations

$$\begin{aligned} a'(t)\cos(2t) + b'(t)\sin(2t) &= 0 \\ a'(t)[\cos(2t) + 2\sin(2t)] + b'(t)[\sin(2t) - 2\cos(2t)] &= -6\sin^2(t) \end{aligned}$$

Here the Cramer rule is probably the best way to go.

$$\begin{aligned} D &= \det\begin{pmatrix} \cos(2t) & \sin(2t) \\ \cos(2t) + 2\sin(2t) & \sin(2t) - 2\cos(2t) \end{pmatrix} = -2\cos^2(2t)] - 2\sin^2(2t) = -2, \\ D_a &= \det\begin{pmatrix} 0 & \sin(2t) \\ -6\sin^2(t) & \sin(2t) - 2\cos(2t) \end{pmatrix} = 6\sin^2(t)\sin(2t), \\ D_b &= \det\begin{pmatrix} \cos(2t) & 0 \\ \cos(2t) + 2\sin(2t) & -6\sin^2(t) \end{pmatrix} = -6\sin^2(t)\cos(2t). \end{aligned}$$

We get $a'(t) = -3\sin^2(t)\sin(2t)$, $b'(t) = 3\sin^2(t)\cos(2t)$. We need to integrate, and integrals of this type are best handled using identities. There are essentially two ways. One is to get rid of the double argument: $a'(t) = -3\sin^2(t)2\sin(t)\cos(t)$, here the integration is easy by substitution and leads to $a(t) = -\frac{3}{2}\sin^4(t)$. Unfortunately, applying this approach to $b'(t)$ leads to a beastly integral, so here it is better to exchange the squaring of sine for doubling of argument:

$$\begin{aligned} b(t) &= \int 3\sin^2(t)\cos(2t)dt = \int \tfrac{3}{2}[1 - \cos(2t)]\cos(2t)dt = \int \tfrac{3}{2}[\cos(2t) - \cos^2(2t)]dt \\ &= \int \tfrac{3}{2}\cos(2t) - \tfrac{3}{4}[1 + \cos(4t)]dt = \int \tfrac{3}{2}\cos(2t) - \tfrac{3}{4} - \tfrac{3}{4}\cos(4t)\,dt = \tfrac{3}{4}\sin(2t) - \tfrac{3}{4}t - \tfrac{3}{16}\sin(4t) \\ &= \tfrac{3}{4}\sin(2t) - \tfrac{3}{4}t - \tfrac{3}{8}\sin(2t)\cos(2t). \end{aligned}$$

It will be better to have a double argument in $a(t)$ as well:

$$a(t) = -\tfrac{3}{2}\big(\sin^2(t)\big)^2 = -\tfrac{3}{2}\big(\tfrac{1}{2}(1-\cos(2t))\big)^2 = \tfrac{3}{4}\cos(2t) - \tfrac{3}{8} - \tfrac{3}{8}\cos^2(2t).$$

We have the following:

$$\begin{aligned}
x_{1p}(t) &= \big[\tfrac{3}{4}\cos(2t) - \tfrac{3}{8} - \tfrac{3}{8}\cos^2(2t)\big]\cos(2t) + \big[\tfrac{3}{4}\sin(2t) - \tfrac{3}{4}t - \tfrac{3}{8}\sin(2t)\cos(2t)\big]\sin(2t),\\
x_{2p}(t) &= \big[\tfrac{3}{4}\cos(2t) - \tfrac{3}{8} - \tfrac{3}{8}\cos^2(2t)\big](\cos(2t) + 2\sin(2t))\\
&\quad + \big[\tfrac{3}{4}\sin(2t) - \tfrac{3}{4}t - \tfrac{3}{8}\sin(2t)\cos(2t)\big](\sin(2t) - 2\cos(2t)),
\end{aligned}$$

that is,

$$\begin{aligned}
x_{1p}(t) &= \tfrac{3}{4}\cos^2(2t) + \tfrac{3}{4}\sin^2(2t) - \tfrac{3}{8}\cos(2t) - \tfrac{3}{4}t\sin(2t) - \tfrac{3}{8}[\cos^2(2t) + \sin^2(2t)]\cos(2t)\\
&= \tfrac{3}{4} - \tfrac{3}{4}\cos(2t) - \tfrac{3}{4}t\sin(2t),\\
x_{2p}(t) &= \tfrac{3}{4}\cos^2(2t) + \tfrac{3}{4}\sin^2(2t) - \tfrac{3}{8}\cos(2t) - \tfrac{3}{4}\sin(2t)\\
&\quad + \tfrac{3}{2}t\cos(2t) - \tfrac{3}{4}t\sin(2t) - \tfrac{3}{8}[\cos^2(2t) + \sin^2(2t)]\cos(2t)\\
&= \tfrac{3}{4} + \tfrac{3}{4}(2t-1)\cos(2t) - \tfrac{3}{4}(t+1)\sin(2t).
\end{aligned}$$

Using $x = x_p + x_h$ we obtain a general solution:

$$\begin{aligned}
x_1(t) &= \tfrac{3}{4} - \tfrac{3}{4}\cos(2t) - \tfrac{3}{4}t\sin(2t) + a\cos(2t) + b\sin(2t),\\
x_2(t) &= \tfrac{3}{4} + \tfrac{3}{4}(2t-1)\cos(2t) - \tfrac{3}{4}(t+1)\sin(2t) + a[\cos(2t) + 2\sin(2t)] + b[\sin(2t) - 2\cos(2t)]
\end{aligned}$$

for $t \in I\!R$.

Now that was not very nice. Is there an **alternative**? Yes, the formula $-6\sin^2(t) = 3\cos(2t) - 3$ offers a better system:

$$\begin{aligned}
\dot{x}_1 &= x_1 - x_2\\
\dot{x}_2 &= 5x_1 - x_2 + 3\cos(2t) - 3
\end{aligned}$$

In this shape the system allows for the guessing method. We see two terms on the right, 3 with $\lambda = 0$ (no correction) and $3\cos(2t)$ with $\lambda = 2i$ (one overlap), and since guessing for systems requires that we use both terms with and without correction, we create the following guess for the particular solution:

$$\begin{aligned}
x_1(t) &= A + B\cos(2t) + Ct\cos(2t) + D\sin(2t) + Et\sin(2t),\\
x_2(t) &= F + G\cos(2t) + Ht\cos(2t) + I\sin(2t) + Jt\sin(2t).
\end{aligned}$$

We substitute into the system:

$$\begin{aligned}
&-2B\sin(2t) + C\cos(2t) - 2Ct\sin(2t) + 2D\cos(2t) + E\sin(2t) + 2Et\cos(2t)\\
&\quad = A + B\cos(2t) + Ct\cos(2t) + D\sin(2t) + Et\sin(2t)\\
&\qquad - F - G\cos(2t) - Ht\cos(2t) - I\sin(2t) - Jt\sin(2t),\\
&-2G\sin(2t) + H\cos(2t) - 2Ht\sin(2t) + 2I\cos(2t) + J\sin(2t) + 2Jt\cos(2t)\\
&\quad = 5A + 5B\cos(2t) + 5Ct\cos(2t) + 5D\sin(2t) + 5Et\sin(2t)\\
&\qquad - F - G\cos(2t) - Ht\cos(2t) - I\sin(2t) - Jt\sin(2t)\\
&\qquad\quad + 3 - 3\cos(2t),
\end{aligned}$$

that is,

$$\begin{aligned}
&(-A+F)+(-C+2E+H)t\cos(2t)+(-B+C+2D+G)\cos(2t)\\
&\quad+(-2C-E+J)t\sin(2t)+(-2B-D+E+I)\sin(2t)=0,\\
&(-5A+F)+(-5C+H+2J)t\cos(2t)+(-5B+G+H+2I)\cos(2t)\\
&\quad+(-5E-2H+J)t\sin(2t)+(-5D-2G+I+J)\sin(2t)\\
&\qquad=3-3\cos(2t).
\end{aligned}$$

We get the equations $\begin{aligned}-A+F&=0\\-5A+F&=-3\end{aligned}$, from which we easily deduce A and F and derive the equations

$$\begin{aligned}
&-C+2E+H=0,\ -B+C+2D+G=0,\ -2C-E+J=0,\ -2B-D+E+I=0\\
&-5C+H+2J=0,\ -5B+G+H+2I=3,\ -5E-2H+J=0,\ -5D-2G+I+J=0.
\end{aligned}$$

There is no simpler subsystem here, we have to solve an 8×8 system and suddenly it seems that variation perhaps was not that bad. While doing the elimination it turns out that some constants are free to be chosen by us. We chose $G=-\frac{3}{4}$, $I=-\frac{3}{4}$ (inspired by the solution we obtained using variation) and obtained

$$\begin{aligned}
&A=\tfrac{3}{4},\ B=-\tfrac{3}{4},\ C=0,\ D=0,\ E=-\tfrac{3}{4},\\
&F=\tfrac{3}{4}, G=-\tfrac{3}{4},\ H=\tfrac{3}{2},\ I=-\tfrac{3}{4},\ J=-\tfrac{3}{4}.
\end{aligned}$$

So we have the same particular solution as in the variation and the rest is the same.
Alternative: **Elimination method**: From the first equations $x_2=x_1-\dot{x}_1 \quad (*)$, we substitute to the second one: $\ddot{x}_1+4x_1=6\sin^2(t)$. It is a non-homogeneous linear equation, hence we first solve homogeneous. We get $\lambda=\pm 2i$ and obtain the solution $x_{1h}(t)=a\sin(2t)+b\cos(2t)$ of homogeneous equation.
Now we need to solve the non-homogeneous equation $\ddot{x}_1+4x_1=6\sin^2(t)$. Right hand-side is not special, so we apply variation of parameter. $x_1(t)=a(t)\sin(2t)+b(t)\cos(2t)$ gives

$$\begin{aligned}
&a'(t)\sin(2t)+b'(t)\cos(2t)=0\\
&2a'(t)\cos(2t)-2b'(t)\sin(2t)=6\sin^2(t)
\end{aligned}
\implies
\begin{aligned}
&a'(t)=3\sin^2(t)\cos(2t)\\
&b'(t)=-3\sin^2(t)\sin(2t)
\end{aligned}$$

These are exactly the same equations like we had when doing variation for the system, so we find the same x_{1p} and substituting into $(*)$ we then obtain x_{2p}.
We also saw the option to rewrite the right-hand side:

$$\ddot{x}_1+4x_1=6\sin^2(t)=3-3\cos(2t).$$

This is a special RHS, we guess the form of solution (beware of overlap and correcting factor t) $x_1(t)=A+Bt\sin(2t)+Ct\cos(2t)$. We substitute into the left-hand side of the equation:

$$4A+4B\cos(2t)-4C\sin(2t)=3-3\cos(2t)\implies A=\tfrac{3}{4},\ B=-\tfrac{3}{4},\ C=0.$$

So we have the particular solution $x_{1p}(t)=\frac{3}{4}-\frac{3}{4}t\sin(2t)$. We find a general solution:

$$x_1(t)=\frac{3}{4}-\frac{3}{4}t\sin(2t)+a\sin(2t)+b\cos(2t),$$

then substitute into $(*)$ and get $x_2(t)$.
Notice that the x_{1p} obtained here is not the same as the one from previous attempts; as expected, it is shifted by a homogeneous solution.