LT: Laplace Transform

$\mathcal{L}$: $f \to F$. For $f \in \mathcal{L}_0$: $\mathcal{L}\{f\}(p) = F(p) = \int_0^\infty f(t)e^{-pt}\,dt$, denote $f(t) \mathrel{\widehat{=}} F(p)$.

We offer two notations, in the left column using $\mathcal{L}$, in the right column using $\widehat{=}$ where we assume $f \mathrel{\widehat{=}} F$. We use $|$ to mark substitution into a function, e.g. $\sin(t)\big|_{2t-1}$ means $\sin(2t-1)$, $\big(p^2+1\big)\big|_{2p}$ means $(2p)^2+1$.

Dictionary

$\mathcal{L}\{e^{\alpha t}\} = \dfrac{1}{p-\alpha}$, $p > \alpha$	$e^{\alpha t} \mathrel{\widehat{=}} \dfrac{1}{p-\alpha}$, $p > \alpha$
$\mathcal{L}\{t^n\} = \dfrac{n!}{p^{n+1}}$, $p > 0$, for $n \in I\!N_0$	$t^n \mathrel{\widehat{=}} \dfrac{n!}{p^{n+1}}$, $p > 0$, for $n \in I\!N_0$
$\mathcal{L}\{\sin(\omega t)\} = \dfrac{\omega}{p^2+\omega^2}$, $p > 0$	$\sin(\omega t) \mathrel{\widehat{=}} \dfrac{\omega}{p^2+\omega^2}$, $p > 0$
$\mathcal{L}\{\cos(\omega t)\} = \dfrac{p}{p^2+\omega^2}$, $p > 0$	$\cos(\omega t) \mathrel{\widehat{=}} \dfrac{p}{p^2+\omega^2}$, $p > 0$

Useful special case: $1 \mathrel{\widehat{=}} \dfrac{1}{p}$, $H(t) \mathrel{\widehat{=}} \dfrac{1}{p}$.

Grammar

$\mathcal{L}\{af(t)+bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$	$af+bg \mathrel{\widehat{=}} aF+bG$
$\mathcal{L}\{f(at)\} = \frac{1}{a}\mathcal{L}\{f\}\big|_{p/a}$, $a > 0$	$f(at) \mathrel{\widehat{=}} \frac{1}{a}F\big(\frac{p}{a}\big)$, $a > 0$
$\mathcal{L}\{e^{at}f(t)\} = \mathcal{L}\{f\}\big|_{p-a}$	$e^{at}f(t) \mathrel{\widehat{=}} F(p-a)$
$\mathcal{L}\{f(t)\cdot H(t-a)\} = e^{-ap}\mathcal{L}\{f(t+a)\}$	
$\mathcal{L}\{f'(t)\} = p\mathcal{L}\{f(t)\} - f(0^+)$	$f'(t) \mathrel{\widehat{=}} pF(p) - f(0^+)$
$\mathcal{L}\{f''(t)\} = p^2\mathcal{L}\{f(t)\} - pf(0^+) - f'(0^+)$	$f''(t) \mathrel{\widehat{=}} p^2F(p) - pf(0^+) - f'(0^+)$
$\mathcal{L}\{t\,f(t)\} = -\frac{\mathrm{d}}{\mathrm{dp}}\mathcal{L}\{f(t)\}$	$t\,f(t) \mathrel{\widehat{=}} -F'(p)$
$\mathcal{L}\big\{\frac{1}{t}f(t)\big\} = \int_p^\infty \mathcal{L}\{f(t)\}(q)\,dq$	$\frac{1}{t}f(t) \mathrel{\widehat{=}} \int_p^\infty F(q)\,dq$
$\mathcal{L}\{t^n f(t)\} = (-1)^n\frac{\mathrm{d}^n}{\mathrm{dp}^n}\mathcal{L}\{f(t)\}$, $n \in I\!N_0$	$t^n f(t) \mathrel{\widehat{=}} (-1)^n F^{(n)}(p)$, $n \in I\!N_0$
$\mathcal{L}\{f^{(n)}(t)\} = p^n\mathcal{L}\{f\} - p^{n-1}f(0^+) - \dots - f^{(n-1)}(0^+)$	$f^{(n)}(t) \mathrel{\widehat{=}} p^nF(p) - p^{n-1}f(0^+) - \dots - f^{(n-1)}(0^+)$
$\mathcal{L}\{\int_0^t f(u)\,du\} = \frac{1}{p}\mathcal{L}\{f(t)\}$	$\int_0^t f(u)\,du \mathrel{\widehat{=}} \frac{1}{p}F(p)$
$\mathcal{L}\{f(t-a)\cdot H(t-a)\} = e^{-ap}\mathcal{L}\{f(t)\}$, $a > 0$	$f(t-a)\cdot H(t-a) \mathrel{\widehat{=}} e^{-ap}F(p)$, $a > 0$
$\mathcal{L}\{f*g\} = \mathcal{L}\{f\}\cdot\mathcal{L}\{g\}$	$f*g \mathrel{\widehat{=}} F\cdot G$

where $(f*g)(t) = \int_0^t f(u)\,g(t-u)\,du = \int_0^t f(t-u)\,g(u)\,du$ (convolution).

If f is a T-periodic function, then $\mathcal{L}\{f(t)\} = \dfrac{\int_0^T f(t)e^{-pt}\,dt}{1-e^{-Tp}} = \dfrac{\mathcal{L}\{f_T(t)\}}{1-e^{-Tp}}$.

Some rules for the inverse LT

$\mathcal{L}^{-1}\{F(ap)\} = \frac{1}{a}\mathcal{L}^{-1}\{F(p)\}\big|_{t/a}$, $a > 0$

$\mathcal{L}^{-1}\{F(p-a)\} = e^{at}\mathcal{L}^{-1}\{F(p)\}$

$\mathcal{L}^{-1}\{e^{-ap}F(p)\} = \mathcal{L}^{-1}\{F\}\big|_{t-a}\cdot H(t-a)$, $a > 0$