Series of functions

0. Review: Series of real numbers

Definition.
A **series** is a symbol $\sum\limits_{k=n_0}^{\infty} a_k = a_{n_0} + a_{n_0+1} + a_{n_0+2} + \dots,$
where $n_0 \in \mathbb{Z}$, $a_k \in \mathbb{R}$ (series of real numbers).

Definition.
Let $\sum\limits_{k=n_0}^{\infty} a_k$ be a series.

We define its **partial sums** by $s_N = \sum\limits_{k=n_0}^{N} a_k$ for $N \geq n_0$.
We say that the given series **converges** if $\{s_N\}_{N=n_0}^{\infty}$ converges.
We say that the given series **converges to** A, denoted $\sum\limits_{k=n_0}^{\infty} a_k = A$, if $\lim\limits_{N\to\infty}(s_N) = A$.
We say that the given series **diverges** if $\{s_N\}_{N=n_0}^{\infty}$ diverges.
We say that the given series **diverges to** ∞, denoted $\sum\limits_{k=n_0}^{\infty} a_k = \infty$, if $\lim\limits_{N\to\infty}(s_N) = \infty$.
We say that the given series **diverges to** $-\infty$, denoted $\sum\limits_{k=n_0}^{\infty} a_k = -\infty$, if $\lim\limits_{N\to\infty}(s_N) = -\infty$.

Example.
$\sum\limits_{k=1}^{\infty} \frac{1}{2^k}$: $s_1 = \frac{1}{2}$, $s_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$, $s_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$, induction: $s_N = 1 - \frac{1}{2^N}$, hence $s_N \to 1$ and $\sum\limits_{k=1}^{\infty} \frac{1}{2^k} = 1$ (series converges).

Example.
$\sum\limits_{k=1}^{\infty} 1$: $s_1 = 1$, $s_2 = 1 + 1 = 2$, $s_3 = 1 + 1 + 1 = 3$, induction: $s_N = N$, hence $s_N \to \infty$ and $\sum\limits_{k=1}^{\infty} 1 = \infty$ (series diverges).

Example.
$\sum\limits_{k=0}^{\infty} (-1)^k$: $s_0 = 1$, $s_1 = 1 - 1 = 0$, $s_2 = 1 - 1 + 1 = 1$, induction: $s_N = \begin{cases} 1, & N \text{ even;} \\ 0, & N \text{ odd,} \end{cases}$ thus $\lim\limits_{N\to\infty}(s_N)$ DNE and $\sum\limits_{k=0}^{\infty} (-1)^k$ diverges.

0.1. Summing up series

Definition.
Let $a, q \in \mathbb{R}$. The series $\sum\limits_{k=n_0}^{\infty} a\,q^k$ is called a **geometric series**.

Fact.
(i) For $N \in \mathbb{N}_0$ we have $\sum\limits_{k=0}^{N} q^k = \dfrac{1 - q^{N+1}}{1 - q}$.

(ii) We have $\sum\limits_{k=0}^{\infty} q^k \begin{cases} = \frac{1}{1-q}, & |q|<1; \\ = \infty \text{ (diverges)}, & q \geq 1; \\ \text{diverges}, & q \leq -1. \end{cases}$

Summing up a series: we can sum up directly only two kinds:
1) geometric series (might be in disguise):
Example.

$$\sum_{k=2}^{\infty} \frac{5\cdot 3^{k-1}}{2^{2k+1}} = \sum_{k=2}^{\infty} \frac{5\cdot 3^{-1}\cdot 3^k}{2^1\cdot (2^2)^k} = \frac{5}{6}\sum_{k=2}^{\infty}\frac{3^k}{4^k} = \frac{5}{6}\sum_{k=2}^{\infty}\left(\tfrac{3}{4}\right)^k = \frac{5}{6}\left(\tfrac{3}{4}\right)^2\sum_{k=0}^{\infty}\left(\tfrac{3}{4}\right)^k$$
$$= \left\langle\!\left\langle \left|\tfrac{3}{4}\right| < 1 \right\rangle\!\right\rangle = \frac{5}{6}\left(\tfrac{3}{4}\right)^2 \frac{1}{1-\frac{3}{4}} = \frac{15}{8}.$$

Note: For a geometric series $\sum\limits_{k=n_0}^{\infty} q^k = q^{n_0}\sum\limits_{k=0}^{\infty} q^k$ is true in general.

Or substitution: $\sum\limits_{k=2}^{\infty}\left(\frac{3}{4}\right)^k = \left| \begin{matrix} n=k-2 \implies k=n+2 \\ 2\mapsto 0,\ \infty\mapsto\infty \end{matrix} \right| = \sum\limits_{n=0}^{\infty}\left(\frac{3}{4}\right)^{n+2} = \left(\frac{3}{4}\right)^2\sum\limits_{n=0}^{\infty}\left(\frac{3}{4}\right)^n.$

This an be used for any series, sometimes I use notation $\left|k-2\mapsto k^*\right|$.
2) telescopic series (might be in disguise):
Example.

$$\sum_{k=3}^{\infty}\frac{1}{k(k-1)} = \sum_{k=3}^{\infty}\left(\tfrac{1}{k-1}-\tfrac{1}{k}\right) = \left(\tfrac{1}{2}-\tfrac{1}{3}\right)+\left(\tfrac{1}{3}-\tfrac{1}{4}\right)+\left(\tfrac{1}{4}-\tfrac{1}{5}\right)+\dots.$$

Induction: $s_N = \frac{1}{2}-\frac{1}{N}\to\frac{1}{2}$, hence $\sum\limits_{k=3}^{\infty}\frac{1}{k(k-1)} = \frac{1}{2}$.

Remark: formulas for finite sums:
$\sum\limits_{k=1}^{n} 1 = n$, $\sum\limits_{k=1}^{n} k = \frac{1}{2}n(n+1)$, $\sum\limits_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$, etc.

Theorem.
Let series $\sum\limits_{k=n_0}^{\infty} a_k$, $\sum\limits_{k=n_0}^{\infty} b_k$ converge.

Then also the series $\sum\limits_{k=n_0}^{\infty}(a_k+b_k)$ converges and $\sum\limits_{k=n_0}^{\infty}(a_k+b_k) = \sum\limits_{k=n_0}^{\infty} a_k + \sum\limits_{k=n_0}^{\infty} a_k$.

For $c\in\mathbb{R}$ also $\sum\limits_{k=n_0}^{\infty}(c\,a_k)$ converges and $\sum\limits_{k=n_0}^{\infty}(c\,a_k) = c\Big(\sum\limits_{k=n_0}^{\infty} a_k\Big)$.

0.2. Convergence of series

Theorem.
Let $n_0<n_1$, consider a series $\sum\limits_{k=n_0}^{\infty} a_k$. $\sum\limits_{k=n_0}^{\infty} a_k$ converges if and only if $\sum\limits_{k=n_1}^{\infty} a_k$ converges.

Then we also have $\sum\limits_{k=n_0}^{\infty} a_k = \sum\limits_{k=n_0}^{n_1-1} a_k + \sum\limits_{k=n_1}^{\infty} a_k$.

If we are only interested in convergence of a series and not its sum (if it exists), then we leave out the index specification.

Theorem. (**necessary condition** for convergence)
If a series $\sum a_k$ converges, then necessarily $\lim\limits_{k\to\infty}(a_k)=0$.
Equivalently: If $\lim\limits_{k\to\infty}(a_k)=0$ is not true, then the series $\sum a_k$ necessarily diverges.

Theorem.
Consider a series $\sum a_k$. If $a_k\geq 0$ for all k, then either $\sum a_k$ converges, or $\sum a_k=\infty$.

0.2.1. Tests for series with non-negative numbers

Theorem. (**integral test**)
Let $f \geq 0$ be a non-increasing function on $[n_0, \infty)$ for $n_0 \in \mathbb{Z}$.
The series $\sum\limits_{k=n_0}^{\infty} f(k)$ converges if and only if $\int\limits_{n_0}^{\infty} f(x)\,dx$ converges.
Moreover we then have $\int\limits_{n_0}^{\infty} f(x)\,dx \leq \sum\limits_{k=n_0}^{\infty} f(k) \leq f(n_0) + \int\limits_{n_0}^{\infty} f(x)\,dx$.

Example.
$\sum\limits_{k=3}^{\infty} \frac{1}{k\ln^2(k)}$: $\int\limits_{x=3}^{\infty} \frac{dx}{x\ln^2(x)} = \left| \begin{matrix} y = \ln(x) \\ dy = \frac{dx}{x} \end{matrix} \right| = \int\limits_{x=\ln(3)}^{\infty} \frac{dy}{y^2} < \infty$. Therefore the series converges.
Morevoer, $\sum\limits_{k=3}^{\infty} \frac{1}{k\ln^2(k)} \in \left[\frac{1}{\ln(3)}, \frac{1}{3\ln^2(3)} + \frac{1}{\ln(3)}\right] \sim [0.91, 1.19]$.
Trick: $\sum\limits_{k=3}^{\infty} \frac{1}{k\ln^2(k)} = \sum\limits_{k=3}^{9} \frac{1}{k\ln^2(k)} + \sum\limits_{k=10}^{\infty} \frac{1}{k\ln^2(k)}$
$\in \left[\sum\limits_{k=3}^{9} \frac{1}{k\ln^2(k)} + \int\limits_{10}^{\infty} \frac{dx}{x\ln^2(x)}, \sum\limits_{k=3}^{9} \frac{1}{k\ln^2(k)} + \frac{1}{10\ln^2(10)} + \int\limits_{10}^{\infty} \frac{dx}{x\ln^2(x)}\right] \sim [1.059, 1.078]$.

Corollary. (**p-test**)
$\sum \frac{1}{k^p}$ converges if and only if $p > 1$.

Example.
$\sum\limits_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$.
$\sum\limits_{k=1}^{\infty} \frac{1}{k} = \infty$ (**harmonic series**).

Theorem. (**comparison test**)
Let $\exists n_0$ so that $0 \leq a_k \leq b_k$ for $k \geq n_0$.
If $\sum b_k$ converges, then also $\sum a_k$ converges.
If $\sum a_k$ diverges, then also $\sum b_k$ diverges (i.e. $\sum a_k = \infty \implies \sum b_k = \infty$).
Remark: Symbolically (and roughly) $a_k \leq b_k \implies \sum a_k \leq \sum b_k$.

Theorem. (**limit comparison test**) Let $\exists n_0 \in \mathbb{Z}$ so that $a_k \geq 0, b_k \geq 0$ for $k \geq n_0$.
If $a_k \sim b_k$, i.e. $\lim\limits_{k\to\infty} \left(\frac{a_k}{b_k}\right) = A > 0$, then $\sum a_k \sim \sum b_k$, i.e. $\sum a_k$ converges if and only if $\sum b_k$ converges.

Example.
$\sum \frac{1}{k^2+1}$: $0 \leq \frac{1}{k^2+1} \leq \frac{1}{k^2}$ and $\sum \frac{1}{k^2}$ converges, therefore by CT also $\sum \frac{1}{k^2+1}$ converges.
Remark: Also IT and LCT would work here.

Example.
$\sum \frac{1}{2k^2-1}$: $\frac{1}{2k^2-1} \geq \frac{1}{2k^2} \geq 0$, $\sum \frac{1}{2k^2} = \frac{1}{2} \sum \frac{1}{k^2}$ converges, but the inequality goes the wrong way, so no conclusion possible.
Guess $\frac{1}{2k^2-1} \sim \frac{1}{2k^2}$ for large k, confirm: $\lim\limits_{k\to\infty} \left(\frac{\frac{1}{2k^2-1}}{\frac{1}{2k^2}}\right) = 1 > 0$,
$\sum \frac{1}{2k^2} = \frac{1}{2} \sum \frac{1}{k^2}$ converges, hence by LCT also $\sum \frac{1}{2k^2-1}$ converges.

Example.
$\sum \frac{1}{k\ln^2(k)}$: Two comparisons seem reasonable, $\frac{1}{k^2} \le \frac{1}{k\ln^2(k)} \le \frac{1}{k}$, but both are in the wrong direction, so nothing here.
Limit comparison: No candidate, $\frac{1}{k\ln^2(k)} \sim \frac{1}{k}$ or $\frac{1}{k\ln^2(k)} \sim \frac{1}{k^2}$ definitely not true.
Thus comparison tests won't help.

Theorem.
Let $a_k \ge 0$ for all k.
ratio test:
(i) If $\exists q < 1$ and $\exists n_0 \in \mathbb{Z}$ such that $\forall k \ge n_0$: $\frac{a_{k+1}}{a_k} \le q$, then $\sum a_k$ converges.
(ii) If $\exists n_0 \in \mathbb{Z}$ such that $\forall k \ge n_0$: $\frac{a_{k+1}}{a_k} \ge 1$, then $\sum a_k$ diverges $(= \infty)$.
limit ratio test: Let $\lambda = \lim\limits_{k\to\infty} \big(\frac{a_{k+1}}{a_k}\big)$, assuming that the limit converges.
(i) If $\lambda < 1$, then $\sum a_k$ converges.
(ii) If $\lambda > 1$, then $\sum a_k$ diverges $(= \infty)$.
root test:
(i) If $\exists q < 1$ and $\exists n_0 \in \mathbb{Z}$ such that $\forall k \ge n_0$: $\sqrt[k]{a_k} \le q$, then $\sum a_k$ converges.
(ii) If $\exists n_0 \in \mathbb{Z}$ such that $\forall k \ge n_0$: $\sqrt[k]{a_k} \ge 1$, then $\sum a_k$ diverges $(= \infty)$.
limit root test: Let $\varrho = \lim\limits_{k\to\infty} \big(\sqrt[k]{a_k}\big)$, assuming that the limit converges.
(i) If $\varrho < 1$, then $\sum a_k$ converges.
(ii) If $\varrho > 1$, then $\sum a_k$ diverges $(= \infty)$.

Example.
$\sum \frac{k!}{2^k}$: Limit ratio test $\lambda = \lim\limits_{k\to\infty} \big(\frac{a_{k+1}}{a_k}\big) = \lim\limits_{k\to\infty} \big(\frac{(k+1)!}{k!}\,\frac{2^k}{2^{k+1}}\big) = \lim\limits_{k\to\infty} \big(\frac{1}{2}(k+1)\big) = \infty > 1$.
Thus $\sum \frac{k!}{2^k}$ diverges.

Example.
$\sum \frac{2}{\ln^k(k+1)}$: Limit root test $\varrho = \lim\limits_{k\to\infty} \big(\sqrt[k]{a_k}\big) = \lim\limits_{k\to\infty} \big(\frac{\sqrt[k]{2}}{\ln(k+1)}\big) = \frac{1}{\infty} = 0 < 1$.
Thus $\sum \frac{2}{\ln^k(k+1)}$ converges.

Example.
$\sum \big(\frac{k}{k+1}\big)^k$: Limit root test $\varrho = \lim\limits_{k\to\infty} \big(\sqrt[k]{a_k}\big) = \lim\limits_{k\to\infty} \big(\frac{k}{k+1}\big) = 1$, no conclusion.
Similarly limit ratio fails. Integral test without chance, comparison as well.
But: $a_k = \big(1 - \frac{1}{k+1}\big)^k \to e^{-1} \neq 0$, hence $\sum \big(\frac{k}{k+1}\big)^k$ diverges.

0.2.2. Tests for alternating series

Theorem. (**Alternating series test** or **Leibniz test**)
Let $b_k \ge 0$ for all k and let $\{b_k\}$ be non-increasing.
The series $\sum (-1)^k b_k$ converges if and only if $\lim\limits_{k\to\infty} (b_k) = 0$.

Example.
$\sum \frac{(-1)^k}{k}$: $b_k = \frac{1}{k} \ge 0$ is decreasing and $\to 0$, hence $\sum \frac{(-1)^k}{k}$ converges (compare with harmonic series).

0.3. Absolute convergence of series

Definition.
We say that a series $\sum a_k$ **converges absolutely** if the series $\sum |a_k|$ converges.

Theorem.
If a series $\sum a_k$ converges absolutely, then it also converges and we have $\Bigl|\sum\limits_{k=n_0}^{\infty} a_k\Bigr| \leq \sum\limits_{k=n_0}^{\infty} |a_k|$.
But not the other way around! Recall that $\sum \frac{(-1)^k}{k}$ converges, but $\sum\bigl|\frac{(-1)^k}{k}\bigr| = \sum \frac{1}{k} = \infty$.

Definition.
We say that a series **converges conditionally** if it converges, but not absolutely.

Thus there are three possibilities now:
—$\sum a_k$ converges, $\sum |a_k|$ converges: absolute convergence (the second implies the first here)
—$\sum a_k$ converges, $\sum |a_k|$ diverges: conditional convergence
—$\sum a_k$ diverges, $\sum |a_k|$ diverges (the first implies the second)

Example.
conditional convergence: $\sum \frac{(-1)^k}{k}$; absolute convergence: $\sum \frac{(-1)^k}{k^2}$; divergence: $\sum(-1)^k$.

Example.
$\sum \frac{\sin(k)}{2^k}$: We do not know how to investigate this series directly. Its terms are not non-negative, therefore the tests won't work. We can't use AST, since the series is not alternating. The necessary condition won't help either, since $a_k \to 0$.
Thus we try the absolute convegence and hope that it will come out true, so that we have some conclusion:
$\sum\bigl|\frac{\sin(k)}{2^k}\bigr| = \sum \frac{|\sin(k)|}{2^k} \leq \sum \frac{1}{2^k}$, this converges, therefore by comparison test also $\sum\bigl|\frac{\sin(k)}{2^k}\bigr|$ converges, hence $\sum \frac{\sin(k)}{2^k}$ converges absolutely.

Example.
$\sum(-1)^k \frac{2^k}{k^3}$: absolute: $\sum\bigl|(-1)^k \frac{2^k}{k^3}\bigr| = \sum \frac{2^k}{k^3}$, ratio test: $\frac{a_{k+1}}{a_k} = 2\bigl(\frac{k}{k+1}\bigr)^3 \to 2 = \lambda > 1$,
thus $\sum\bigl|(-1)^k \frac{2^k}{k^3}\bigr|$ diverges, hence $\sum(-1)^k \frac{2^k}{k^3}$ does not converge absolutely. But we do not know whether it by itself does converge (then it would be conditional convergence) or not.
However, $\frac{2^k}{k^3} \to \infty$, thus $a_k = (-1)^k \frac{2^k}{k^3} \not\to 0$, so the series diverges.

Theorem.
Consider a series $\sum\limits_{k=2n_0}^{\infty} a_k$.
If $\sum a_k$ converges absolutely, then also $\sum a_{2k}$ and $\sum a_{2k+1}$ converge and
$\sum\limits_{k=2n_0}^{\infty} a_k = \sum\limits_{k=n_0}^{\infty} a_{2k} + \sum\limits_{k=n_0}^{\infty} a_{2k+1}$.

Not true for conditional convergence, see $\sum\limits_{k=1}^{\infty} \frac{(-1)^k}{k}$.

Theorem.
Consider a series $\sum\limits_{k=n_0}^{\infty} a_k$.
If $\sum a_k$ converges absolutely, then for every choice of signs $\varepsilon_k = \pm 1$ also $\sum \varepsilon_k a_k$ converges.
If $\sum a_k$ converges conditionally, then there is a choice of signs $\varepsilon_k = \pm 1$ such that $\sum \varepsilon_k a_k = \infty$.

Definition.
Consider a series $\sum\limits_{k=n_0}^{\infty} a_k$.

By a **rearrangement** of $\sum\limits_{k=n_0}^{\infty} a_k$ we mean any series $\sum\limits_{k=n_0}^{\infty} a_{\pi(k)}$, where π is an arbitrary bijective mapping of $\{n_0, n_0+1, n_0+2, \dots\} \subset \mathbb{Z}$ onto $\{n_0, n_0+1, n_0+2, \dots\}$, i.e. π is a permutation of $\{n_0, n_0+1, n_0+2, \dots\}$.

Theorem.
Consider a series $\sum\limits_{k=n_0}^{\infty} a_k$.
If $\sum a_k$ converges absolutely, then also all its rearrangements $\sum a_{\pi(k)}$ converge and we have $\sum\limits_{k=n_0}^{\infty} a_{\pi(k)} = \sum\limits_{k=n_0}^{\infty} a_k$.
If $\sum\limits_{k=n_0}^{\infty} a_k$ converges conditionally, then $\forall c \in \mathbb{R} \cup \{\pm\infty\}$ there exists its rearrangement such that $\sum\limits_{k=n_0}^{\infty} a_{\pi(k)} = c$.

Series of complex numbers.
Theory is similar, as suggested by the following statement.

Theorem.
Let a_k be complex numbers.
Then $a_k \to A$ if and only if $\text{Re}(a_k) \to \text{Re}(A)$ and $\text{Im}(a_k) \to \text{Im}(A)$.
Similarly a series $\sum\limits_{k=n_0}^{\infty} a_k$ converges if and only if $\sum\limits_{k=n_0}^{\infty} \text{Re}(a_k)$ and $\sum\limits_{k=n_0}^{\infty} \text{Im}(a_k)$ converge.
Then also $\sum\limits_{k=n_0}^{\infty} a_k = \Big(\sum\limits_{k=n_0}^{\infty} \text{Re}(a_k)\Big) + j\Big(\sum\limits_{k=n_0}^{\infty} \text{Im}(a_k)\Big)$.

In this way we can work with plain convergence. However, in the world of complex numbers the absolute convergence becomes a very strong weapon, as it allows us to move from series of complex numbers to series with non-negative numbers and is still stronger than the usual convergence.

1. Sequences and series of functions

Definition.
By a **sequence of functions** we mean an ordered set $\{f_k\}_{k=n_0}^{\infty} = \{f_{n_0}, f_{n_0+1}, f_{n_0+2}, \dots\}$, where f_k are functions.
Remark: Given a sequence of functions $\{f_k\}_{k=n_0}^{\infty}$ and $x \in \bigcap D(f_k)$, then $\{f_k(x)\}$ is a standard sequence of real (complex) numbers.

Definition.
Let $\{f_k\}_{k\geq n_0}$, f be functions on a set M.
We say that $\{f_k\}$ **converges (pointwise)** to f on M, denoted $f_k \to f$ or $f = \lim\limits_{k\to\infty}(f_k)$, if $\forall x \in M$: $\lim\limits_{k\to\infty}\big(f_k(x)\big) = f(x)$.

Example.

Consider $f_k(x) = \arctan(kx)$. Then $\lim\limits_{k\to\infty}\big(f_k(x)\big) = \begin{cases} 0, & x = 0; \\ \frac{\pi}{2}, & x > 0; \\ -\frac{\pi}{2}, & x < 0. \end{cases}$

Definition.
Let $\{f_k\}_{k\geq n_0}$, f be functions on a set M.
We say that $\{f_k\}$ **converges uniformly** to f on M, denoted $f_k \rightrightarrows f$,
if $\forall \varepsilon > 0 \exists N_0 \in \mathbb{N}$ such that $\forall k \geq N_0 \forall x \in M : |f(x) - f_k(x)| < \varepsilon$.

Theorem.
Let $f_k \rightrightarrows f$ on M.
(i) If all f_k are continuous on M, then also f is continuous there.
(ii) If all f_k have a continuous derivative on M and $\{f_k'\}$ converges uniformly on M, then also f is differentiable there and $f' = \lim\limits_{k\to\infty}(f_k')$ on M.
(iii) If all f_k are continuous and have an antiderivative on M, then also f has an antiderivative there and $\int_{x_0}^{x} f\,dx = \lim\limits_{k\to\infty}\big(\int_{x_0}^{x} f_k\,dx\big)$ for $\overline{x_0, x} \subseteq M$.

Definition.
A **series of functions** is a symbol $\sum\limits_{k=n_0}^{\infty} f_k = f_{n_0} + f_{n_0+1} + f_{n_0+2} + \dots$, where f_k are functions.
Remark: Given a series of functions $\sum f_k$ and $x \in \bigcap D(f_k)$, then $\sum f_k(x)$ is a standard series of real (complex) numbers.

Definition.
Consider a series of functions $\sum\limits_{k=n_0}^{\infty} f_k$.
The **region of convergence** of this series is the set $\big\{x \in \bigcap D(f_k);\ \sum f_k(x) \text{ converges}\big\}$. By defining $f(x) = \sum\limits_{k=n_0}^{\infty} f_k(x)$ we then obtain a function f on this set called the **sum of the series**, denoted $\sum\limits_{k=n_0}^{\infty} f_k = f$.
The **region of absolute convergence** of this series is the set
$\big\{x \in \bigcap D(f_k);\ \sum f_k(x) \text{ converges absolutely}\big\}$.
We say that this series **converges uniformly** to f on M, denoted $\sum f_k \rightrightarrows f$ on M, if the sequence of partial sums $\Big\{\sum\limits_{k=n_0}^{N} f_k(x)\Big\}$ converges uniformly to f on M.

Theorem.
Consider series of functions $\sum f_k$ and $\sum g_k$.
If $\sum\limits_{k=n_0}^{\infty} f_k = f$ on M and $\sum\limits_{k=n_0}^{\infty} g_k = g$ on M, then $\forall a,b \in \mathbb{R}$: $\sum\limits_{k=n_0}^{\infty} (af_k + bg_k) = af + bg$ on M.

Theorem. (Weierstrass criterion)
Let f_k for $k \geq n_0$ be functions on M. Let $a_k \geq 0$ satisfy $\forall x \in M \forall k \geq n_0$: $|f_k(x)| \leq a_k$.
If $\sum a_k$ converges, then $\sum f_k$ converges uniformly on M.

Example.
$\sum x^k = \frac{1}{1-x}$ on $(-1,1)$, but the convergence is not uniform. It will be uniform if we restrict our attention to $[-\varrho,\varrho]$ for $\varrho \in (0,1)$.

Theorem.
Let $\sum f_k \rightrightarrows f$ on M.
(i) If all f_k are continuous on M, then also f is continuous there.
(ii) If all f_k have a derivative on M and $\sum f_k'$ converges univormly on M, then also f has derivative on M and $f' = \sum\limits_{k=n_0}^{\infty} f_k'$ on M.
(iii) If all f_k are continuous and have an antideriative on M, then also f has an antiderivative on M and $\int_{x_0}^{x} f\,dx = \sum\limits_{k=n_0}^{\infty} \int_{x_0}^{x} f_k\,dx$ for $\overline{x_0,x} \subseteq M$.

None of this is true in general for ordinary (pointwise) convergence.

2. Power series

Definition.
Let $z_0 \in \mathbb{R}$.
By a **power series with center** x_0 we mean any series of functions of the form $\sum\limits_{k=0}^{\infty} a_k(x-x_0)^k$, where $a_k \in \mathbb{R}$.

Theorem.
Consider a power series $\sum\limits_{k=0}^{\infty} a_k(x-x_0)^k$.
There exists $r \in \mathbb{R}_0^+ \cup \{\infty\}$ such that $\sum\limits_{k=0}^{\infty} a_k(x-x_0)^k$ converges absolutely on $U_r(x_0) = (x_0 - r, x_0 + r)$ and diverges for $|x - x_0| > r$. Moreover, $r = \dfrac{1}{\limsup\limits_{k\to\infty}(\sqrt[k]{|a_k|})}$.
Remark: We also have $r = \dfrac{1}{\lim\limits_{k\to\infty}(\frac{|a_{k+1}|}{|a_k|})}$, assuming that this limit exists.
Remark: A power series always converges (absolutely) at $x = x_0$.

Definition.
Consider a power series $\sum\limits_{k=0}^{\infty} a_k(x-x_0)^k$.
The number r with properties as in the previous theorem is called the **radius of convergence** of this series.

Example.
$\sum \frac{(2x)^k}{k\,3^k} = \sum \frac{2^k}{k\,3^k}(x-0)^k$, hence $x_0 = 0$.
Absolute convergence by limit root test: $\sqrt[k]{|a_k|} = \frac{2|x|}{3\sqrt[k]{k}} \to \frac{2|x|}{3} = \varrho$.
$\varrho < 1 \iff \frac{2|x|}{3} < 1 \iff |x| < \frac{3}{2}$, thus the radius of convergence $r = \frac{3}{2}$. Endpoints $x_0 \pm r = \pm\frac{3}{2}$:
$x = \frac{3}{2}$: $\sum \frac{1}{k} = \infty$.
$x = -\frac{3}{2}$: $\sum \frac{(-1)^k}{k}$ converges.
Region of convergence $\big[-\frac{3}{2}, \frac{3}{2}\big)$, region of absolute convergence $\big(-\frac{3}{2}, \frac{3}{2}\big)$.

Example.
$\sum \frac{(2x-4)^k}{k!} = \sum \frac{2^k}{k!}(x-2)^k$, hence $x_0 = 2$.
Absolute convergence by limit ratio test: $\frac{|a_{k+1}|}{|a_k|} = \frac{2}{k+1}|x-2| \to 0 = \lambda$.
$\lambda < 1$ is true $\forall x$, hence radius of convergence $r = \infty$.
Region of convergence and region of absolute convergence $\mathbb{R}$.

Example.
$\sum k^k(2x+3)^k = \sum k^k 2^k\big(x - \big(-\frac{3}{2}\big)\big)^k$, hence $x_0 = -\frac{3}{2}$.
Absolute convergence by limit root test:
$$\sqrt[k]{|a_k|} = 2k\big|x + \tfrac{3}{2}\big| \to \begin{cases} \infty, & x \neq -\frac{3}{2}; \\ 0, & x = -\frac{3}{2} \end{cases} = \varrho.$$
$\varrho < 1 \iff x = -\frac{3}{2}$, hence radius of convergence $r = 0$.
Region of convergence ans radius of absolute convergence $\big\{-\frac{3}{2}\big\}$.

Theorem.

Let $x_0 \in \mathbb{R}$, assume that $\sum\limits_{k=0}^{\infty} a_k(x-x_0)^k = f$, $\sum\limits_{k=0}^{\infty} b_k(x-x_0)^k = g$ have radii of convergence r_f and r_g.

(i) Then $\forall a,b \in \mathbb{R}$: $\sum\limits_{k=0}^{\infty} (aa_k + bb_k)(x-x_0)^k = af + bg$ has radius of convergence $r = \min(r_f, r_g)$.

(ii) The series $\sum\limits_{k=0}^{\infty} \Big(\sum\limits_{i=0}^{k} a_i b_{k-i}\Big)(x-x_0)^k = \Big(\sum\limits_{k=0}^{\infty} a_k(x-x_0)^k\Big) \cdot \Big(\sum\limits_{k=0}^{\infty} b_k(x-x_0)^k\Big) = f \cdot g$ has radius of convergence $r = \min(r_f, r_g)$.

Theorem.

Let $\sum\limits_{k=0}^{\infty} a_k(x-x_0)^k = f$ have radius of convergence $r > 0$.

(i) For any $\varrho \in (0,r)$: $\sum\limits_{k=0}^{\infty} a_k(x-x_0)^k \rightrightarrows f$ on $U_\varrho(x_0)$.

(ii) f is continuous, it has the derivative $f'(x) = \sum\limits_{k=1}^{\infty} k\, a_k(x-x_0)^{k-1}$ with radius of convergence r and an antiderivative $F(x) = \sum\limits_{k=0}^{\infty} \frac{a_k}{k+1}(x-x_0)^{k+1}$ with radius of convergence r.

Corollary.

Let $f(x) = \sum\limits_{k=0}^{\infty} a_k(x-x_0)^k$ on $U_r(x_0)$.

Then on $U_r(x_0)$ we have for $n \in \mathbb{N}$ also derivatives

$f^{(n)}(x) = \sum\limits_{k=n}^{\infty} k(k-1)\cdot\ldots\cdot(k-n+1)\, a_k(x-x_0)^{k-n}$.

Remark: At endpoints $x_0 \pm r$ anything can happen, there is no theorem that would also include behaviour there, so we can lose properties there (convergence for instance).

Example.

$f(x) = -\ln(1-x) = \sum\limits_{k=1}^{\infty} \frac{1}{k+1} x^{k+1}$ converges on $[-1,1)$, but $f'(x) = \frac{1}{1-x} = \sum\limits_{k=0}^{\infty} x^k$ converges only on $(-1,1)$.

Remark: If $f(x) = \sum\limits_{k=0}^{\infty} a_k(x-x_0)^k$ on $U_r(x_0)$, then $f^{(n)}(x_0) = n!\, a_n$ for $n \in \mathbb{N}_0$.

Definition.

Let f have derivatives of all orders at x_0.

We define the **Taylor series** of f with center at x_0 by the formula $\sum\limits_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$.

Corollary. (**uniqueness of expansion**)

If $f(x) = \sum\limits_{k=0}^{\infty} a_k(x-x_0)^k$ on $U_r(x_0)$, then this series is necessarily the Taylor series.

The finding of this series is called **expanding a given function into a power/Taylor series (with center x_0)**. Partial sum is a Taylor polynomial, so we know all that we need, even what is the difference between T_n and f by Lagrange.

Example.
$e^x=\sum\limits_{k=0}^{\infty}\dfrac{x^k}{k!}$ for $x\in I\!R$.

Proof: Denote $T_N=\sum\limits_{k=0}^{N}\dfrac{x^k}{k!}$. Then by the Lagrange form of remainder $e^x-T_N(x)=R_N(x)$, where $R_N(x)=\frac{[e^x]^{(N+1)}(c)}{(N+1)!}x^{N+1}$. Fix some x and estimate $|R_N(x)|\le\frac{1}{(N+1)!}\max\limits_{x\in\overline{0x}}|e^x||x|^{N+1}$, thus for $x\ge 0$ we have $|R_N(x)|\le\frac{e^x|x|^{N+1}}{(N+1)!}$, for $x\le 0$ we have $|R_N(x)|\le\frac{|x|^{N+1}}{(N+1)!}$. In any case $R_N(x)\to 0$, so $T_N(x)\to e^x$.

Theorem.
Let a function f have derivatives of all orders on some $U_r(x_0)$ and let $\exists M>0$ such that $|f^{(k)}(x_0)|\le M$ for all $k\in\mathbb{N}_0$ and $x\in U_r(x_0)$. Then for $x\in U_r(x_0)$ we have $f(x)=\sum\limits_{k=0}^{\infty}\dfrac{f^{(k)}(x_0)}{k!}(x-x_0)^k$.

Fact.

$$\frac{1}{1-x}=\sum_{k=0}^{\infty}x^k=1+x+x^2+x^3+x^4+\ldots,\qquad x\in(-1,1);$$

$$e^x=\sum_{k=0}^{\infty}\frac{x^k}{k!}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots,\qquad x\in\mathbb{R};$$

$$\sin(x)=\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!}=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots,\qquad x\in\mathbb{R};$$

$$\cos(x)=\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!}=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots,\qquad x\in\mathbb{R};$$

$$\ln(1+x)=\sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^k}{k}=1-x+\frac{x^2}{2}-\frac{x^3}{3}+\frac{x^4}{4}-\ldots,\qquad x\in(-1,1];$$

$$(1+x)^\alpha=\sum_{k=0}^{\infty}\binom{\alpha}{k}x^k=1+\alpha x+\frac{\alpha(\alpha-1)}{2}x^2+\frac{\alpha(\alpha-1)(\alpha-3)}{3!}x^3+\ldots,\qquad x\in(-1,1).$$

Here $\dbinom{\alpha}{k}=\dfrac{\alpha\cdot(\alpha-1)\cdot\ldots\cdot(\alpha-k+2)\cdot(\alpha-k+1)}{k!}$.

Other functions are done using these and algebra, substitution etc.

Example.
Expand $f(x)=(x+3)e^{4x}$ into a series with center $x_0=0$:

$$\begin{aligned}(x+3)e^{4x}&=xe^{4x}+3e^{4x}=\langle\!\langle y=4x\rangle\!\rangle=x\sum_{k=0}^{\infty}\tfrac{(4x)^k}{k!}+3\sum_{k=0}^{\infty}\tfrac{(4x)^k}{k!}=\sum_{k=0}^{\infty}\tfrac{4^k}{k!}x^{k+1}+\sum_{k=0}^{\infty}\tfrac{3\cdot 4^k}{k!}x^k\\&=\langle\!\langle k+1\mapsto k^*\rangle\!\rangle=\sum_{k=1}^{\infty}\tfrac{4^{k-1}}{(k-1)!}x^k+\sum_{k=0}^{\infty}\tfrac{3\cdot 4^k}{k!}x^k=\sum_{k=1}^{\infty}\tfrac{k4^{k-1}}{k!}x^k+3+\sum_{k=1}^{\infty}\tfrac{3\cdot 4^k}{k!}x^k\\&=3+\sum_{k=1}^{\infty}\tfrac{(k+12)4^{k-1}}{k!}x^k,\ x\in\mathbb{R}.\end{aligned}$$

Example.
Expand $f(x)=\frac{1}{1+3x^2}$ into a series with center $x_0=0$:

$$\tfrac{1}{1+3x^2}=\tfrac{1}{1-(-3x^2)}=\langle\!\langle y=-3x^2,\ |y|<1\rangle\!\rangle=\sum_{k=0}^{\infty}(-3x^2)^k=\sum_{k=0}^{\infty}(-1)^k3^kx^{2k},\ |x|<\tfrac{1}{\sqrt{3}}.$$

Example.
Expand $f(x)=(x-1)\sin(\pi x)$ into a series with center $x_0=-1$:
$$\begin{aligned}(x-1)\sin(\pi x)&=(x-(-1)-2)\sin(\pi(x-(-1)-1))\\
&=(x-(-1))\sin(\pi(x-(-1))-\pi)-2\sin(\pi(x-(-1))-\pi)\\
&=-(x+1)\sin(\pi(x+1))+2\sin(\pi(x+1))=\langle\!\langle y=\pi(x+1)\rangle\!\rangle\\
&=-(x+1)\sum_{k=0}^{\infty}\tfrac{[\pi(x+1)]^{2k+1}}{(2k+1)!}+2\sum_{k=0}^{\infty}\tfrac{[\pi(x+1)]^{2k+1}}{(2k+1)!}\\
&=\sum_{k=0}^{\infty}\tfrac{2\pi^{2k+1}}{(2k+1)!}(x-(-1))^{2k+1}-\sum_{k=0}^{\infty}\tfrac{\pi^{2k+1}}{(2k+1)!}(x-(-1))^{2k+2},\ x\in\mathbb{R}.\end{aligned}$$

Example.
Expand $f(x)=\ln(1+x)$ into a series with center $x_0=0$:
$$\begin{aligned}\ln(1+x)&=\int\tfrac{dx}{x+1}=\int\tfrac{1}{1-(-x)}\,dx=\langle\!\langle y=-x,\ |y|<1\rangle\!\rangle=\int\sum_{k=0}^{\infty}(-x)^k\,dx=\sum_{k=0}^{\infty}\int(-1)^kx^k\,dx\\
&=\sum_{k=0}^{\infty}(-1)^k\tfrac{x^{k+1}}{k+1}+C=\langle\!\langle k+1\mapsto k^*\rangle\!\rangle=\sum_{k=1}^{\infty}(-1)^{k-1}\tfrac{x^k}{k}+C;\end{aligned}$$
How much is C? Substitute $x=0$: $\ln(1+0)=\sum 0+C$, hence $C=0$.
Thus $\ln(1+x)=\sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^k}{k}$, $|x|<1$.
This works also for $x=1$, but not for $x=-1$.

Example.
Expand $f(x)=\frac{1}{(2x-5)^2}$ into a series with center $x_0=1$:
$$\begin{aligned}\tfrac{1}{(2x-5)^2}&=\Big[-\tfrac12\,\tfrac{1}{2x-5}\Big]'=-\tfrac12\Big[\tfrac{1}{2(x-1)-3}\Big]'=\tfrac16\Big[\tfrac{1}{1-\frac23(x-1)}\Big]'=\langle\!\langle y=\tfrac23(x-1),\ |y|<1\rangle\!\rangle\\
&=\tfrac16\Big[\sum_{k=0}^{\infty}\big(\tfrac23(x-1)\big)^k\Big]'=\tfrac16\sum_{k=0}^{\infty}\big(\tfrac23\big)^k[(x-1)^k]'=\sum_{k=1}^{\infty}\tfrac16\big(\tfrac23\big)^kk(x-1)^{k-1}=\langle\!\langle k-1\mapsto k^*\rangle\!\rangle\\
&=\sum_{k=0}^{\infty}\tfrac{2^k}{3^{k+2}}(k+1)(x-1)^k,\ |x-1|<\tfrac32.\end{aligned}$$

Series of complex numbers.
Complex ∞, $\frac{1}{\infty}=0$, $\frac10=\infty$, e^{∞} DNE!
Neighborhood in complex plane: $U_\varepsilon(z_0)=\{z\in\mathbb{C};\ |z-z_0|<\varepsilon\}$, $U_\varepsilon(\infty)=\{z\in\mathbb{C};\ |z|>1/\varepsilon\}$.
Definitions that are done using neighborhoods are the same in $\mathbb{R}$ and in $\mathbb{C}$ (limit, sum of a series). Therefore also theorems that use neighborhoods and do not use comparison between terms (but they can use comparison of absolute values of terms) are valid in complex case, for instance the ratio and root tests of convergence when applied to absolute convergence.
Power series then work the same, including the fact that if r is the ratio of convergence of a power series, then the series converges absolutely on $U_r(z_0)=\{z\in\mathbb{C};\ |z-z_0|<r\}$.

3. Fourier series

Definition.
By a **trigonometric series** we mean any series of the form $\frac{a_0}{2}+\sum\limits_{k=1}^{\infty}\bigl[a_k\cos(k\omega t)+b_k\sin(k\omega t)\bigr]$.

By a **trigonometric polynomial** of degree N we mean $\frac{a_0}{2}+\sum\limits_{k=1}^{N}\bigl[a_k\cos(k\omega t)+b_k\sin(k\omega t)\bigr]$.

Remark: These are special cases of **Fourier series**, we will call them this way. Which functions can be expanded into Fourier series?

Fact.
Functions $\sin(k\omega t)$, $\cos(k\omega t)$ are periodic with period $T=\frac{2\pi}{\omega}$.
Thus also trigonometric polynomials and trigonometric series (if it converges) are T-periodic.

Hence only periodic functions can be sums of Fourier series. If we have such a function, which series is the best candidate for expansion?

Theorem.
Let $\omega>0$, $T=\frac{2\pi}{\omega}$. The following are true:
(i) $\int\limits_0^T\sin^2(k\omega t)\,dt=\int\limits_0^T\cos^2(k\omega t)\,dt=\frac{T}{2}$ for $k\in\mathbb{N}$,
(ii) $\int\limits_0^T\sin(k\omega t)\sin(m\omega t)\,dt=\int\limits_0^T\cos(k\omega t)\cos(m\omega t)\,dt=0$ for $k\neq m\in\mathbb{N}$,
(iii) $\int\limits_0^T\sin(k\omega t)\cos(m\omega t)\,dt=0$ for $k,m\in\mathbb{N}$.
Remark on (i): $\int\limits_0^T\sin^2(k\omega t)\,dt=0$ and $\int\limits_0^T\cos^2(k\omega t)\,dt=T$ for $k=0$.

Theorem.
Let f be a T-periodic function, denote $\omega=\frac{2\pi}{T}$.
If $\frac{a_0}{2}+\sum\limits_{k=1}^{\infty}\bigl[a_k\cos(k\omega t)+b_k\sin(k\omega t)\bigr]\rightrightarrows f$ on $[0,T]$, then
$a_k=\frac{2}{T}\int\limits_0^T f(t)\cos(k\omega t)\,dt$ for $k\in\mathbb{N}_0$ and $b_k=\frac{2}{T}\int\limits_0^T f(t)\sin(k\omega t)\,dt$ for $k\in\mathbb{N}$.

Remark: $a_0=\frac{2}{T}\int\limits_0^T f(t)\,dt$.

Definition.
Let f be a function that is T-periodic and integrable on $[0,T]$.
We define its **Fourier series** as $\frac{a_0}{2}+\sum\limits_{k=1}^{\infty}\bigl[a_k\cos(k\omega t)+b_k\sin(k\omega t)\bigr]$, where
$a_k=\frac{2}{T}\int\limits_0^T f(t)\cos(k\omega t)\,dt$ for $k\in\mathbb{N}_0$ and $b_k=\frac{2}{T}\int\limits_0^T f(t)\sin(k\omega t)\,dt$ for $k\in\mathbb{N}$.
We denote $f\sim\frac{a_0}{2}+\sum\limits_{k=1}^{\infty}\bigl[a_k\cos(k\omega t)+b_k\sin(k\omega t)\bigr]$.

Remark: If a function g is T-periodic, then $\forall a \in \mathbb{R}$ we have $\int\limits_0^T g(t)\,dt = \int\limits_a^{a+T} g(t)\,dt$. This can be applied to functions and integrals in formulas for Fourier transform, popular versions are e.g. $a_k = \frac{2}{T}\int\limits_{-T/2}^{T/2} f(t)\cos(k\omega t)\,dt$ for $k \in \mathbb{N}_0$ a $b_k = \frac{2}{T}\int\limits_{-T/2}^{T/2} f(t)\sin(k\omega t)\,dt$ for $k \in \mathbb{N}$.

Definition.
Let f be a function defined on an interval $I = [a, a+T)$ for some $a \in \mathbb{R}$, $T > 0$.
We define its **periodic extension on** $\mathbb{R}$ as the function
$f(t) = f(t - kT)$ for $t \in [a + kT, a + (k+1)T)$.
Remark: We obtain a T-periodic function on $\mathbb{R}$.

Definition.
Let f be a function defined on an interval $I = [a, a+T)$ for some $a \in \mathbb{R}$, $T > 0$.
We define its Fourier series as the Fourier series of its periodic extension.

Example.
Fourier series of function $f(t) = t^2$ on $[-1, 1)$. $T = 2$, $\omega = \pi$.
$a_0 = \frac{2}{2}\int\limits_0^2 f(t)\,dt = \int\limits_{-1}^1 f(t)\,dt = \int\limits_{-1}^1 t^2 dt = \frac{2}{3}$.
$a_k = \frac{2}{2}\int\limits_{-1}^1 t^2\cos(k\pi t)\,dt = \frac{4\cos(k\pi)}{\pi^2 k^2} = \frac{4(-1)^k}{\pi^2 k^2}$.
$b_k = \frac{2}{2}\int\limits_{-1}^1 t^2\sin(k\pi t)\,dt = 0$.
Thus $f \sim \frac{1}{3} + \sum\limits_{k=1}^{\infty}\frac{4(-1)^k}{\pi^2 k^2}\cos(k\pi t)$.

Theorem. (**Jordan criterion implied by derivative**)
Let f be a T-periodic function that is piecewise continuous on some interval I of length T, assume that it has a derivative f' piecewise continuous on I.
Let $f \sim \frac{a_0}{2} + \sum\limits_{k=1}^{\infty}\big[a_k\cos(k\omega t) + b_k\sin(k\omega t)\big]$. Then for every $t \in \mathbb{R}$ we have

$$\lim_{N\to\infty}\Big(\frac{a_0}{2} + \sum_{k=1}^{N}\big[a_k\cos(k\omega t) + b_k\sin(k\omega t)\big]\Big) = \frac{1}{2}[f(t^-) + f(t^+)].$$

If moreover f is continuous on $\mathbb{R}$, then $\frac{a_0}{2} + \sum\limits_{k=1}^{\infty}\big[a_k\cos(k\omega t) + b_k\sin(k\omega t)\big] \rightrightarrows f$.

Example.
For every $t \in [-1, 1]$ we have $t^2 = \frac{1}{3} + \sum\limits_{k=1}^{\infty}\frac{4(-1)^k}{\pi^2 k^2}\cos(k\pi t)$.
We use it for $t = 0$ to get $\sum\limits_{k=1}^{\infty}\frac{(-1)^k}{k^2} = -\frac{\pi^2}{12}$.
For $t = 1$ we get $\sum\limits_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}$.

Definition.
Let f be a function defined and continuous on $[0, L)$.
We define its **sine series** as the Fourier series of its odd periodic extension.
We define its **cosine series** as the Fourier series of its even periodic extension.

Theorem.

Let f be a T-periodic function that is integrable on $[0,T)$, let $\omega = \frac{2\pi}{T}$.

(i) If f is odd, then $a_k = 0$ and $b_k = \frac{4}{T}\int\limits_0^{T/2} f(t)\sin(k\omega t)\,dt$.

(ii) If f is even, then $b_k = 0$ and $a_k = \frac{4}{T}\int\limits_0^{T/2} f(t)\cos(k\omega t)\,dt$.

Fact.

Let f be a function defined and continuous on $[0,L)$.

Its sine Fourier series can be obtained as a Fourier series with $a_k = 0$, $b_k = \frac{2}{L}\int\limits_0^{L} f(t)\sin(k\omega t)\,dt$ and $\omega = \frac{\pi}{L}$.

Its cosine Fourier series can be obtained as a Fourier series with $b_k = 0$, $a_k = \frac{2}{L}\int\limits_0^{L} f(t)\cos(k\omega t)\,dt$ and $\omega = \frac{\pi}{L}$.

Remark: The sum of the sine series is a $T - 2L$-periodic extension of f into an odd function. The sum of the cosine series is a $T = 2L$-periodic extension of f into an even function. Both sums must be also modified using the Jordan criterion.

Example.

$f(t) = \begin{cases} 1, & t \in [0,1); \\ 0, & t \in [1,2). \end{cases}$

Fourier series: $T = 2$, $\omega = \pi$.

$a_0 = \frac{2}{2}\int\limits_0^2 f(t)\,dt = \int\limits_0^1 dt = 1.$

$a_k = \frac{2}{2}\int\limits_0^1 \cos(k\pi t)\,dt = \big[\frac{1}{k\pi}\sin(k\pi t)\big]_0^1 = 0.$

$b_k = \frac{2}{2}\int\limits_0^1 \sin(k\pi t)\,dt = \big[-\frac{1}{k\pi}\cos(k\pi t)\big]_0^1 = \frac{1}{k\pi}[1-\cos(k\pi)] = \frac{1}{k\pi}[1-(-1)^k] = \begin{cases} 0, & k \text{ even}; \\ \frac{2}{k\pi}, & k \text{ odd}. \end{cases}$

Thus $f \sim \frac{1}{2} + \sum\limits_{k=1}^{\infty} \frac{1}{k\pi}[1-(-1)^k]\sin(k\pi t) = \frac{1}{2} + \sum\limits_{k=0}^{\infty} \frac{2}{(2k+1)\pi}\sin((2k+1)\pi t).$

Sine Fourier series: $L = 2$, $T = 4$, $\omega = \frac{\pi}{2}$, $a_k = 0$.

$b_k = \frac{2}{2}\int\limits_0^1 \sin\big(k\frac{\pi}{2}t\big)\,dt = \big[-\frac{2}{k\pi}\cos\big(k\frac{\pi}{2}t\big)\big]_0^1 = \frac{2}{k\pi}[\cos(k\pi) - \cos\big(k\frac{\pi}{2}\big)].$

Thus $\sum\limits_{k=1}^{\infty} \frac{2}{k\pi}[(-1)^k - \cos\big(k\frac{\pi}{2}\big)]\sin\big(k\frac{\pi}{2}t\big).$

Cosine Fourier series: $L = 2$, $T = 4$, $\omega = \frac{\pi}{2}$, $b_k = 0$.

$a_0 = \frac{2}{2}\int\limits_0^1 dt = 1.$

$a_k = \frac{2}{2}\int\limits_0^1 \cos\big(k\frac{\pi}{2}t\big)\,dt = \big[\frac{2}{k\pi}\sin\big(k\frac{\pi}{2}t\big)\big]_0^1 = \frac{2}{k\pi}\sin\big(k\frac{\pi}{2}\big).$

Thus $\frac{1}{2} + \sum\limits_{k=1}^{\infty} \frac{2}{k\pi}\sin\big(k\frac{\pi}{2}\big)\cos\big(k\frac{\pi}{2}t\big).$

Here $a_{2k} = 0$, $a_{2k+1} = (-1)^{k+1}\frac{2}{(2k+1)\pi}$, so $\frac{1}{2} + \sum\limits_{k=0}^{\infty} (-1)^{k+1}\frac{2}{(2k+1)\pi}\cos\big((2k+1)\frac{\pi}{2}t\big).$

Remark: If f is T-periodic, then also its derivative f' is T-periodic, but this is not true for its antiderivative $F(t)=\int\limits_0^t f(u)\,du$. This one is T-periodic if $\int\limits_0^T f(u)\,du=0$, i.e. $a_0=0$.

Theorem.
Let f be a T-periodic function that is piecewise continuous on $[0,T)$ and it has a piecewise continuous derivative on $[0,T)$. Let $f\sim\frac{a_0}{2}+\sum\limits_{k=1}^{\infty}\big[a_k\cos(k\omega t)+b_k\sin(k\omega t)\big]$.

(i) $f'\sim\frac{a_0}{2}+\sum\limits_{k=1}^{\infty}\big[-a_k\sin(k\omega t)k\omega+b_k\cos(k\omega t)k\omega\big]=\frac{a_0}{2}+\sum\limits_{k=1}^{\infty}\big[b_k k\omega\cos(k\omega t)-a_k k\omega\sin(k\omega t)\big]$.

(ii) If $\int\limits_0^T f(u)\,du=0$ (that is, $a_0=0$), then

$$F(t)=\int\limits_0^t f(u)\,du\sim\sum\limits_{k=1}^{\infty}\big[a_k\sin(k\omega t)\tfrac{1}{k\omega}-b_k\cos(k\omega t)\tfrac{1}{k\omega}\big]=\sum\limits_{k=1}^{\infty}\big[\tfrac{-b_k}{k\omega}\cos(k\omega t)+\tfrac{a_k}{k\omega}\sin(k\omega t)\big].$$

Remark: If we know that Fourier series converges to f on some interval $[a,b]$, then we unfortunately cannot claim that the convergence is uniform there. At the ends of the interval we have the so-called Gibbs problem.

Definition.
Let $f\sim\frac{a_0}{2}+\sum\limits_{k=1}^{\infty}\big[a_k\cos(k\omega t)+b_k\sin(k\omega t)\big]$. Denote $A_k=\sqrt{a_k^2+b_k^2}$, find φ_k so that

$b_k=A_k\cos(\varphi_k)$ and $a_k=A_k\sin(\varphi_k)$. Then $f\sim\frac{a_0}{2}+\sum\limits_{k=1}^{\infty}A_k\sin(k\omega t+\varphi_k)$.
This series is called the **Fourier series in amplitude-phase form**.
Denote $c_0=\frac{a_0}{2}$ a $c_k=\frac{1}{2}(a_k-j\,b_k)$, $c_{-k}=\frac{1}{2}(a_k+j\,b_k)$ for $k\in\mathbb{N}$. Then $f\sim\sum\limits_{k=-\infty}^{\infty}c_k e^{jk\omega t}$.
This series is called the **Fourier series in complex form**.

Remark $\varphi_k=\operatorname{arctg}\big(\frac{a_k}{b_k}\big)$, or $\varphi_k=\operatorname{arccotg}\big(\frac{b_k}{a_k}\big)$, or some shifts, see transformation of Cartesian coordinates to polar.

Fact.
We have $c_k=\frac{1}{T}\int\limits_0^T f(t)e^{-jk\omega t}dt$.

4. Application of series

Example.
$y'' + y = \begin{cases} 1, & t \in [2k, 2k+1); \\ 0, & t \in [2k-1, 2k). \end{cases}$
Expand the right hand-side $f = \frac{1}{2} + \sum\limits_{k=1}^{\infty} \frac{1-(-1)^k}{k\pi} \sin(k\pi t)$.

We assume that a solution can be found of the form $y = \frac{a_0}{2} + \sum\limits_{k=1}^{\infty} [a_k \cos(k\pi t) + b_k \sin(k\pi t)]$.
We substitute into the equation and obtain
$\frac{a_0}{2} + \sum\limits_{k=1}^{\infty} [a_k(1-k^2\pi^2)\cos(k\pi t) + b_k(1-k^2\pi^2)\sin(k\pi t)] = \frac{1}{2} + \sum\limits_{k=1}^{\infty} \frac{1-(-1)^k}{k\pi} \sin(k\pi t)$.
Comparing both sides we get $a_0 = 1$, $a_k = 0$ for $k \geq 1$ and $b_k = \frac{1-(-1)^k}{k\pi(1-k^2\pi^2)}$
and thus $y = \frac{1}{2} + \sum\limits_{k=1}^{\infty} \frac{1-(-1)^k}{k\pi(1-k^2\pi^2)} \sin(k\pi t)] = \frac{1}{2} + \sum\limits_{k=0}^{\infty} \frac{2}{(2k+1)\pi(1-(2k+1)^2\pi^2)} \sin((2k+1)\pi t)]$.

Example.
$y'' - x^3 y = 24x^2$ around $x_0 = 0$.
On the right we have a power series with $x_0 = 0$, we will try to find a solution of the form $y = \sum\limits_{k=0}^{\infty} a_k x^k$.

We substitute into the equation and get $\sum\limits_{k=2}^{\infty} a_k k(k-1)x^{k-2} - \sum\limits_{k=0}^{\infty} a_k x^{k+3} = 24x^2$, hence

$2a_2 + 6a_3x_12a_4x^2 + \sum\limits_{k=3}^{\infty} [a_{k+2}(k+1)(k+2) - a_{k-3}]x^k = 24x^2$.
Comparing both sides we get $a_2 = 0$, $a_3 = 0$, $a_4 = 2$, and equations $3 \cdot 5a_5 - a_0 = 0$, $5 \cdot 6a_6 - a_1 = 0$, $6 \cdot 7a_7 - a_2 = 0$, $7 \cdot 8a_8 - a_3 = 0$, $8 \cdot 9a_9 - a_4 = 0$, $9 \cdot 10a_{10} - a_5 = 0$, $10 \cdot 11a_{11} - a_6 = 0$, etc.
We choose $a_0 = a$, $a_1 = b$, then $a_5 = \frac{a}{20}$, $a_6 = \frac{b}{30}$, $a_7 = a_8 = 0$, $a_9 = \frac{1}{36}$, $a_{10} = \frac{a}{1800}$, $a_{11} = \frac{b}{3300}$, $a_{12} = a_{13} = 0$, etc.
and thus $y(x) = a + bx + 2x^4 + \frac{a}{20}x^5 + \frac{b}{30}x^6 + \frac{1}{36}x^9 + \frac{a}{1800}x^{10} + \frac{b}{3300}x^{11} + \dots$

Example.
$\lim\limits_{x\to 0}\left(\frac{1-\cos(x)}{x^2}\right) = \lim\limits_{x\to 0}\left(\frac{1-[1-\frac{x^2}{2}+\frac{x^4}{4!}-\dots]}{x^2}\right) = \lim\limits_{x\to 0}\left(\frac{\frac{x^2}{2}-\frac{x^4}{4!}+\dots}{x^2}\right) = \lim\limits_{x\to 0}\left(\frac{1}{2} - \frac{x^2}{4!} + \dots\right) = \frac{1}{2}$.

Example.
$\int \frac{1}{x}\sin(x)\,dx = \int \frac{1}{x} \sum\limits_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}\,dx = \sum\limits_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \int x^{2k}\,dx = \sum\limits_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \frac{x^{2k+1}}{2k+1} + C$
$= \sum\limits_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!(2k+1)} x^{2k+1} + C.$