Laplace transform

Definition.
For $f\colon [0,\infty)\mapsto\mathbb{R}$ we define its **Laplace transform** $\mathcal{L}\{f(t)\}$ by

$$\mathcal{L}\{f(t)\}\colon p\mapsto \int_0^\infty f(t)e^{-pt}dt,$$

assuming that the integral converges for at least one p.
Notation: $\mathcal{L}\{f(t)\}$, $\mathcal{L}\{f\}$, F, alternative $f(t) \mathrel{\widehat{=}} F(p)$.

Example.
$\mathcal{L}$ of $f(t)=e^{\alpha t}$ for $t\geq 0$ is the function F given by formula $F(p)=\int_0^\infty e^{\alpha t}e^{-pt}dt=\dfrac{1}{p-\alpha}$ for $p>\alpha$.

If we want to apply $\mathcal{L}$ to functions f defined on a larger set, for instance on $\mathbb{R}$, then we will consider them equal to zero for $t<0$.

Definition.
Heaviside function is defined $H(t)=\begin{cases}1, & t\geq 0;\\ 0, & t<0.\end{cases}$

Fact.
Let f be a function on $\mathbb{R}$, $a\in\mathbb{R}$. Then $f(t)H(t-a)=\begin{cases}f(t), & t\geq a;\\ 0, & t<a.\end{cases}$

Notation: If f is a function defined by a formula and we write $\mathcal{L}\{f(t)\}$, then by this we automatically understand $\mathcal{L}\{f(t)H(t)\}$.

Example.
The previous example can be written as $\mathcal{L}\{e^{\alpha t}\}=\frac{1}{p-\alpha}$ or for instance $e^{\alpha t} \mathrel{\widehat{=}} \frac{1}{p-\alpha}$.

Example.
$\mathcal{L}\{e^{t^2}\}=\mathcal{L}\{e^{t^2}H(t)\}$ DNE.

Definition.
We say that a function f is **piecewise continuous** on an interval I if there are $x_0<x_1<\ldots\in\overline{I}$ such that $\{x_k\}$ is either finite or a sequence going to infinity as $k\to\infty$, $\overline{I}=\bigcup[x_{k-1},x_k]$ and for every $k=1,2,\dots$ the function f is continuous on (x_{k-1},x_k) and it has one-sided limits $f({x_{k-1}}^+)$, $f({x_k}^-)$.
We say that a function f is of **at most exponential growth** if $\exists\alpha,M>0$ such that $\forall t$: $|f(t)|\leq Me^{\alpha t}$.

Definition.
We define the space $\mathcal{L}_0$ by
$\mathcal{L}_0=\{f\colon [0,\infty)\mapsto\mathbb{R};\ f$ is of at most exponential growth and piecewise continuous on $[0,\infty)\}$.

Theorem.
If $f\in\mathcal{L}_0$ then $\mathcal{L}\{f\}$ exists on some (p_f,∞).
Moreover, $\lim\limits_{p\to\infty}\big(\mathcal{L}\{f\}(p)\big)=0$.

The space $\mathcal{L}_0$ contains for instance $e^{\alpha t}$, t^n for $n \geq 0$ and all (piecewise) continuous functions are there as well. For most functions we find their Laplace transform algoritmically.

1. Calculating Laplace transform

Theorem. (**dictionary**)
(i) $\forall\alpha\in\mathbb{R}$: $e^{\alpha t}\in\mathcal{L}_0$ and $\mathcal{L}\{e^{\alpha t}\}=\frac{1}{p-\alpha}$, $p>\alpha$;
(ii) $\forall n\in\mathbb{N}_0$: $t^n\in\mathcal{L}_0$ and $\mathcal{L}\{t^n\}=\frac{n!}{p^{n+1}}$, $p>0$;
(iii) $\forall\omega\in\mathbb{R}$: $\sin(\omega t)\in\mathcal{L}_0$ and $\mathcal{L}\{\sin(\omega t)\}=\frac{\omega}{p^2+\omega^2}$, $p\in\mathbb{R}$;
(iv) $\forall\omega\in\mathbb{R}$: $\cos(\omega t)\in\mathcal{L}_0$ and $\mathcal{L}\{\cos(\omega t)\}=\frac{p}{p^2+\omega^2}$, $p\in\mathbb{R}$.

Theorem. (**linearity**)
Let $f,g\in\mathcal{L}_0$. Then $\forall a,b\in\mathbb{R}$: $af+bg\in\mathcal{L}_0$ and $\mathcal{L}\{af(t)+bg(t)\}=a\mathcal{L}\{f(t)\}+b\mathcal{L}\{g(t)\}$.

Theorem. (**grammar**)
Let $f\in\mathcal{L}_0$. Then the following are true:
(i) (change of scale) $\forall a>0$: $f(at)\in\mathcal{L}_0$ and
$$\mathcal{L}\{f(at)\}=\tfrac{1}{a}\mathcal{L}\{f(t)\}\big|_{p/a};$$
(ii) (shift in image) $\forall a\in\mathbb{R}$: $e^{at}f(t)\in\mathcal{L}_0$ and
$$\mathcal{L}\{e^{at}f(t)\}=\mathcal{L}\{f(t)\}\big|_{p-a};$$
(iii) (shift in preimage) $\forall a>0$: $f(t-a)H(t-a)\in\mathcal{L}_0$ and
$$\mathcal{L}\{f(t-a)H(t-a)\}=e^{-ap}\mathcal{L}\{f(t)H(t)\};$$
(iv) (derivative of image) $\forall n\in\mathbb{N}$: $t^nf(t)\in\mathcal{L}_0$ and
$$\mathcal{L}\{t^nf(t)\}=(-1)^n\tfrac{\mathrm{d}^n}{\mathrm{d}p^n}\mathcal{L}\{f(t)\};$$
(v) (integration of image) If $\lim\limits_{t\to0^+}\big(\frac{f(t)}{t}\big)$ converges, then $\frac{f(t)}{t}\in\mathcal{L}_0$ and
$$\mathcal{L}\big\{\tfrac{1}{t}f(t)\big\}=\int_p^\infty\mathcal{L}\{f(t)\}(q)\,dq.$$
(vi) (derivative of preimage) If $f^{(n)}\in\mathcal{L}_0$, then
$$\mathcal{L}\{f^{(n)}(t)\}=p^n\mathcal{L}\{f(t)\}(p)-p^{n-1}f(0^+)-p^{n-2}f'(0^+)-\ldots-pf^{(n-2)}(0^+)-f^{(n-1)}(0^+);$$
(vii) (integration of preimage) $\int_0^t f(s)\,ds\in\mathcal{L}_0$ and
$$\mathcal{L}\big\{\textstyle\int_0^t f(s)\,ds\big\}=\tfrac{1}{p}\mathcal{L}\{f(t)\}.$$

Remark: Instead of (iii) we usually prefer $\mathcal{L}\{f(t)H(t-a)\}=e^{-ap}\mathcal{L}\{f(t+a)H(t)\}$.

Example.
$\mathcal{L}\{t\,e^{3t}\}=-[\mathcal{L}\{e^{3t}\}]'=-\big[\frac{1}{p-3}\big]'=\frac{1}{(p-3)^2}$.
$\mathcal{L}\{t\,e^{3t}\}=\mathcal{L}\{e^{3t}t\}=\mathcal{L}\{t\}\big|_{p-3}=\frac{1}{p^2}\big|_{p-3}=\frac{1}{(p-3)^2}$.
$\mathcal{L}\big\{\frac{\sin(t)}{t}\big\}=\int_p^\infty\mathcal{L}\{\sin(t)\}(q)\,dq=\int_p^\infty\frac{1}{q^2+1}\,dq=\big[\arctan(q)\big]_p^\infty=\frac{\pi}{2}-\arctan(p)$.
Remark: $\frac{\cos(t)}{t}\notin\mathcal{L}_0$.
$\mathcal{L}\big\{\sin(2t)H\big(t-\frac{\pi}{2}\big)\big\}=e^{-\frac{\pi}{2}p}\mathcal{L}\big\{\sin\big(2\big(t+\frac{\pi}{2}\big)\big)\big\}=e^{-\frac{\pi}{2}p}\mathcal{L}\{\sin(2t+\pi)\}$
$\quad=e^{-\frac{\pi}{2}p}\mathcal{L}\{-\sin(2t)\}=-\frac{2e^{-\frac{\pi}{2}p}}{p^2+4}$.

Definition.
By a **finite impuls** we mean any function defined on $[0,\infty)$ that is non-zero only on some bounded closed interval.

Definition.
Let M be a subset of $\mathbb{R}$. We define its **characteristic function** $\chi_M = \begin{cases} 1, & x \in M; \\ 0, & x \notin M. \end{cases}$

Fact.
Let M be a subset of $\mathbb{R}$, f a function on $\mathbb{R}$. Then $f(t)\chi_M = \begin{cases} f(t), & t \in M; \\ 0, & t \notin M. \end{cases}$

Fact.
Let $a < b \in \mathbb{R}$. Then $\chi_{[a,b)} = H(t-a) - H(t-b)$.

Example.
Laplace transform of one hill of sine of $2t$:
$\mathcal{L}\bigl\{\sin(2t)\bigl[H(t) - H\bigl(t - \frac{\pi}{2}\bigr)\bigr]\bigr\} = \mathcal{L}\{\sin(2t)\} - \mathcal{L}\bigl\{\sin(2t)H\bigl(t - \frac{\pi}{2}\bigr)\bigr]\bigr\} = \frac{2}{p^2+4} + \frac{2e^{-\frac{\pi}{2}p}}{p^2+4}$.

Theorem. (on **periodic** function)
Let f be a function that is T-periodic on $[0,\infty)$. We mark one period by $f_T = f \cdot \chi_{[0,T)}$. Then $\mathcal{L}\{f(t)\} = \dfrac{\mathcal{L}\{f_T(t)\}}{1 - e^{-pT}}$.

Example.

$$\mathcal{L}\{|\sin(2t)|\} = \frac{\mathcal{L}\{f_T(t)\}}{1-e^{-p\pi}} = \frac{\mathcal{L}\{\sin(2t)[H(t)-H(t-\frac{\pi}{2})]\}}{1-e^{-\pi p}} = \frac{\frac{2}{p^2+4} + \frac{2e^{-\frac{\pi}{2}p}}{p^2+4}}{1-e^{-\pi p}} = \frac{2}{p^2+4}\,\frac{1+e^{-\frac{\pi}{2}p}}{1-e^{-\pi p}}$$
$$= \frac{2}{p^2+4}\,\frac{1+e^{-\frac{\pi}{2}p}}{(1-e^{-\frac{\pi}{2}p})(1+e^{-\frac{\pi}{2}p})} = \frac{2}{p^2+4}\,\frac{1}{1-e^{-\frac{\pi}{2}p}}.$$

2. Inverse Laplace transform

There is a problem with Laplace transform not being one-to-one.

Theorem.
If $f, g \in \mathcal{L}_0$ have $\mathcal{L}\{f\} = \mathcal{L}\{g\}$ on some $[p_0,\infty)$, then $f = g$ with exception of a countable set of isolated points.
If moreover f and g are continuous from the right everywhere, then $f = g$.

Corollary.
Consider the linear space $V = \{f \in \mathcal{L}_0;\ f \text{ continuous from the right on } \mathbb{R}_0^+\}$. On this space the Laplace transform is one-to-one, therefore we can consider its inverse $\mathcal{L}^{-1}$.

Theorem. (**dictionary** for $\mathcal{L}^{-1}$)
$\mathcal{L}^{-1}\bigl\{\frac{1}{p-\alpha}\bigr\} = e^{\alpha t}$, $\mathcal{L}^{-1}\bigl\{\frac{1}{p^n}\bigr\} = \frac{1}{(n-1)!}t^{n-1}$, $\mathcal{L}^{-1}\bigl\{\frac{\omega}{p^2+\omega^2}\bigr\} = \sin(\omega t)$, $\mathcal{L}^{-1}\bigl\{\frac{p}{p^2+\omega^2}\bigr\} = \cos(\omega t)$.

Theorem. (**grammar** for $\mathcal{L}^{-1}$)
(0) $\mathcal{L}^{-1}$ is linear;
(1) $\mathcal{L}^{-1}\{e^{-ap}F(p)\} = \mathcal{L}^{-1}\{F(p)\}\big|_{t-a} \cdot H(t-a)$;
(2) $\mathcal{L}^{-1}\{F(p-a)\} = e^{at}\mathcal{L}^{-1}\{F(p)\}$;
(3) $\mathcal{L}^{-1}\{F(ap)\} = \frac{1}{a}\mathcal{L}^{-1}\{F(p)\}\big|_{t/a}$;
(4) $\mathcal{L}^{-1}\{F'(p)\} = -t\mathcal{L}^{-1}\{F(p)\}$;
(5) $\mathcal{L}^{-1}\{pF(p)\} = \bigl[\mathcal{L}^{-1}\{F(p)\}\bigr]' + \mathcal{L}^{-1}\{F(p)\}(0^+)$.

Example.

$$\mathcal{L}^{-1}\left\{\tfrac{pe^{-\pi p}}{p^2+1}\right\} = \mathcal{L}^{-1}\left\{\tfrac{p}{p^2+1}\right\}\Big|_{t-\pi} H(t-\pi) = \cos(t)\Big|_{t-\pi} H(t-\pi) = \cos(t-\pi)H(t-\pi)$$
$$= -\cos(t)H(t-\pi) = \begin{cases} 0, & t \in [0,\pi); \\ -\cos(t), & t \geq \pi. \end{cases}$$

Theorem.
If $F(p)$ is a proper rational function, then $\mathcal{L}^{-1}\{F(p)\}$ exists and it can be found using partial fractions decomposition.

3. Laplace transform and differential equations

Solving differential equations (Cauchy problems) using LT; Laplace the equation, solve the resulting algebraic equation, unlaplace it.

Example.

$$\ddot{x} - x = \begin{cases} 2, & t \in [0,1); \\ 0, & \text{elsewhere} \end{cases} = 2\chi_{[0,1)}, \qquad x(0^+) = \dot{x}(0^+) = 0.$$

We denote $\mathcal{L}\{x\} = X$, then $[p^2X - 0p - 0] - X = \mathcal{L}\{2[H(t) - H(t-1)]\}$, $(p^2-1)X = \frac{2}{p} - e^{-p}\frac{2}{p}$,
so $X(p) = \frac{2}{(p^2-1)p} - e^{-p}\frac{2}{(p^2-1)p} = \left(\frac{1}{p-1} + \frac{1}{p+1} - \frac{2}{p}\right) - e^{-p}\left(\frac{1}{p-1} + \frac{1}{p+1} - \frac{2}{p}\right)$,
hence $x(t) = e^t + e^{-t} - 2 - (e^t + e^{-t} - 2)\big|_{t-1} H(t-1) = e^t + e^{-t} - 2 - (e^{t-1} + e^{1-t} - 2)H(t-1)$

$$= \begin{cases} e^t + e^{-t} - 2, & t \in [0,1); \\ e^t(1-e^{-1}) + e^{-t}(1-e), & t \geq 1 \end{cases}$$

or $x(t) = 2\cosh(t) - 2 - (2\cosh(t-1) - 2)H(t-1) = \begin{cases} 2\cosh(t) - 2, & t \in [0,1); \\ 2\cosh(t) - 2\cosh(t-1), & t \geq 1. \end{cases}$

Finding a general solution using LT: Two possibilities.
1) Choose null initial conditions, find one particular solution using LT, then add to it a general homogeneous solution (most likely found via characteristic numbers).
2) Choose general initial conditions $y(0^+) = a$ etc., solve the problem using LT, we get a solution with parameters, that is, a general one.

Example.

General solution of $\dot{x} + 9\int\limits_0^t x(u)\,du = 0$.
Choice $x(0^+) = a$, then $pX - a + 9\frac{1}{p}X = 0$, $X(p) = \frac{ap}{p^2+9}$, $x(t) = a\cos(3t)$, $t \geq 0$.

Using LT one can also solve systems of equations.

Example.

$$\begin{matrix} y_1' = 2y_1 + y_2 \\ y_2' = y_1 + 2y_2 \end{matrix}, \qquad y_1(0) = 1,\ y_2(0) = 1.$$

Denote $\begin{matrix} \mathcal{L}\{y_1\} = Y_1 \\ \mathcal{L}\{y_1\} = Y_1 \end{matrix}$, then $\begin{matrix} pY_1 - 1 = 2Y_1 + Y_2 \\ pY_2 - 1 = Y_1 + 2Y_2 \end{matrix}$, hence $\begin{matrix} (p-2)Y_1 - Y_2 = 1 \\ -Y_1 + (p-2)Y_2 = 1 \end{matrix}$, from this (by elimination or Cramer) $Y_1(p) = \frac{1}{p-3}$, $Y_2(p) = \frac{1}{p-3}$, thus $y_1(x) = y_2(x) = e^{3x}$, $x \in \mathbb{R}$.

Definition.
Let f,g be functions defined on $\mathbb{R}$. We define their **convolution** as the function $f*g$ on $\mathbb{R}$ given by $(f*g)(t)=\int\limits_{-\infty}^{\infty} f(t-s)g(s)\,ds=\int\limits_{-\infty}^{\infty} f(s)g(t-s)\,ds$.

If f,g are zero on $(-\infty,0)$, for instance if $f,g\in\mathcal{L}_0$, then $(f*g)(t)=\int\limits_0^t f(t-s)g(s)\,ds$.

Fact.
$f*g=g*f$, $f*(g*h)=(f*g)*h$, $a(f*g)=(af)*g$, $f*(g+h)=f*g+f*h$.

Theorem.
Let $f,g\in\mathcal{L}_0$. Then $f*g\in\mathcal{L}_0$ and $\mathcal{L}\{f*g\}=\mathcal{L}\{f\}\cdot\mathcal{L}\{g\}$.
From this $\mathcal{L}^{-1}\{F\cdot G\}=\mathcal{L}^{-1}\{F\}*\mathcal{L}^{-1}\{G\}$.

Example.
$y'+\int\limits_0^t \cosh(t-u)y(u)\,du=e^{-t}, \qquad y(0^+)=0.$
It is $y'(t)+\cosh(t)*y(t)=e^{-t}$, hence $pY-0+\mathcal{L}\{\cosh(t)\}\cdot Y=\frac{1}{p+1}$, here
$\mathcal{L}\{\cosh(t)\}=\frac{1}{2}\mathcal{L}\{e^t\}+\frac{1}{2}\mathcal{L}\{e^{-t}\}=\frac{1}{2}\big(\frac{1}{p-1}+\frac{1}{p+1}\big)=\frac{p}{p^2-1}$, thus
$pY+\frac{p}{p^2-1}Y=\frac{1}{p+1}$, $p^3Y=p-1$, $Y(p)=\frac{1}{p^2}-\frac{1}{p^3}$, therefore $y(t)=t-\frac{1}{2}t^2$, $t\geq 0$.