Solved problems on Laplace transform

1. Using Laplace transform solve the equation

$$y'' + 3y' + 2y = 2e^{-3x}\chi_{[1,2)} = \begin{cases} 2e^{-3x}, & x \in [1,2); \\ 0, & \text{elsewhere}, \end{cases} \qquad y(0^+) = 0,\ y'(0^+) = 1.$$

2. Using Laplace transform solve the equation

$$y' + y = (2t+3)\chi_{[0,1)} = \begin{cases} 2t+3, & t \in [0,1); \\ 0, & \text{elsewhere}, \end{cases} \qquad y(0^+) = 1.$$

3. Using Laplace transform solve the equation

$$\dot{x} + 2\int_0^t x(u)\,du = 2t + 3, \qquad x(0^+) = 0.$$

4. Find the Laplace transform of the function

$$f(t) = t\,e^{-3t}\sin(2t) + \frac{\mathrm{d}}{\mathrm{dt}}\big[1 + \sinh(t)\big] + \int_0^t \big(u^2 + \sin(u)e^{3(t-u)}\big)\,du.$$

5. Find an inverse Laplace transform of the function

$$F(p) = \frac{p + 1 - e^{-\pi p}}{p^2 + 4p + 13}.$$

Solutions

1. We transform the equation denoting $\mathcal{L}\{y\} = Y$, the right hand-side is a finite signal, we handle it using Heaviside function:

$$\begin{aligned} p^2Y - 0p - 1 + 3(pY - 0) + 2Y &= \mathcal{L}\{2e^{-3x}[H(x-1) - H(x-2)]\} \\ &= e^{-p}\mathcal{L}\{2e^{-3(x+1)}\} - e^{-2p}\mathcal{L}\{2e^{-3(x+2)}\} = e^{-p}2e^{-3}\mathcal{L}\{e^{-3x}\} - e^{-2p}2e^{-6}\mathcal{L}\{e^{-3x}\} \\ &= e^{-p}2e^{-3}\tfrac{1}{p+3} - e^{-2p}2e^{-6}\tfrac{1}{p+3}. \end{aligned}$$

Thus $Y(p^2 + 3p + 2) = \frac{2e^{-3}}{p+3}e^{-p} - \frac{2e^{-6}}{p+3}e^{-2p} + 1$, hence

$$Y = \tfrac{2e^{-3}}{(p+1)(p+2)(p+3)}e^{-p} - \tfrac{2e^{-6}}{(p+1)(p+2)(p+3)}e^{-2p} + \tfrac{1}{(p+1)(p+2)}.$$

Partial fractions yield

$$Y(p) = \tfrac{1}{p+1} - \tfrac{1}{p+2} + \big(\tfrac{e^{-3}}{p+1} - \tfrac{2e^{-3}}{p+2} + \tfrac{e^{-3}}{p+3}\big)e^{-p} - \big(\tfrac{e^{-6}}{p+1} - \tfrac{2e^{-6}}{p+2} + \tfrac{e^{-6}}{p+3}\big)e^{-2p},$$

so

$$\begin{aligned} y(x) = \mathcal{L}^{-1}\{Y(p)\} &= e^{-x} - e^{-2x} + \mathcal{L}^{-1}\big\{\tfrac{e^{-3}}{p+1} - \tfrac{2e^{-3}}{p+2} + \tfrac{e^{-3}}{p+3}\big\}\big|_{x-1}H(x-1) \\ &\quad - \mathcal{L}^{-1}\big\{\tfrac{e^{-6}}{p+1} - \tfrac{2e^{-6}}{p+2} + \tfrac{e^{-6}}{p+3}\big\}\big|_{x-2}H(x-2) \\ &= e^{-x} - e^{-2x} + (e^{-3}e^{-x} - 2e^{-3}e^{-2x} + e^{-3}e^{-3x})\big|_{x-1}H(x-1) \end{aligned}$$

$$
\begin{aligned}
&\quad - (e^{-6}e^{-x} - 2e^{-6}e^{-2x} + e^{-6}e^{-3x})\big|_{x-2} H(x-2)\\
&= e^{-x} - e^{-2x} + (e^{-3}e^{1-x} - 2e^{-3}e^{2-2x} + e^{-3}e^{3-3x})H(x-1)\\
&\quad - (e^{-6}e^{2-x} - 2e^{-6}e^{4-2x} + e^{-6}e^{6-3x})H(x-2)\\
&= \begin{cases} e^{-x} - e^{-2x}, & x \in [0,1),\\ e^{-x} - e^{-2x} + (e^{-2}e^{-x} - 2e^{-1}e^{-2x} + e^{-3x}) \\ \qquad = (1+e^{-2})e^{-x} - (1+2e^{-1})e^{-2x} + e^{-3x}, & x \in [1,2),\\ (1+e^{-2})e^{-x} - (1+2e^{-1})e^{-2x} + e^{-3x} - (e^{-4}e^{-x} - 2e^{-2}e^{-2x} + e^{-3x}) \\ \qquad = (1+e^{-2}-e^{-4})e^{-x} - (1+2e^{-1}-2e^{-2})e^{-2x}, & x \geq 2. \end{cases}
\end{aligned}
$$

Remark: It is also easy to transform the right hand-side by definition:

$$
\begin{aligned}
\mathcal{L}\{2e^{-3x}\chi_{[1,2)}\} &= \int_1^2 2e^{-3x}e^{-px}\,dx = 2\int_1^2 e^{-(p+3)x}\,dx = 2\Big[\frac{e^{-(p+3)x}}{-(p+3)}\Big]_{x=1}^{x=2}\\
&= \frac{2e^{-(p+3)}}{p+3} - \frac{2e^{-2(p+3)}}{p+3} = \frac{2e^{-3}e^{-p}}{p+3} - \frac{2e^{-6}e^{-2p}}{p+3}.
\end{aligned}
$$

2. We transform the equation denoting $\mathcal{L}\{y\} = Y$, the right hand-side is a finite signal, we handle it using Heaviside function:

$$
\begin{aligned}
pY - 1 + Y &= \mathcal{L}\{(2t+3)[H(t) - H(t-1)]\} = \tfrac{2}{p^2} + \tfrac{3}{p} - e^{-p}\mathcal{L}\{2(t+1)+3\}\\
&= \tfrac{2}{p^2} + \tfrac{3}{p} - e^{-p}\mathcal{L}\{2t+5\} = \tfrac{2}{p^2} + \tfrac{3}{p} - \big(\tfrac{2}{p^2} + \tfrac{5}{p}\big)e^{-p}.
\end{aligned}
$$

Thus $Y(p+1) = 1 + \frac{2}{p^2} + \frac{3}{p} - \big(\frac{2}{p^2} + \frac{5}{p}\big)e^{-p}$, hence $Y = \frac{1}{p+1} + \frac{2}{p^2(p+1)} + \frac{3}{p(p+1)} - \big(\frac{2}{p^2(p+1)} + \frac{5}{p(p+1)}\big)e^{-p}$. Partial fractions yield

$$
Y(p) = \tfrac{1}{p+1} + \big(\tfrac{2}{p+1} - \tfrac{2}{p} + \tfrac{2}{p^2}\big) + \big(\tfrac{3}{p} - \tfrac{3}{p+1}\big) - \big(\tfrac{2}{p+1} - \tfrac{2}{p} + \tfrac{2}{p^2} + \tfrac{5}{p} - \tfrac{5}{p+1}\big)e^{-p} = \tfrac{1}{p} + \tfrac{2}{p^2} - \big(\tfrac{3}{p} + \tfrac{2}{p^2} - \tfrac{3}{p+1}\big)e^{-p},
$$

so

$$
\begin{aligned}
y(t) &= \mathcal{L}^{-1}\{Y(p)\} = 1 + 2t - \mathcal{L}^{-1}\big\{\tfrac{3}{p} + \tfrac{2}{p^2} - \tfrac{3}{p+1}\big\}\big|_{t-1} H(t-1)\\
&= 1 + 2t - (3 + 2t - 3e^{-t})\big|_{t-1} H(t-1) = 1 + 2t - (3 + 2(t-1) - 3e^{1-t})H(t-1)\\
&= 1 + 2t - (1 + 2t - 3e^{1-t})H(t-1) = \begin{cases} 1+2t, & t \in [0,1),\\ 3e^{1-t}, & t \geq 1. \end{cases}
\end{aligned}
$$

3. We transform the equation denoting $\mathcal{L}\{x\} = X$:

$$
pX + \tfrac{2}{p}X = \tfrac{2}{p^2} + \tfrac{3}{p}.
$$

Thus

$$
p^2X + 2X = \tfrac{2+3p}{p} \implies (p^2+2)X = \tfrac{2+3p}{p},
$$

hence $X = \frac{2+3p}{(p^2+2)p}$. Partial fractions yield

$$
X(p) = \tfrac{1}{p} - \tfrac{p}{p^2+2} + \tfrac{3}{p^2+2},
$$

so

$$
\begin{aligned}
x(t) &= \mathcal{L}^{-1}\{X(p)\} = 1 + \mathcal{L}^{-1}\big\{-\tfrac{p}{p^2+(\sqrt{2})^2} + \tfrac{3}{\sqrt{2}}\,\tfrac{\sqrt{2}}{p^2+(\sqrt{2})^2}\big\}\\
&= 1 - \cos(\sqrt{2}t) + \tfrac{3}{\sqrt{2}}\sin(\sqrt{2}t).
\end{aligned}
$$

4.

$$\begin{aligned}
\mathcal{L}\{f(t)\} &= \mathcal{L}\{t\,e^{-3t}\sin(2t)\} + \mathcal{L}\bigl\{\tfrac{\mathrm{d}}{\mathrm{d}t}\bigl[1+\sinh(t)\bigr]\bigr\} + \mathcal{L}\Bigl\{\int_0^t u^2 du\Bigr\} + \mathcal{L}\Bigl\{\int_0^t \sin(u)e^{3(t-u)}du\Bigr\}\\
&= -\tfrac{\mathrm{d}}{\mathrm{d}p}\mathcal{L}\{e^{-3t}\sin(2t)\} + p\mathcal{L}\{1+\sinh(t)\} - [1+\sinh(t)]\big|_{t=0+} + \tfrac{1}{p}\mathcal{L}\{u^2\} + \mathcal{L}\{\sin(t)*e^{3t}\}\\
&= -\tfrac{\mathrm{d}}{\mathrm{d}p}\bigl(\mathcal{L}\{\sin(2t)\}\big|_{p+3}\bigr) + p\mathcal{L}\bigl\{1+\tfrac{1}{2}(e^t-e^{-t})\bigr\} - [1+\sinh(0)] + \tfrac{1}{p}\,\tfrac{2}{p^3} + \mathcal{L}\{\sin(t)\}\cdot\mathcal{L}\{e^{3t}\}\\
&= -\tfrac{\mathrm{d}}{\mathrm{d}p}\bigl(\tfrac{2}{p^2+4}\big|_{p+3}\bigr) + p\bigl(\tfrac{1}{p}+\tfrac{1}{2}\,\tfrac{1}{p-1}-\tfrac{1}{2}\,\tfrac{1}{p+1}\bigr) - 1 + \tfrac{2}{p^4} + \tfrac{1}{p^2+1}\,\tfrac{1}{p-3}\\
&= -\tfrac{\mathrm{d}}{\mathrm{d}p}\bigl(\tfrac{2}{(p+3)^2+4}\bigr) + 1 + \tfrac{p/2}{p-1} - \tfrac{p/2}{p+1} - 1 + \tfrac{2}{p^4} + \tfrac{1}{(p^2+1)(p-3)}\\
&= \tfrac{4(p+3)}{[(p+3)^2+4]^2} + \tfrac{\frac{1}{2}p}{p-1} - \tfrac{\frac{1}{2}p}{p+1} + \tfrac{2}{p^4} + \tfrac{1}{(p^2+1)(p-3)}.
\end{aligned}$$

5.

$$\begin{aligned}
\mathcal{L}^{-1}\{F(p)\} &= \mathcal{L}^{-1}\bigl\{\tfrac{p+1-e^{-\pi p}}{p^2+4p+13}\bigr\} = \mathcal{L}^{-1}\bigl\{\tfrac{p+1}{(p+2)^2+9}\bigr\} - \mathcal{L}^{-1}\bigl\{\tfrac{1}{(p+2)^2+9}e^{-\pi p}\bigr\}\\
&= \mathcal{L}^{-1}\bigl\{\tfrac{p+2-1}{(p+2)^2+9}\bigr\} - \mathcal{L}^{-1}\bigl\{\tfrac{1}{(p+2)^2+9}\bigr\}\big|_{t-\pi}H(t-\pi)\\
&= e^{-2t}\mathcal{L}^{-1}\bigl\{\tfrac{p-1}{p^2+9}\bigr\} - \bigl(e^{-2t}\mathcal{L}^{-1}\bigl\{\tfrac{1}{p^2+9}\bigr\}\bigr)\big|_{t-\pi}H(t-\pi)\\
&= e^{-2t}\mathcal{L}^{-1}\bigl\{\tfrac{p}{p^2+9}-\tfrac{1}{3}\,\tfrac{3}{p^2+9}\bigr\} - \bigl(e^{-2t}\mathcal{L}^{-1}\bigl\{\tfrac{1}{3}\,\tfrac{3}{p^2+9}\bigr\}\bigr)\big|_{t-\pi}H(t-\pi)\\
&= e^{-2t}\bigl(\cos(3t)-\tfrac{1}{3}\sin(3t)\bigr) - \bigl(e^{-2t}\tfrac{1}{3}\sin(3t)\bigr)\big|_{t-\pi}H(t-\pi)\\
&= e^{-2t}\cos(3t) - \tfrac{e^{-2t}}{3}\sin(3t) - \tfrac{e^{-2(t-\pi)}}{3}\sin(3(t-\pi))H(t-\pi)\\
&= e^{-2t}\cos(3t) - \tfrac{e^{-2t}}{3}\sin(3t) - \tfrac{e^{2\pi-2t}}{3}\sin(3t-3\pi)H(t-\pi)\\
&= e^{-2t}\cos(3t) - \tfrac{e^{-2t}}{3}\sin(3t) + \tfrac{e^{2\pi-2t}}{3}\sin(3t)H(t-\pi)\\
&= \begin{cases} e^{-2t}\cos(3t)-\frac{1}{3}e^{-2t}\sin(3t), & t\in[0,\pi),\\ e^{-2t}\cos(3t)+\frac{1}{3}(e^{2\pi}-1)e^{-2t}\sin(3t), & t\geq\pi.\end{cases}
\end{aligned}$$

Remark: $\cos(3t-3\pi)$ is done using $\cos(\alpha-3\pi)=-\cos(\alpha)$ with choice $\alpha=3t$. Similarly $\cos(\alpha\pm\pi)=-\cos(\alpha)$, $\cos\bigl(\alpha\pm\frac{\pi}{2}\bigr)=\mp\sin(\alpha)$, $\sin(\alpha\pm\pi)=-\sin(\alpha)$, $\sin\bigl(\alpha\pm\frac{\pi}{2}\bigr)=\cos(\alpha)$.