Solved problems on Fourier series

1. Find the Fourier series for (periodic extension of)

$$f(t) = \begin{cases} 1, & t \in [0,2); \\ -1, & t \in [2,4). \end{cases}$$

Determine the sum of this series.

2. Find the Fourier series for (periodic extension of)

$$f(t) = \begin{cases} t-1, & t \in [0,2); \\ 3-t, & t \in [2,4). \end{cases}$$

Determine the sum of this series.

3. Find the sine Fourier series for (periodic extension of)

$$f(t) = \begin{cases} t-1, & t \in [0,2); \\ 3-t, & t \in [2,4). \end{cases}$$

Determine the sum of this series.

4. Find the cosine Fourier series for (periodic extension of)

$$f(t) = \begin{cases} 1, & t \in [0,1); \\ 0, & t \in [1,4). \end{cases}$$

Determine the sum of this series.

5. Find the Fourier series for (periodic extension of)

$$f(t) = 1 - t^2,\ t \in [-1,1).$$

Determine the sum of this series.

Solutions

1. Parameters: The period length is $T = 4$, frequency $\omega = \frac{2\pi}{T} = \frac{\pi}{2}$.

$$a_0 = \frac{2}{T}\int_0^T f(t)\,dt = \frac{1}{2}\Big(\int_0^2 1\,dt - \int_2^4 1\,dt\Big) = 0,$$

$$a_k = \frac{2}{T}\int_0^T f(t)\cos(k\omega t)\,dt = \frac{1}{2}\Big(\int_0^2 \cos\big(k\tfrac{\pi}{2}t\big)\,dt - \int_2^4 \cos\big(k\tfrac{\pi}{2}t\big)\,dt\Big)$$

$$= \frac{1}{2}\Big[\frac{2}{k\pi}\sin\big(k\tfrac{\pi}{2}t\big)\Big]_0^2 - \frac{1}{2}\Big[\frac{2}{k\pi}\sin\big(k\tfrac{\pi}{2}t\big)\Big]_2^4 = 0,$$

$$b_k = \frac{2}{T}\int_0^T f(t)\sin(k\omega t)\,dt = \frac{1}{2}\Big(\int_0^2 \sin\big(k\tfrac{\pi}{2}t\big)\,dt - \int_2^4 \sin\big(k\tfrac{\pi}{2}t\big)\,dt\Big)$$

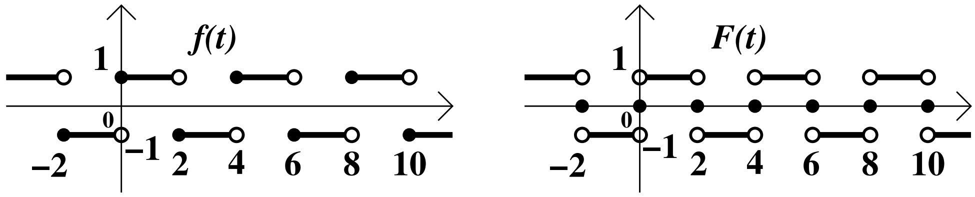
$$= \frac{1}{2}\Big[-\frac{2}{k\pi}\cos\big(k\tfrac{\pi}{2}t\big)\Big]_0^2 - \frac{1}{2}\Big[-\frac{2}{k\pi}\cos\big(k\tfrac{\pi}{2}t\big)\Big]_2^4 = \frac{1}{k\pi}\big[-\cos(k\pi) + \cos(0) + \cos(2k\pi) - \cos(k\pi)\big]$$

$$= \frac{1}{k\pi}\big[-(-1)^k + 1 + 1 - (-1)^k\big] = \frac{2}{k\pi}[1-(-1)^k] = \begin{cases} 0, & k \text{ even}, \\ \frac{4}{k\pi}, & k \text{ odd}. \end{cases}$$

Odd numbers can be expressed as $k = 2i+1$, to numbers $k = 1,3,5,7,\dots$ correspond indeces $i = 0,1,2,3,\dots$. For those we then have $a_k = \dfrac{4}{(2i+1)\pi}$. We rewrite the resulting series accordingly, and since the index k is traditional, we pass from i to k at the end.
Thus

$$f \sim \frac{a_0}{2} + \sum_{k=1}^{\infty}\big(a_k\cos(k\omega t) + b_k\sin(k\omega t)\big) = \sum_{k=1}^{\infty}\frac{2}{k\pi}[1-(-1)^k]\sin\big(k\tfrac{\pi}{2}t\big) = \sum_{k=0}^{\infty}\frac{4}{(2k+1)\pi}\sin\big((2k+1)\tfrac{\pi}{2}t\big).$$

What is the sum of this series? First we draw a periodic extension of the function f (on the left). To this we then apply the Jordan criterion. According to it, the resulting series converges to f at all points where f (or rather its periodic extension) is continuous. At points of discontinuity of f the series converges to the average $\frac{1}{2}\big(f(t^+)+f(t^-)\big)$. Result: On the right is the function to which our Fourier series converges, i.e. its sum.



2. Parameters: The period length is $T=4$, frequency $\omega=\frac{2\pi}{T}=\frac{\pi}{2}$.

$$a_0=\frac{2}{T}\int_0^T f(t)\,dt=\frac{1}{2}\Big(\int_0^2 t-1\,dt+\int_2^4 3-t\,dt\Big)=0.$$

$$a_k=\frac{2}{T}\int_0^T f(t)\cos(k\omega t)\,dt=\frac{1}{2}\Big(\int_0^2 (t-1)\cos\big(k\tfrac{\pi}{2}t\big)\,dt+\int_2^4 (3-t)\cos\big(k\tfrac{\pi}{2}t\big)\,dt\Big)=\langle\!\langle \text{ by parts } \rangle\!\rangle$$

$$=\frac{1}{2}\Big[(t-1)\frac{2}{k\pi}\sin\big(k\tfrac{\pi}{2}t\big)\Big]_0^2-\frac{1}{2}\frac{2}{k\pi}\int_0^2\sin\big(k\tfrac{\pi}{2}t\big)\,dt+\frac{1}{2}\Big[(3-t)\frac{2}{k\pi}\sin\big(k\tfrac{\pi}{2}t\big)\Big]_2^4+\frac{1}{2}\frac{2}{k\pi}\int_2^4\sin\big(k\tfrac{\pi}{2}t\big)\,dt$$

$$=0+\Big[\frac{2}{k^2\pi^2}\cos\big(k\tfrac{\pi}{2}t\big)\Big]_0^2+0-\Big[\frac{2}{k^2\pi^2}\cos\big(k\tfrac{\pi}{2}t\big)\Big]_2^4$$

$$=\frac{2}{k^2\pi^2}[\cos(k\pi)-\cos(0)]-\frac{2}{k^2\pi^2}[\cos(2k\pi)-\cos(k\pi)]$$

$$=\frac{2}{k^2\pi^2}[(-1)^k-1]-\frac{2}{k^2\pi^2}[1-(-1)^k]=\frac{4}{k^2\pi^2}[(-1)^k-1]=\begin{cases}0, & k \text{ even},\\ -\frac{8}{k^2\pi^2}, & k \text{ odd}.\end{cases}$$

$$b_k=\frac{2}{T}\int_0^T f(t)\sin(k\omega t)\,dt=\frac{1}{2}\Big(\int_0^2 (t-1)\sin\big(k\tfrac{\pi}{2}t\big)\,dt+\int_2^4 (3-t)\sin\big(k\tfrac{\pi}{2}t\big)\,dt\Big)=\langle\!\langle \text{ by parts } \rangle\!\rangle$$

$$=\frac{1}{2}\Big[-(t-1)\frac{2}{k\pi}\cos\big(k\tfrac{\pi}{2}t\big)\Big]_0^2+\frac{1}{2}\frac{2}{k\pi}\int_0^2\cos\big(k\tfrac{\pi}{2}t\big)\,dt$$

$$-\frac{1}{2}\Big[(3-t)\frac{2}{k\pi}\cos\big(k\tfrac{\pi}{2}t\big)\Big]_2^4-\frac{1}{2}\frac{2}{k\pi}\int_2^4\cos\big(k\tfrac{\pi}{2}t\big)\,dt$$

$$=-\frac{1}{k\pi}[\cos(k\pi)+\cos(0)]+\Big[\frac{2}{k^2\pi^2}\sin\big(k\tfrac{\pi}{2}t\big)\Big]_0^2+\frac{1}{k\pi}[\cos(2k\pi)+\cos(k\pi)]-\Big[\frac{2}{k^2\pi^2}\sin\big(k\tfrac{\pi}{2}t\big)\Big]_2^4=0.$$

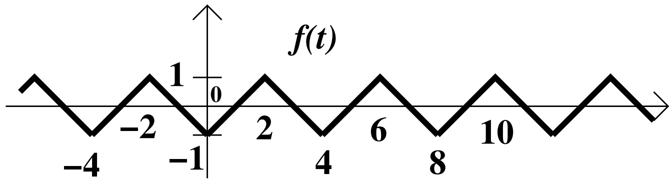
Odd numbers can be expressed as $k=2i+1$, to numbers $k=1,3,5,7,\dots$ correspond indeces $i=0,1,2,3,\dots$. For those we then have $a_k=-\dfrac{8}{(2i+1)\pi}$. We rewrite the resulting series accordingly, and since the index k is traditional, we pass from i to k at the end.
Thus

$$f\sim\frac{a_0}{2}+\sum_{k=1}^{\infty}\big(a_k\cos(k\omega t)+b_k\sin(k\omega t)\big)=\sum_{k=1}^{\infty}\frac{4}{k^2\pi^2}[(-1)^k-1]\cos\big(k\tfrac{\pi}{2}t\big)$$

$$=\sum_{k=0}^{\infty}\frac{-8}{(2k+1)\pi}\cos\big((2k+1)\tfrac{\pi}{2}t\big).$$

What is the sum of this series? First we draw a periodic extension of the function f. To this we then apply the Jordan criterion. According to it, the resulting series converges to f at all points where f

(or rather its periodic extension) is continuous. Since our extension is continuosu everywhere, this functions is also the sum of the series.



Since the extension of f is an even function, we should get a cosine series, which we did indeed.

3. Parameters: The length of the given segment is $L=4$, after creating an odd function by flipping the shape about both axes we eventually obtain a function with period $T=8$, for sine series we use the special frequency $\omega=\frac{\pi}{2T}=\frac{\pi}{L}=\frac{\pi}{4}$ and classical formulas with L in place of T.
Sine series has $a_0=a_k=0$.

$$b_k=\frac{2}{L}\int_0^L f(t)\sin(k\omega t)\,dt=\frac{1}{2}\Big(\int_0^2(t-1)\sin\big(k\tfrac{\pi}{4}t\big)\,dt+\int_2^4(3-t)\sin\big(k\tfrac{\pi}{4}t\big)\,dt\Big)=\langle\!\langle \text{ by parts } \rangle\!\rangle$$

$$=\frac{1}{2}\Big[-(t-1)\frac{4}{k\pi}\cos\big(k\tfrac{\pi}{4}t\big)\Big]_0^2+\frac{1}{2}\frac{4}{k\pi}\int_0^2\cos\big(k\tfrac{\pi}{4}t\big)\,dt$$

$$-\frac{1}{2}\Big[(3-t)\frac{4}{k\pi}\cos\big(k\tfrac{\pi}{4}t\big)\Big]_2^4-\frac{1}{2}\frac{4}{k\pi}\int_2^4\cos\big(k\tfrac{\pi}{4}t\big)\,dt$$

$$=-\frac{2}{k\pi}\big[\cos\big(k\tfrac{\pi}{2}\big)+\cos(0)\big]+\Big[\tfrac{8}{k^2\pi^2}\sin\big(k\tfrac{\pi}{4}t\big)\Big]_0^2+\frac{2}{k\pi}\big[\cos(k\pi)+\cos\big(k\tfrac{\pi}{2}\big)\big]-\Big[\tfrac{8}{k^2\pi^2}\sin\big(k\tfrac{\pi}{4}t\big)\Big]_2^4$$

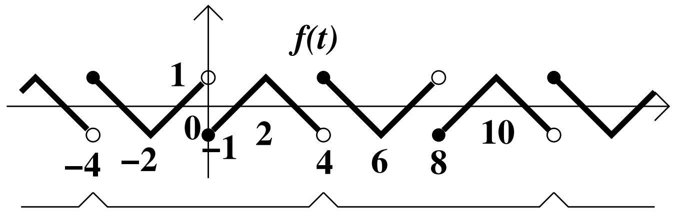
$$=\frac{2}{k\pi}[(-1)^k-1]+\tfrac{16}{k^2\pi^2}\sin\big(k\tfrac{\pi}{2}\big).$$

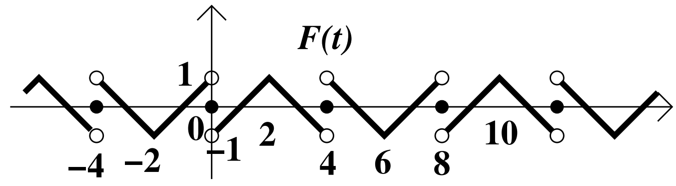
For k even ve get 0. If k is odd, the first term gives $\frac{-4}{k\pi}$, while the second one is $(-1)^i\frac{16}{k^2\pi^2}$ for $k=2i+1$. Thus for k odd, $k=2i+1$ we get $b_k=\frac{-4}{(2i+1)\pi}+(-1)^i\frac{16}{(2i+1)^2\pi^2}$. As usual we use k instead of i.
Thus

$$f\sim\frac{a_0}{2}+\sum_{k=1}^{\infty}\big(a_k\cos(k\omega t)+b_k\sin(k\omega t)\big)=\sum_{k=1}^{\infty}\Big[\frac{2}{k\pi}[(-1)^k-1]+\frac{16}{k^2\pi^2}\sin\big(k\tfrac{\pi}{2}\big)\Big]\sin\big(k\tfrac{\pi}{4}t\big)$$

$$=\sum_{k=0}^{\infty}\Big[\tfrac{-4}{(2k+1)\pi}+(-1)^k\frac{16}{(2k+1)^2\pi^2}\Big]\sin\big((2k+1)\tfrac{\pi}{4}t\big).$$

What is the sum of this series? First we flip the given shape about both axes, thus creating and odd function, extending this basic shape we obtain the **odd** periodic extension of the function f (on the left). To this we then apply the Jordan criterion. According to it, the resulting series converges to f at all points where f (or rather its periodic extension) is continuous. At points of discontinuity of f the series converges to the average $\frac{1}{2}\big(f(t^+)+f(t^-)\big)$. Result: On the right is the function to which our Fourier series converges, i.e. its sum.





Alternative: It is possible not to memorize the special formula for sine/cosine Fourier, but apply the usual Fourier series to that extended basic shape of f to an odd function (see picture on the left). In this way we get $T=8$, $\omega=\frac{2\pi}{T}=\frac{\pi}{4}$. Then we need to find formulas for the segments that give the basic period of odd extension and we can go, for b_k we get

$$b_k=\frac{2}{T}\int_{-L}^{L}f(t)\sin(k\omega t)\,dt$$
$$=\frac{1}{4}\Big(\int_{-4}^{-2}(-t-3)\sin\big(k\tfrac{\pi}{4}t\big)\,dt+\int_{-2}^{0}(t+1)\sin\big(k\tfrac{\pi}{4}t\big)\,dt+\int_{0}^{2}(t-1)\sin\big(k\tfrac{\pi}{4}t\big)\,dt+\int_{2}^{4}(3-t)\sin\big(k\tfrac{\pi}{4}t\big)\,dt\Big).$$

This looks tough, perhaps it is better to remember that special formula for sine/cosine series. This alternative can be made a bit easier by the following reasoning: If $f(t)$ is odd on $[-4,4)$, then $f(t)\sin\big(k\frac{\pi}{4}t\big)$ is even on $[-4,4)$, thus it is enough to integrate over its right half and take it twice:

$$b_k=2\cdot\frac{2}{T}\int_{0}^{L}f(t)\sin(k\omega t)\,dt=\frac{1}{2}\Big(\int_{0}^{2}(t-1)\sin\big(k\tfrac{\pi}{4}t\big)\,dt+\int_{2}^{4}(3-t)\sin\big(k\tfrac{\pi}{4}t\big)\,dt\Big).$$

But that's exactly the formula we got from the special version right away, so it is probably really best to simply remember the special frequency $\omega=\frac{\pi}{L}$ for sine/cosine series.

4. Parameters: The length of the given part is $L=4$, we see that the specification $f(t)=0$ on $[1,4)$ is important since it tells us how long the period is.
For the cosine series we first create by flipping the shape an even function with period $T=8$, the we use the special frequency $\omega=\frac{\pi}{2T}=\frac{\pi}{L}=\frac{\pi}{4}$ and classical formulas with L in place of T.

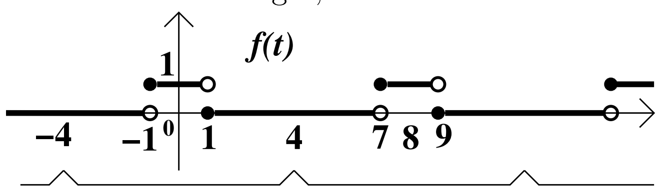
Cosine series has $b_k=0$. $a_0=\frac{2}{L}\int_{0}^{L}f(t)\,dt=\frac{1}{2}\int_{0}^{1}1\,dt=\frac{1}{2}$.

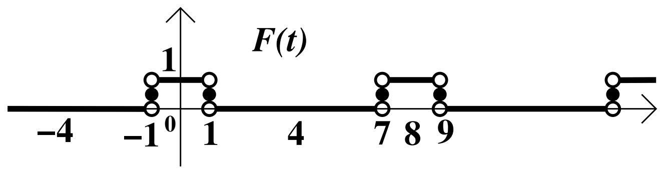
$$a_k=\frac{2}{L}\int_{0}^{L}f(t)\cos(k\omega t)\,dt=\frac{1}{2}\int_{0}^{1}\cos\big(k\tfrac{\pi}{4}t\big)\,dt=\frac{1}{2}\Big[\frac{4}{k\pi}\sin\big(k\tfrac{\pi}{4}t\big)\Big]_0^1=\frac{2}{k\pi}\sin\big(k\tfrac{\pi}{4}\big).$$

It is not possible to write this somehow better, since when we try to substitute $k=0$, 1, 2, 3, 4, 5, 6, 7, we get 0, $\frac{\sqrt{2}}{2}$, 1, $\frac{\sqrt{2}}{2}$, 0, $-\frac{\sqrt{2}}{2}$, -1, $-\frac{\sqrt{2}}{2}$, 0, which is too irregular.
Thus

$$f\sim\frac{a_0}{2}+\sum_{k=1}^{\infty}\big(a_k\cos(k\omega t)+b_k\sin(k\omega t)\big)=\frac{1}{4}+\sum_{k=1}^{\infty}\frac{2}{k\pi}\sin\big(k\tfrac{\pi}{4}\big)\cos\big((2k+1)\tfrac{\pi}{4}t\big).$$

What is the sum of this series? First we flip the given shape about the y-axis, thus obtaining an even function, by extending it we arrive at the **even** periodic extension of the function f (on the left). To this we then apply the Jordan criterion. According to it, the resulting series converges to f at all points where f (or rather its periodic extension) is continuous. At points of discontinuity of f the series converges to the average $\frac{1}{2}\big(f(t^+)+f(t^-)\big)$. Result: On the right is the function to which our Fourier series converges, i.e. its sum.





5. Parameters: The period length is $T=2$. This function is not given on an interval of the form $[0,T)$, but somewhere else, however, a shift in an interval is no problem, we find the Fourier series as usual. Frequency is $\omega = \frac{2\pi}{T} = \pi$.

$$a_0 = \frac{2}{T}\int\limits_{-T/2}^{T/2} f(t)\,dt = \int\limits_{-1}^{1} 1 - t^2 dt = \frac{4}{3}.$$

$$a_k = \frac{2}{T}\int\limits_{-T/2}^{T/2} f(t)\cos(k\omega t)\,dt = \int\limits_{-1}^{1} (1-t^2)\cos(k\pi t)\,dt = \langle\!\langle \text{ by parts } \rangle\!\rangle$$

$$= \Big[\frac{1}{k\pi}(1-t^2)\sin(k\pi t)\Big]_{-1}^{1} + \frac{2}{k\pi}\int\limits_{-1}^{1} t\sin(k\pi t)\,dt = 0 + \Big[-\frac{2}{k^2\pi^2}t\cos(k\pi t)\Big]_{-1}^{1} + \frac{2}{k^2\pi^2}\int\limits_{-1}^{1}\cos(k\pi t)\,dt$$

$$= -\frac{2}{k^2\pi^2}[\cos(k\pi) + \cos(-k\pi)] + \Big[\frac{2}{k^3\pi^3}\sin(k\pi t)\Big]_{-1}^{1} = -\frac{2}{k^2\pi^2}[\cos(k\pi) + \cos(k\pi)] + 0$$

$$= -(-1)^k\frac{4}{k^2\pi^2}.$$

$$b_k = \frac{2}{T}\int\limits_{-T/2}^{T/2} f(t)\sin(k\omega t)\,dt = \int\limits_{-1}^{1} (1-t^2)\sin(k\pi t)\,dt = \langle\!\langle \text{ by parts } \rangle\!\rangle$$

$$= \Big[-\frac{1}{k\pi}(1-t^2)\cos(k\pi t)\Big]_{-1}^{1} - \frac{2}{k\pi}\int\limits_{-1}^{1} t\cos(k\pi t)\,dt = 0 - \Big[\frac{2}{k^2\pi^2}t\sin(k\pi t)\Big]_{-1}^{1} + \frac{2}{k^2\pi^2}\int\limits_{-1}^{1}\sin(k\pi t)\,dt$$

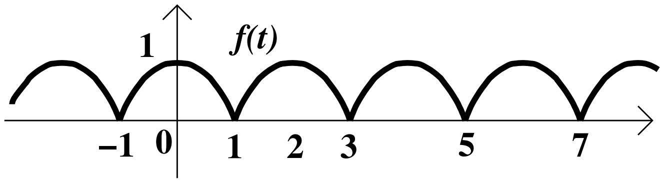
$$= -\frac{2}{k^2\pi^2}[\sin(k\pi) + \sin(-k\pi)] + \Big[-\frac{2}{k^3\pi^3}\cos(k\pi t)\Big]_{-1}^{1} = 0 - \frac{2}{k^3\pi^3}[\cos(k\pi) - \cos(-k\pi)]$$

$$= -\frac{2}{k^3\pi^3}[\cos(k\pi) - \cos(k\pi)] = 0.$$

Thus

$$f \sim \frac{a_0}{2} + \sum_{k=1}^{\infty}\big(a_k\cos(k\omega t) + b_k\sin(k\omega t)\big) = \frac{2}{3} + \sum_{k=1}^{\infty}(-1)^{k+1}\frac{4}{k^2\pi^2}\cos(k\pi t).$$

What is the sum of this series? First we draw a periodic extension of the function f. To this we then apply the Jordan criterion. According to it, the resulting series converges to f at all points where f (or rather its periodic extension) is continuous. Since our extension is continuosu everywhere, this functions is also the sum of the series.



Notice that the given function is even after we extended it, so the resulting series is naturally a cosine series. If we drew the extension first, we need not have calculated b_k, we could have just written $b_k = 0$.

Since the interval $[-1, 1)$ extends to both sides from the origin, the symmetry is decided right from the start. We see that the cosine series is possible, since the function $f(t) = 1 - t^2$ is even on the interval $[-1, 1)$, but we cannot make an odd function out of f and thus sine series is not possible.