EM2 Solved problems—Laplace & Fourier transform

Find the Laplace transform of the following functions:

1. $f(t) = t\sin(2t)$; **2.** $f(t) = e^{-3t}\cos(3t)$; **3.** $f(t) = \int\limits_0^t e^{3s}\,ds$;

4. $f(t) = (t+2)\cdot \mathbf{1}(t-3)$; **5.** $f(t) = \begin{cases} \sin(t), & t \in [0,\pi); \\ 0, & \text{elsewhere}; \end{cases}$ **6.** $f(t) = |\sin(t)|$.

Find the inverse Laplace transform of the following functions:

7. $F(p) = \frac{2p}{(p^2+1)(p+1)}$; **8.** $F(p) = \frac{8e^{-3p}}{p^2+2p+5}$;

9. $F(p) = \frac{1-e^{-p}}{p^2} - \frac{e^{-p}}{p} + \frac{e^{-p}}{p-1}$, sketch the graph of this $f(t) = \mathcal{L}^{-1}\{F(p)\}$.

Use the Laplace transform to solve the following problems:

10. $y' - y = (x+2)e^{2x}$, $y(0) = 3$.

11. $\ddot{y} + y = \begin{cases} 1, & t \in [0,\pi); \\ 0, & \text{elsewhere}, \end{cases}$ $y(0_+) = 2$, $\dot{y}(0_+) = 1$.

12. $y' + 4\int\limits_0^t y(s)\,ds = 6$, $y(0_+) = 1$.

13. $y' - y = \begin{cases} \sin(t), & t \in [0,\frac{\pi}{2}); \\ 0, & \text{elsewhere}, \end{cases}$ $y(0) = 4$.

14. $\dot{x} - x = \begin{cases} t, & t \in [0,1); \\ 0, & \text{elsewhere}, \end{cases}$ $x(0+) = -1$.

15. Use the Laplace transform to find the general solution of $y'' + y = 8e^{2t}\sin(t)$.

16. Consider the function $f(t) = \begin{cases} t, & t \in [0,1); \\ 1, & t \in [1,2). \end{cases}$

Find its Fourier series, its sine Fourier series, and its cosine Fourier series.
For each series determine its sum.

Solutions

1. We first use the rule for $\mathcal{L}\{t\,f(t)\}$ to get rid of t:

$$\mathcal{L}\{t\sin(2t)\} = -\frac{\mathrm{d}}{\mathrm{dp}}\mathcal{L}\{\sin(2t)\} = -\Big[\frac{2}{p^2+4}\Big]' = \frac{4p}{(p^2+4)^2}.$$

2. We first use the rule for $\mathcal{L}\{e^{at}\,f(t)\}$ to get rid of the exponential:

$$\mathcal{L}\{e^{-3t}\cos(3t)\} = \mathcal{L}\{\cos(3t)\}\big|_{p-(-3)} = \frac{p}{p^2+9}\Big|_{p+3} = \frac{p+3}{(p+3)^2+9} = \frac{p+3}{p^2+6p+18}.$$

It is also possible to first get rid of $3t$, since it is everywhere in the function. We use the choice $f(t)=e^{-t}\cos(t)$, then $f(3t)$ is the given function. Using the appropriate formula we get rid of $3t$, then we continue as above (get rid of exponential, transform cosine). When doing two operations, one has to be careful about the proper order of grammar rules:

$$\mathcal{L}\{e^{-3t}\cos(3t)\} = \frac{1}{3}\mathcal{L}\{e^{-t}\cos(t)\}\big|_{p/3} = \frac{1}{3}\Big(\mathcal{L}\{\cos(t)\}\big|_{p-(-1)}\Big)\Big|_{p/3} = \frac{1}{3}\Big(\frac{p}{p^2+1}\Big|_{p+1}\Big)\Big|_{p/3}$$

$$= \frac{1}{3}\Big(\frac{p+1}{(p+1)^2+1}\Big)\Big|_{p/3} = \frac{1}{3}\frac{p/3+1}{(p/3+1)^2+1} = \frac{\frac{p}{9}+\frac{1}{3}}{\frac{p^2}{9}+\frac{2p}{3}+2} = \frac{p+3}{p^2+6p+18}.$$

3. We use the rule for integrals:

$$\mathcal{L}\Big\{\int_0^t e^{3s}\,ds\Big\} = \frac{1}{p}\mathcal{L}\{e^{3t}\} = \frac{1}{p}\,\frac{1}{p-3} = \frac{1}{p(p-3)}.$$

4. We have to use the shift rule. There are two ways to do it:

a) We change the given function $t+2$ to feature the expression $t-3$ just like the jump function, then use the shift rule:

$$\mathcal{L}\{(t+2)\cdot H(t-3)\} = \mathcal{L}\{([t-3]+5)\cdot H(t-3)\} = e^{-3p}\mathcal{L}\{(t+5)\cdot H(t)\} = e^{-3p}\big(\tfrac{1}{p^2}+\tfrac{5}{p}\big).$$

b) It is also possible to use the shift rule and think of it as a substitution, for $t-3$ one puts, say, s, so $t=s+3$:

$$\mathcal{L}\{(t+2)\cdot H(t-3)\} = e^{-3p}\mathcal{L}\{([s+3]+2)\cdot H(s)\} = e^{-3p}\mathcal{L}\{(s+5)\cdot H(s)\} = e^{-3p}\big(\tfrac{1}{p^2}+\tfrac{5}{p}\big).$$

5. There are two possibilities. First, one can use the definition:

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-pt}\,dt = \int_0^\pi \sin(t)e^{-pt}\,dt.$$

This integral is standard (two integrations by parts, then solving an equation), but it is a long calculation. It might be easier to express the given function as a product of sine and a discrete unit signal on the interval $[0,\pi]$:

$$f(t) = \sin(t)\cdot[H(t)-H(t-\pi)] = \sin(t)H(t) - \sin(t)H(t-\pi).$$

Thus (see the previous problem)

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\sin(t)H(t) - \sin(t)H(t-\pi)\} = \mathcal{L}\{\sin(t)H(t)\} - \mathcal{L}\{\sin(t)H(t-\pi)\}$$

$$= \frac{1}{p^2+1} - e^{-\pi p}\mathcal{L}\{\sin(s+\pi)H(s)\} = \frac{1}{p^2+1} - e^{-\pi p}\mathcal{L}\{-\sin(s)H(s)\}$$

$$= \frac{1}{p^2+1} + e^{-\pi p}\frac{1}{p^2+1} = \frac{e^{-\pi p}+1}{p^2+1}.$$

6. There is no rule for the absolute value. One possibility is to use the definition, but integrating absolute value is a problem, such an integral would have to be split depending on whether the sine is positive or negative.

It will be therefore easier to think of f as a function that is periodic with $T=\pi$, since indeed, when you draw the graph, you find out that the function f is actually just one "hill" of the sine repeated over and over. In fact, the first period, f_T, is exactly the function from part d). By the theorem on periodic functions and the previous part,

$$\mathcal{L}\{f(t)\} = \frac{\mathcal{L}\{f_T\}}{1-e^{-\pi p}} = \frac{e^{-\pi p}+1}{(p^2+1)(1-e^{-\pi p})}.$$

7. To find the inverse, it is enough to decompose the fraction into partial fractions:

$$\mathcal{L}^{-1}\Big\{\frac{2p}{(p^2+1)(p+1)}\Big\} = \mathcal{L}^{-1}\Big\{\frac{p+1}{p^2+1} - \frac{1}{p+1}\Big\}$$

$$= \mathcal{L}^{-1}\Big\{\frac{p}{p^2+1}\Big\} + \mathcal{L}^{-1}\Big\{\frac{1}{p^2+1}\Big\} - \mathcal{L}^{-1}\Big\{\frac{1}{p-(-1)}\Big\} = \cos(t)+\sin(t)-e^{-t}.$$

8. First we will use the appropriate rule to get rid of the exponential. The inverse Laplace of the remaining fraction is done first by completing a square, then by using the shift rule for the inverse Laplace and finally using the dictionary:

$$\begin{aligned}\mathcal{L}^{-1}\Big\{\tfrac{8e^{-3p}}{p^2+2p+5}\Big\} &= \mathcal{L}^{-1}\Big\{\tfrac{8}{p^2+2p+5}\Big\}\Big|_{t-3}\cdot H(t-3) = \mathcal{L}^{-1}\Big\{\tfrac{8}{(p+1)^2+4}\Big\}\Big|_{t-3}\cdot H(t-3)\\ &= \big(e^{-t}\mathcal{L}^{-1}\big\{\tfrac{8}{p^2+2^2}\big\}\big)\Big|_{t-3}\cdot H(t-3) = \big(e^{-t}4\sin(2t)\big)\Big|_{t-3}\cdot H(t-3)\\ &= 4e^{3-t}\sin\big(2(t-3)\big)\cdot H(t-3) = \begin{cases} 0, & t\in[0,3);\\ 4e^{3-t}\sin\big(2(t-3)\big), & t\geq 3.\end{cases}\end{aligned}$$

When rewriting the answer as a split function, we used the fact that $H(t-3)=0$ for $t<3$ and $H(t-3)=1$ for $t\geq 3$.

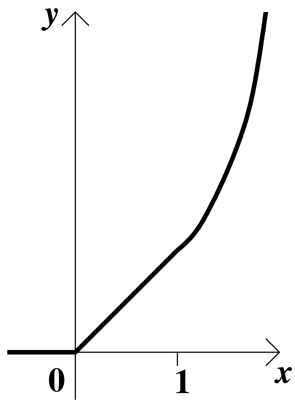
Note how $t-3$ was substituted back: It also became a part of the exponential, due to the order in which the rules were used. It is possible to also first handle the shift, then the rest:

$$\begin{aligned}\mathcal{L}^{-1}\Big\{\tfrac{8e^{-3p}}{p^2+2p+5}\Big\} &= \mathcal{L}^{-1}\Big\{\tfrac{8e^{-3(p+1)+3}}{(p+1)^2+4}\Big\} = e^{-t}\mathcal{L}^{-1}\Big\{\tfrac{8e^{-3p+3}}{p^2+4}\Big\} = e^{-t}\mathcal{L}^{-1}\Big\{\tfrac{4\cdot 2e^3e^{-3p}}{p^2+4}\Big\}\\ &= e^{-t}4e^3\mathcal{L}^{-1}\Big\{\tfrac{2e^{-3p}}{p^2+4}\Big\} = 4e^{3-t}\mathcal{L}^{-1}\Big\{\tfrac{2}{p^2+2^2}\Big\}\Big|_{t-3}\cdot H(t-3)\\ &= 4e^{3-t}\big(\sin(2t)\big)\Big|_{t-3}\cdot H(t-3) = 4e^{3-t}\sin\big(2(t-3)\big)\cdot H(t-3).\end{aligned}$$

Now the substitution of $t-3$ applied only to the sine part itself, because e^{3-t} was already done, it was not a part of the inverse Laplace when the "get rid of exponential" rule was applied.

9. We just have to use the rule for getting rid of exponential and then the dictionary:

$$\begin{aligned}\mathcal{L}^{-1}\Big\{\tfrac{1-e^{-p}}{p^2}-\tfrac{e^{-p}}{p}+\tfrac{e^{-p}}{p-1}\Big\} &= \mathcal{L}^{-1}\Big\{\tfrac{1}{p^2}\Big\}-\mathcal{L}^{-1}\Big\{e^{-p}\big(\tfrac{1}{p^2}+\tfrac{1}{p}-\tfrac{1}{p-1}\big)\Big\}\\ &= t-\mathcal{L}^{-1}\Big\{\tfrac{1}{p^2}+\tfrac{1}{p}-\tfrac{1}{p-1}\Big\}\Big|_{t-1}\cdot H(t-1) = t-\big(t+1-e^t\big)\Big|_{t-1}\cdot H(t-1)\\ &= t-([t-1]+1-e^{t-1})\cdot H(t-1) = t+(e^{t-1}-t)\cdot H(t-1)\\ &= \begin{cases} 0, & t<0;\\ t, & t\in[0,1);\\ e^{t-1}, & t\geq 1.\end{cases}\end{aligned}$$



10. We apply the Laplace transform to both sides of the equation, denoting $Y=\mathcal{L}\{y\}$:

$$\begin{aligned}\mathcal{L}\{y'-y\} = \mathcal{L}\{(x+2)e^{2x}\} \quad&\Longrightarrow\quad \mathcal{L}\{y\}'-\mathcal{L}\{y\} = \mathcal{L}\{x+2\}\big|_{p-2}\\ \Longrightarrow\quad& [p\mathcal{L}\{y\}-y(0_+)]-\mathcal{L}\{y\} = \big(\tfrac{1}{p^2}+\tfrac{2}{p}\big)\big|_{p-2}\\ \Longrightarrow\quad& pY-3-Y = \tfrac{1}{(p-2)^2}+\tfrac{2}{p-2}\\ \Longrightarrow\quad& (p-1)Y = 3+\tfrac{1}{(p-2)^2}+\tfrac{2}{p-2}.\end{aligned}$$

Therefore $Y(p)=\frac{3}{p-1}+\frac{1}{(p-1)(p-2)^2}+\frac{2}{(p-1)(p-2)}$.

To find $y(t)$ we use the inverse Laplace transform. First we need to replace the rational functions in Y with partial fractions:

$$\frac{2}{(p-1)(p-2)} = \frac{2}{p-2} - \frac{2}{p-1}$$
$$\frac{1}{(p-1)(p-2)^2} = \frac{1}{p-1} - \frac{1}{p-2} + \frac{1}{(p-2)^2}.$$

Thus $Y(p) = \frac{2}{p-1} + \frac{1}{p-2} + \frac{1}{(p-2)^2}$.
Consequently

$$y(t) = \mathcal{L}^{-1}\{\tfrac{2}{p-1} + \tfrac{1}{p-2} + \tfrac{1}{(p-2)^2}\} = \mathcal{L}^{-1}\{\tfrac{2}{p-1}\} + \mathcal{L}^{-1}\{\tfrac{1}{p-2}\} + \mathcal{L}^{-1}\{\tfrac{1}{(p-2)^2}\}$$
$$= 2e^t + e^{2t} + e^{2t}\mathcal{L}^{-1}\{\tfrac{1}{p^2}\} = 2e^t + e^{2t} + e^{2t}t,\ t \geq 0.$$

Since the equation and the solution exist on the whole real line, it is very likely that the solution actually works on $I\!R$. This would have to be checked by substituting y into the equation and indeed it works.

11. We apply the Laplace transform to both sides. It might be a good idea to first figure out the Laplace transform on the right. We can do it by definition, since this should be easy:

$$\mathcal{L}\{f(t)\} = \int_0^\pi e^{-pt}\,dt = \big[\tfrac{e^{-pt}}{-p}\big]_0^\pi = \frac{1-e^{-\pi p}}{p}.$$

Alternative:

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{H(t) - H(t-\pi)\} = \tfrac{1}{p} - e^{-\pi p}\mathcal{L}\{H(t)\} = \tfrac{1}{p} - e^{-\pi p}\tfrac{1}{p}.$$

The left hand-side transforms (denoting $Y = \mathcal{L}\{y\}$) like this:

$$\mathcal{L}\{\ddot{y} + y\} = [p^2Y - 2p - 1] + Y = Y(p^2+1) - 2p - 1.$$

Therefore the equation becomes

$$Y(p^2+1) = 1 + 2p + \tfrac{1}{p} - e^{-\pi p}\tfrac{1}{p} \quad\Longrightarrow\quad Y = \tfrac{1}{p^2+1} + \tfrac{2p}{p^2+1} + \tfrac{1}{p(p^2+1)} - e^{-\pi p}\tfrac{1}{p(p^2+1)}.$$

Thus $y(t) = \mathcal{L}^{-1}\{\frac{1}{p^2+1} + \frac{2p}{p^2+1} + \frac{1}{p(p^2+1)} - e^{-\pi p}\frac{1}{p(p^2+1)}\}$. The inverse Laplace of the first two terms is easy, for the other two we use the partial fractions decomposition:

$$Y(p) = \tfrac{1}{p^2+1} + \tfrac{2p}{p^2+1} + \big(\tfrac{1}{p} - \tfrac{p}{p^2+1}\big) - e^{-\pi p}\big(\tfrac{1}{p} - \tfrac{p}{p^2+1}\big)$$
$$= \tfrac{1}{p} + \tfrac{1}{p^2+1} + \tfrac{p}{p^2+1} - e^{-\pi p}\big(\tfrac{1}{p} - \tfrac{p}{p^2+1}\big)$$

Thus

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\{\tfrac{1}{p} + \tfrac{1}{p^2+1} + \tfrac{p}{p^2+1} - e^{-\pi p}\big(\tfrac{1}{p} - \tfrac{p}{p^2+1}\big)\}\\
&= 1 + \sin(t) + \cos(t) - \mathcal{L}^{-1}\{\tfrac{1}{p} - \tfrac{p}{p^2+1}\}\big|_{t-\pi} \cdot H(t-\pi)\\
&= 1 + \sin(t) + \cos(t) - \big(1 - \cos(t)\big)\big|_{t-\pi} \cdot H(t-\pi)\\
&= 1 + \sin(t) + \cos(t) - \big(1 - \cos(t-\pi)\big) \cdot H(t-\pi)\\
&= 1 + \sin(t) + \cos(t) - \big(1 + \cos(t)\big) \cdot H(t-\pi)\\
&= \begin{cases} 1 + \sin(t) + \cos(t), & t \in [0,\pi);\\ \sin(t), & t \geq \pi. \end{cases}
\end{aligned}$$

12. We again (surprise surprise!) use the Laplace transform denoting $Y = \mathcal{L}\{y\}$:

$$\mathcal{L}\{y' + 4\int_0^t y(s)\,ds\} = \mathcal{L}\{6\} \quad\Longrightarrow\quad \mathcal{L}\{y'\} + 4\mathcal{L}\{\int_0^t y(s)\,ds\} = 6\mathcal{L}\{1\}$$
$$\Longrightarrow\quad pY - 1 + 4\tfrac{Y}{p} = 6\tfrac{1}{p} \quad\Longrightarrow\quad p^2Y - p + 4Y = 6.$$

Thus $Y(p^2+4) = 6 + p$ and therefore $Y(p) = \frac{6}{p^2+4} + \frac{p}{p^2+4}$.
Consequently

$$y(t) = \mathcal{L}^{-1}\{\tfrac{6}{p^2+4} + \tfrac{p}{p^2+4}\} = 3\mathcal{L}^{-1}\{\tfrac{2}{p^2+2^2}\} + \mathcal{L}^{-1}\{\tfrac{p}{p^2+2^2}\} = 3\sin(2t) + \cos(2t),\ t \geq 0.$$

Again, the solution probably works for all $t \in I\!R$.

13. How do we transform the right-hand side (denoted by $f(t)$)?
By definition: $\mathcal{L}\{f(t)\} = \int_0^{\pi/2} \sin(t)e^{-pt}\,dt$. This is not an easy integral. We try another way and write $f(t) = \sin(t)\big[H(t) - H\big(t - \frac{\pi}{2}\big)\big]$. Then

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\sin(t)H(t)\} - \mathcal{L}\{\sin(t)H\big(t-\tfrac{\pi}{2}\big)\} = \tfrac{1}{p^2+1} - e^{-\frac{\pi}{2}p}\mathcal{L}\{\sin\big(s+\tfrac{\pi}{2}\big)H(s)\}$$
$$= \tfrac{1}{p^2+1} - e^{-\frac{\pi}{2}p}\mathcal{L}\{\cos(s)H(s)\} = \tfrac{1}{p^2+1} - e^{-\frac{\pi}{2}p}\tfrac{p}{p^2+1}.$$

Thus the whole equation transforms as

$$pY - 4 - Y = \tfrac{1}{p^2+1} - e^{-\frac{\pi}{2}p}\tfrac{p}{p^2+1} \implies Y = \tfrac{4}{p-1} + \tfrac{1}{(p^2+1)(p-1)} - e^{-\frac{\pi}{2}p}\tfrac{p}{(p^2+1)(p-1)}.$$

Partial fractions decomposition yields

$$\tfrac{1}{(p^2+1)(p-1)} = \tfrac{\frac{1}{2}}{p-1} - \tfrac{\frac{1}{2}p+\frac{1}{2}}{p^2+1}, \qquad \tfrac{p}{(p^2+1)(p-1)} = \tfrac{\frac{1}{2}}{p-1} + \tfrac{-\frac{1}{2}p+\frac{1}{2}}{p^2+1}.$$

Thus

$$Y = \tfrac{\frac{9}{2}}{p-1} - \tfrac{\frac{1}{2}p+\frac{1}{2}}{p^2+1} - e^{-\frac{\pi}{2}p}\Big(\tfrac{\frac{1}{2}}{p-1} + \tfrac{-\frac{1}{2}p+\frac{1}{2}}{p^2+1}\Big).$$

Consqeuently

$$y(t) = \mathcal{L}^{-1}\big\{\tfrac{9}{2}\tfrac{1}{p-1} - \tfrac{1}{2}\tfrac{p}{p^2+1} - \tfrac{1}{2}\tfrac{1}{p^2+1} - e^{-\frac{\pi}{2}p}\big(\tfrac{1}{2}\tfrac{1}{p-1} - \tfrac{1}{2}\tfrac{p}{p^2+1} + \tfrac{1}{2}\tfrac{1}{p^2+1}\big)\big\}$$
$$= \tfrac{9}{2}e^t - \tfrac{1}{2}\cos(t) - \tfrac{1}{2}\sin(t) - \mathcal{L}^{-1}\big\{\tfrac{1}{2}\tfrac{1}{p-1} - \tfrac{1}{2}\tfrac{p}{p^2+1} + \tfrac{1}{2}\tfrac{1}{p^2+1}\big\}\Big|_{t-\pi/2} H\big(t-\tfrac{\pi}{2}\big)$$
$$= \tfrac{9}{2}e^t - \tfrac{1}{2}\cos(t) - \tfrac{1}{2}\sin(t) - \tfrac{1}{2}\big(e^{t-\frac{\pi}{2}} - \cos\big(t-\tfrac{\pi}{2}\big) + \sin\big(t-\tfrac{\pi}{2}\big)\big)H\big(t-\tfrac{\pi}{2}\big)$$
$$= \tfrac{9}{2}e^t - \tfrac{1}{2}\cos(t) - \tfrac{1}{2}\sin(t) - \tfrac{1}{2}\big(e^{t-\frac{\pi}{2}} - \sin(t) - \cos(t)\big)H\big(t-\tfrac{\pi}{2}\big)$$
$$= \begin{cases} \tfrac{9}{2}e^t - \tfrac{1}{2}\cos(t) - \tfrac{1}{2}\sin(t), & t \in [0, \tfrac{\pi}{2}); \\ \tfrac{9-e^{-\frac{\pi}{2}}}{2}\, e^t, & t \geq \tfrac{\pi}{2}. \end{cases}$$

14. We denote the function on the right by $f(t)$ and first calculate its Laplace transform. One possibility is the definition, using integration by parts:

$$\mathcal{L}\{f\} = \int_0^1 te^{-pt}\,dt = \Big[t\tfrac{e^{-pt}}{-p}\Big]_0^1 + \tfrac{1}{p}\int_0^1 e^{-pt}\,dt = -\tfrac{e^{-p}}{p} - \tfrac{e^{-p}}{p^2} + \tfrac{1}{p^2}.$$

Alternatively, one can write $f(t) = t\cdot H(t) - t\cdot H(t-1)$ and use the grammar:

$$\mathcal{L}\{f\} = \mathcal{L}\{t\} - \mathcal{L}\{tH(t-1)\} = \tfrac{1}{p^2} - e^{-p}\mathcal{L}\{(s+1)H(s)\} = \tfrac{1}{p^2} - e^{-p}\Big(\tfrac{1}{p^2} + \tfrac{1}{p}\Big).$$

Either way, applying the Laplace transform to both sides of the given equations and denoting $\mathcal{L}\{x\} = X$ we get

$$[pX+1] - X = \tfrac{1}{p^2} - \tfrac{e^{-p}}{p^2} - \tfrac{e^{-p}}{p} \implies X = \tfrac{-1}{p+1} + \tfrac{1}{p^2(p+1)} - \tfrac{e^{-p}}{p^2(p+1)} - \tfrac{e^{-p}}{p(p+1)}.$$

Partial fractions decomposition yields $\frac{1}{p(p+1)} = \frac{1}{p} - \frac{1}{p+1}$ and $\frac{1}{p^2(p+1)} = \frac{1}{p+1} + \frac{1}{p^2} - \frac{1}{p}$. We substitute into X and get $X(p) = \frac{1}{p^2} - \frac{1}{p} - e^{-p}\,\frac{1}{p^2}$.

Thus (using the inverse Laplace transform and a bit of grammar)

$$x(t) = t - 1 - \mathcal{L}^{-1}\big\{\tfrac{1}{p^2}\big\}\Big|_{t-1}\cdot H(t-1) = t - 1 - (t-1)\cdot H(t-1) = \begin{cases} t-1, & t \in (0,1]; \\ 0, & \text{elsewhere.} \end{cases}$$

15. Since we want a general solution, we need to introduce parameters, namely we need two, a and b. On the other hand, we need to know two initial conditions to be able to apply LT to y''. Thus we decide to set $y(0^+) = a$, $y'(0^+) = b$.

Transforming the equation and using $\mathcal{L}\{e^{2t}\sin(t)\} = \mathcal{L}\{\sin(t)\}\big|_{p-2}$ leads to

$$[p^2Y - ap - b] + Y = \tfrac{8}{(p-2)^2+1} \implies Y = \tfrac{ap}{p^2+1} + \tfrac{b}{p^2+1} + \tfrac{8}{(p^2+1)(p^2-4p+5)}.$$

We apply partial fractions:

$$\tfrac{8}{(p^2+1)(p^2-4p+5)} = \tfrac{Ap+B}{p^2+1} + \tfrac{Cp+D}{p^2-4p+5} \implies A=1,\ B=1,\ C=-1,\ D=3.$$

Thus

$$y(t) = \mathcal{L}^{-1}\big\{\tfrac{ap}{p^2+1} + \tfrac{b}{p^2+1} + \tfrac{p+1}{p^2+1} + \tfrac{-p+3}{p^2-4p+5}\big\}$$
$$= \mathcal{L}^{-1}\big\{\tfrac{(a+1)p}{p^2+1} + \tfrac{b+1}{p^2+1} - \tfrac{p-2}{(p-2)^2+1} + \tfrac{1}{(p-2)^2+1}\big\}$$
$$= (a+1)\cos(t) + (b+1)\sin(t) - e^{2t}\mathcal{L}^{-1}\big\{\tfrac{p}{p^2+1}\big\} + e^{2t}\mathcal{L}^{-1}\big\{\tfrac{1}{p^2+1}\big\}$$
$$= A\cos(t) + B\sin(t) - e^{2t}\cos(t) + e^{2t}\sin(t),\ t \geq 0.$$

We check that this solution in fact works on $\mathbb{R}$.

For solutions using "classical" approach see Solved problems—ODE of order 2.

16. a) The Fourier series: $T=2$, $\omega=\frac{2\pi}{T}=\pi$.

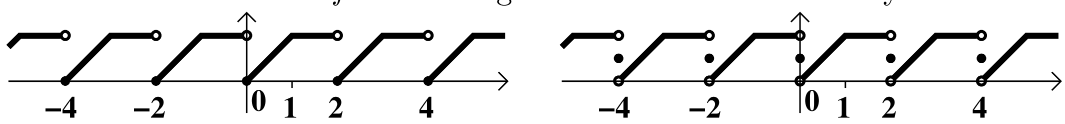
$$a_0=\frac{2}{T}\int\limits_0^T f(t)\,dt=\int\limits_0^1 t\,dt+\int\limits_1^2 1\,dt=\frac{3}{2};$$

$$\begin{aligned}a_k&=\frac{2}{T}\int\limits_0^T f(t)\cos(k\omega t)\,dt=\int\limits_0^1 t\cos(k\pi t)\,dt+\int\limits_1^2\cos(k\pi t)\,dt\\&=\Big[t\tfrac{1}{k\pi}\sin(k\pi t)\Big]_0^1-\tfrac{1}{k\pi}\int\limits_0^1\sin(k\pi t)\,dt+\Big[\tfrac{1}{k\pi}\sin(k\pi t)\Big]_1^2\\&=\tfrac{1}{k\pi}\sin(k\pi)-\tfrac{1}{k\pi}\Big[\tfrac{-1}{k\pi}\cos(k\pi t)\Big]_0^1+\big(\tfrac{1}{k\pi}\sin(2k\pi)-\tfrac{1}{k\pi}\sin(k\pi)\big)\\&=0+\tfrac{1}{k^2\pi^2}(\cos(k\pi)-\cos(0))+0=\tfrac{1}{k^2\pi^2}((-1)^k-1).\end{aligned}$$

$$\begin{aligned}b_k&=\frac{2}{T}\int\limits_0^T f(t)\sin(k\omega t)\,dt=\int\limits_0^1 t\sin(k\pi t)\,dt+\int\limits_1^2\sin(k\pi t)\,dt\\&=\Big[t\tfrac{-1}{k\pi}\cos(k\pi t)\Big]_0^1+\tfrac{1}{k\pi}\int\limits_0^1\cos(k\pi t)\,dt+\Big[\tfrac{-1}{k\pi}\cos(k\pi t)\Big]_1^2\\&=\tfrac{-1}{k\pi}\cos(k\pi)+\tfrac{1}{k\pi}\Big[\tfrac{1}{k\pi}\sin(k\pi t)\Big]_0^1-\tfrac{1}{k\pi}\big(\cos(2k\pi)-\cos(k\pi)\big)\\&=\tfrac{-1}{k\pi}(-1)^k+\tfrac{1}{k^2\pi^2}\big(\sin(k\pi)-\sin(0)\big)-\tfrac{1}{k\pi}\big(1-(-1)^k\big)=-\tfrac{1}{k\pi}.\end{aligned}$$

Thus $f\sim\frac{1}{2}\cdot\frac{3}{2}+\sum\limits_{k=1}^{\infty}\big[\frac{1}{k^2\pi^2}((-1)^k-1)\cos(k\pi t)-\frac{1}{k\pi}\sin(k\pi t)\big]$.

On the left: Periodic extension of f. On the right: The sum of this series by the Jordan criterion.



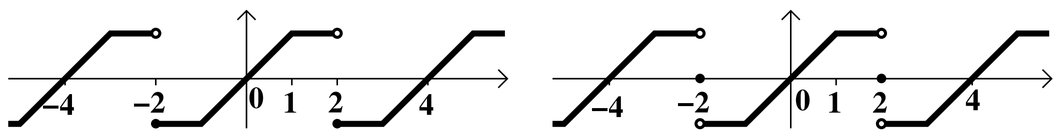
b) The sine Fourier series: $T=2$, $\omega=\frac{\pi}{T}=\frac{\pi}{2}$.

$a_0=a_n=0;$

$$\begin{aligned}b_k&=\frac{2}{T}\int\limits_0^T f(t)\sin(k\omega t)\,dt=\int\limits_0^1 t\sin\big(k\tfrac{\pi}{2}t\big)\,dt+\int\limits_1^2\sin\big(k\tfrac{\pi}{2}t\big)\,dt\\&=\Big[t\tfrac{-2}{k\pi}\cos\big(k\tfrac{\pi}{2}t\big)\Big]_0^1+\tfrac{2}{k\pi}\int\limits_0^1\cos\big(k\tfrac{\pi}{2}t\big)\,dt+\Big[\tfrac{-2}{k\pi}\cos\big(k\tfrac{\pi}{2}t\big)\Big]_1^2\\&=\tfrac{-2}{k\pi}\cos\big(k\tfrac{\pi}{2}\big)+\tfrac{2}{k\pi}\Big[\tfrac{2}{k\pi}\sin\big(k\tfrac{\pi}{2}t\big)\Big]_0^1-\tfrac{2}{k\pi}\big(\cos(k\pi)-\cos\big(k\tfrac{\pi}{2}\big)\big)\\&=\tfrac{-2}{k\pi}\cos\big(k\tfrac{\pi}{2}\big)+\tfrac{4}{k^2\pi^2}\big(\sin\big(k\tfrac{\pi}{2}\big)-\sin(0)\big)-\tfrac{2}{k\pi}\big((-1)^k-\cos\big(k\tfrac{\pi}{2}\big)\big)=\tfrac{4}{k^2\pi^2}\sin\big(k\tfrac{\pi}{2}\big)-\tfrac{2}{k\pi}(-1)^k.\end{aligned}$$

Thus $f\sim\sum\limits_{k=1}^{\infty}\big[\frac{4}{k^2\pi^2}\sin\big(k\frac{\pi}{2}\big)-\frac{2}{k\pi}(-1)^k\big]\sin\big(k\frac{\pi}{2}t\big)$.

On the left: Odd periodic extension of f. On the right: The sum of this series by the Jordan criterion.



b) The cosine Fourier series: $T=2$, $\omega=\frac{\pi}{T}=\frac{\pi}{2}$.

$$b_n=0;\ a_0=\frac{2}{T}\int\limits_0^T f(t)\,dt=\int\limits_0^1 t\,dt+\int\limits_1^2 1\,dt=\frac{3}{2};$$

$$\begin{aligned}a_k&=\frac{2}{T}\int\limits_0^T f(t)\cos(k\omega t)\,dt=\int\limits_0^1 t\cos\big(k\tfrac{\pi}{2}t\big)\,dt+\int\limits_1^2\cos\big(k\tfrac{\pi}{2}t\big)\,dt\\&=\Big[t\tfrac{2}{k\pi}\sin\big(k\tfrac{\pi}{2}t\big)\Big]_0^1-\tfrac{2}{k\pi}\int\limits_0^1\sin\big(k\tfrac{\pi}{2}t\big)\,dt+\Big[\tfrac{2}{k\pi}\sin\big(k\tfrac{\pi}{2}t\big)\Big]_1^2\\&=\tfrac{2}{k\pi}\sin\big(k\tfrac{\pi}{2}\big)-\tfrac{2}{k\pi}\Big[\tfrac{-2}{k\pi}\cos\big(k\tfrac{\pi}{2}t\big)\Big]_0^1+\tfrac{2}{k\pi}\big(\sin(k\pi)-\sin\big(k\tfrac{\pi}{2}\big)\big)\end{aligned}$$

$$= \frac{2}{k\pi}\sin\left(k\frac{\pi}{2}\right) + \frac{4}{k^2\pi^2}\left(\cos\left(k\frac{\pi}{2}\right) - \cos(0)\right) - \frac{2}{k\pi}\sin\left(k\frac{\pi}{2}\right) = \frac{4}{k^2\pi^2}\left(\cos\left(k\frac{\pi}{2}\right) - 1\right).$$

Thus $f \sim \frac{3}{4} + \sum\limits_{k=1}^{\infty}\left[\frac{4}{k^2\pi^2}\left(\cos\left(k\frac{\pi}{2}\right) - 1\right)\right]\cos\left(k\frac{\pi}{2}t\right)$.

Even periodic extension of f, it is also the sum of this series by the Jordan criterion.