Practice problems on Laplace transform
Brief solutions

1. $= \mathcal{L}\{t^5\}\big|_{p-2} = \frac{5!}{p^6}\big|_{p-2} = \frac{5!}{(p-2)^6}$ or $= -\big[\frac{1}{p-2}\big]'''''$.

2. $= -\big[\frac{1}{2}\mathcal{L}\{\sin(\omega t)\}\big]' = -\frac{1}{2}\big[\frac{\omega}{p^2+\omega^2}\big]' = \frac{\omega p}{(p^2+\omega^2)^2}$.

3. $= e^{-\frac{\pi}{2}p}\mathcal{L}\big\{\sin\big(3\big(t+\frac{\pi}{2}\big)\big)\big\} = e^{-\frac{\pi}{2}p}\mathcal{L}\big\{\sin\big(3t+\frac{3\pi}{2}\big)\big\} = -e^{-\frac{\pi}{2}p}\mathcal{L}\{\cos(3t)\} = -e^{-\frac{\pi}{2}p}\frac{p}{p^2+9}$.

4. $= \mathcal{L}\big\{t[H(t)-H(t-1)]\big\} + \mathcal{L}\{H(t-1)\} = \frac{1}{p^2} - e^{-p}\mathcal{L}\{t+1\} + e^{-p}\mathcal{L}\{1\} = \frac{1}{p^2} - e^{-p}\frac{1}{p^2}$.

5. $= \mathcal{L}\big\{\big(1-\frac{t}{2}\big)[H(t)-H(t-2)]\big\} = \big(\frac{1}{p}-\frac{1}{2}\frac{1}{p^2}\big) - e^{-2p}\mathcal{L}\big\{1-\frac{t+2}{2}\big\} = \big(\frac{1}{p}-\frac{1}{2p^2}\big) + e^{-2p}\frac{1}{2}\mathcal{L}\{t\} = \big(\frac{1}{p}-\frac{1}{2p^2}\big) + e^{-2p}\frac{1}{2p^2}$.

6. $= \mathcal{L}\big\{H(t)-2H(t-1)+H(t-2)\big\} = \frac{1}{p} - 2e^{-p}\mathcal{L}\{1\} + e^{-2p}\mathcal{L}\{1\} = \frac{1}{p}(1-2e^{-p}+e^{-2p}) = \frac{1}{p}(1-e^{-p})^2$.

7. $= \mathcal{L}^{-1}\big\{\frac{p-1-1}{(p-1)^2+4}\big\} = e^t\mathcal{L}^{-1}\big\{\frac{p-1}{p^2+4}\big\} = e^t\big(\cos(2t)-\frac{1}{2}\sin(2t)\big)$.

8. $= \frac{1}{2}\sin(2t)\big|_{t-\frac{\pi}{2}}H\big(t-\frac{\pi}{2}\big) = \frac{1}{2}\sin\big(2\big(t-\frac{\pi}{2}\big)\big)H\big(t-\frac{\pi}{2}\big) = \frac{1}{2}\sin(2t-\pi)H\big(t-\frac{\pi}{2}\big) = -\frac{1}{2}\sin(2t)H\big(t-\frac{\pi}{2}\big)$.

9. $= \mathcal{L}^{-1}\big\{\frac{1}{p-1} - \frac{p-4}{p^2+4}\big\} = e^t - \cos(2t) + 2\sin(2t)$.

10. $= \mathcal{L}^{-1}\big\{\frac{1}{(p-2)^2}\big\}\big|_{t-3}H(t-3) = \Big(e^{2t}\mathcal{L}^{-1}\big\{\frac{1}{p^2}\big\}\Big)\Big|_{t-3}H(t-3) = \big(e^{2t}t\big)\big|_{t-3}H(t-3) = (t-3)e^{2(t-3)}H(t-3)$.

11. $p^2Y - 3p - 3 - (pY-3) - 2Y = \frac{3}{p+2}$, $Y(p) = \frac{3p}{(p+1)(p-2)} + \frac{3}{(p+1)(p-2)(p+2)} = \frac{9/4}{p-2} + \frac{3/4}{p+2}$, hence
$y(x) = \frac{9}{4}e^{2x} + \frac{3}{4}e^{-2x}$.
Alternative: Guess for right hand-side: $\lambda^2-\lambda-2=0$, $\lambda=-1,2$, $y_h(x) = a\,e^{-x} + b\,e^{2x}$, guess $y(x) = Ae^{-2x}$, after substituting $A=\frac{1}{4}$, $y_p(x) = \frac{1}{4}e^{-2x}$, $y(x) = y_h(x) + y_p(x) = a\,e^{-x} + b\,e^{2x} + \frac{1}{4}e^{-2x}$, initial conditions give $y(x) = \frac{9}{4}e^{2x} + \frac{3}{4}e^{-2x}$.
Alternative: Variation of parameter: $y(x) = a(x)e^{-x} + b(x)e^{2x}$, equations $a'(x)e^{-x} + b'(x)e^{2x} = 0$ and $-a'(x)e^{-x} + 2b'(x)e^{2x} = 3e^{-2x}$. Solutions are $a'(x) = -e^{-x}$ and $b'(x) = e^{-4x}$, hence $a(x) = e^{-x}$ and $b(x) = -\frac{1}{4}e^{-4x}$, thus $y_p(x) = e^{-2x} - \frac{1}{4}e^{-2x} = \frac{3}{4}e^{-2x}$, $y = y_p + y_h$, initial conditions give $y(x) = \frac{9}{4}e^{2x} + \frac{3}{4}e^{-2x}$.

12. $p^2Y - 0p - (-1) - (pY - 0) = \mathcal{L}\{4x\}\big|_{p+1} = \frac{4}{p^2}\big|_{p+1} = \frac{4}{(p+1)^2}$,
$Y(p) = \frac{-1}{p(p-1)} + \frac{4}{(p+1)^2p(p-1)} = \big(\frac{1}{p} - \frac{1}{p-1}\big) + \big(\frac{3}{p+1} + \frac{2}{(p+1)^2} - \frac{4}{p} + \frac{1}{p-1}\big) = \frac{3}{p+1} + \frac{2}{(p+1)^2} - \frac{3}{p}$, hence
$y(x) = 3e^{-x} + 2e^{-x}\mathcal{L}^{-1}\big\{\frac{2}{p^2}\big\} - 3 = 3e^{-x} + 2x\,e^{-x} - 3$.

13. $p^2X - 0p - 0 + X = \frac{6}{p^2+4} - \frac{1}{p}$,
$X(p) = \frac{6}{(p^2+1)(p^2+4)} - \frac{1}{p(p^2+1)} = \big(\frac{2}{p^2+1} - \frac{2}{p^2+4}\big) - \big(\frac{1}{p} - \frac{p}{p^2+1}\big) = \frac{2}{p^2+1} + \frac{p}{p^2+1} - \frac{2}{p^2+4} - \frac{1}{p}$, hence
$x(t) = 2\sin(t) + \cos(t) - \sin(2t) - 1$.

14. $pY - Y = e^{-p}\frac{1}{p} - e^{-2p}\frac{1}{p}$, $Y(p) = \frac{1}{p(p-1)}e^{-p} - \frac{1}{p(p-1)}e^{-2p} = \big(\frac{1}{p-1} - \frac{1}{p}\big)e^{-p} - \big(\frac{1}{p-1} - \frac{1}{p}\big)e^{-2p}$, hence
$y(t) = (e^t-1)\big|_{t-1}H(t-1) - (e^t-1)\big|_{t-2}H(t-2) = (e^{t-1}-1)H(t-1) - (e^{t-2}-1)H(t-2)$

$$= \begin{cases} 0, & t<1; \\ e^{t-1}-1, & t\in[1,2); \\ (e^{-1}-e^{-2})e^t, & t\ge 2. \end{cases}$$

15. $pX - 1 + 4X + 13\frac{1}{p}X = \frac{1}{p}$, $p^2X - p + 4pX + 13X = 1$, $X(p) = \frac{p+1}{p^2+4p+13} = \frac{(p+2)-1}{(p+2)^2+9}$, hence
$x(t) = e^{-2t}\mathcal{L}^{-1}\big\{\frac{p-1}{p^2+9}\big\} = e^{-2t}\big[\cos(3t) - \frac{1}{3}\sin(3t)\big]$.

16. $pY - 0 - 2Y = \frac{4}{p^2} - \mathcal{L}\{4(x+1)\}e^{-p} = \frac{4}{p^2} - \mathcal{L}\{4x+4\}e^{-p} = \frac{4}{p^2} - \big(\frac{4}{p^2} + \frac{4}{p}\big)e^{-p}$,
$Y(p) = \frac{4}{p^2(p-2)} - \big(\frac{4}{p^2(p-2)} + \frac{4}{p(p-2)}\big)e^{-p} = \big(\frac{1}{p-2} - \frac{2}{p^2} - \frac{1}{p}\big) - \big(\frac{1}{p-2} - \frac{2}{p^2} - \frac{1}{p} - \frac{2}{p} + \frac{2}{p-2}\big)e^{-p}$
$= \big(\frac{1}{p-2} - \frac{2}{p^2} - \frac{1}{p}\big) - \big(\frac{3}{p-2} - \frac{2}{p^2} - \frac{3}{p}\big)e^{-p}$, hence
$y(x) = e^{2x} - 2x - 1 - \big(3e^{2x} - 2x - 3\big)\big|_{x-1}H(x-1)$

$$= e^{2x} - 2x - 1 - \big(3e^{2x-2} - 2(x-1) - 3\big)H(x-1) = \begin{cases} e^{2x} - 2x - 1, & x\in[0,1); \\ e^{2x}(1-3/e^2), & x\ge 1. \end{cases}$$

17. $pY-0+\frac{1}{p}Y=\frac{2}{p-1}+\frac{2}{p+1}$, $p^2Y+Y=\frac{2p}{p-1}+\frac{2p}{p+1}$,
$Y(p)=\frac{2p}{(p-1)(p^2+1)}+\frac{2p}{(p+1)(p^2+1)}=\big(\frac{1}{p-1}-\frac{p}{p^2+1}+\frac{1}{p^2+1}\big)+\big(\frac{-1}{p+1}+\frac{p}{p^2+1}+\frac{1}{p^2+1}\big)=\frac{1}{p-1}-\frac{1}{p+1}+\frac{2}{p^2+1}$.
Thus $y(x)=e^x-e^{-x}+2\sin(x)=2\sinh(x)+2\sin(x)$.

18. $pY+1+3Y=-\frac{13p}{p^2+4}+13\mathcal{L}\{\cos(2(x+\pi/4))\}e^{-\frac{\pi}{4}p}=-\frac{13p}{p^2+4}+13\mathcal{L}\{\cos(2x+\pi/2)\}e^{-\frac{\pi}{4}p}$
$=-\frac{13p}{p^2+4}+13\mathcal{L}\{-\sin(2x)\}e^{-\frac{\pi}{4}p}=-\frac{13p}{p^2+4}-\frac{26}{p^2+4}e^{-\frac{\pi}{4}p}$, here we used $\cos(\alpha+\pi/2)=-\sin(\alpha)$ for $\alpha=2x$,
thus $Y(p)=\frac{-1}{p+3}-\frac{13p}{(p+3)(p^2+4)}-\frac{26}{(p+3)(p^2+4)}e^{-\frac{\pi}{4}p}=\frac{-1}{p+3}+\frac{3}{p+3}-\frac{3p+4}{p^2+4}-\big(\frac{2}{p+3}+\frac{-2p+6}{p^2+4}\big)e^{-\frac{\pi}{4}p}$
$=\frac{2}{p+3}-\frac{3p}{p^2+4}-\frac{4}{p^2+4}-\big(\frac{2}{p+3}-\frac{2p}{p^2+4}+\frac{6}{p^2+4}\big)e^{-\frac{\pi}{4}p}$.
Hence $y(x)=2e^{-3x}-3\cos(2x)-2\sin(2x)-\big(2e^{-3x}-2\cos(2x)+3\sin(2x)\big)\big|_{x-\pi/4}H(x-\pi/4)$
$=2e^{-3x}-3\cos(2x)-2\sin(2x)-\big(2e^{3\pi/4-3x}-2\cos(2(x-\pi/4))+3\sin(2(x-\pi/4))\big)H(x-\pi/4)$
$=2e^{-3x}-3\cos(2x)-2\sin(2x)-\big(2e^{3\pi/4-3x}-2\cos(2x-\pi/2)+3\sin(2x-\pi/2)\big)H(x-\pi/4)$
$=2e^{-3x}-3\cos(2x)-2\sin(2x)-\big(2e^{3\pi/4-3x}-2\sin(2x)-3\cos(2x)\big)H(x-\pi/4)$
$=\begin{cases}2e^{-3x}-3\cos(2x)-2\sin(2x), & x\in[0,\pi/4);\\ 2e^{-3x}(1-e^{3\pi/4}), & x\ge\pi/4.\end{cases}$

19. $pX+1-X=\frac{2}{p}-\frac{1}{p^2}-\mathcal{L}\{2-(t+2)\}e^{-2p}=\frac{2}{p}-\frac{1}{p^2}+\mathcal{L}\{t\}e^{-2p}=\frac{2}{p}-\frac{1}{p^2}+\frac{1}{p^2}\,e^{-2p}$,
$X(p)=-\frac{1}{p-1}+\frac{2p-1}{p^2(p-1)}+\frac{1}{p^2(p-1)}\,e^{-2p}=-\frac{1}{p-1}+\frac{1}{p-1}-\frac{1}{p}+\frac{1}{p^2}+\big(\frac{1}{p-1}-\frac{1}{p}-\frac{1}{p^2}\big)e^{-2p}$
$=\frac{1}{p^2}-\frac{1}{p}+\big(\frac{1}{p-1}-\frac{1}{p}-\frac{1}{p^2}\big)e^{-2p}$.
Thus $x(t)=t-1+\big(e^t-1-t\big)\big|_{t-2}H(t-2)=t-1+\big(e^{t-2}-1-(t-2)\big)H(t-2)$
$=t-1+\big(e^{t-2}+1-t\big)H(t-2)=\begin{cases}t-1, & t\in[0,2);\\ e^{t-2}, & t\ge 2.\end{cases}$

20. $p^2X-0-0+3X=\frac{3}{p}e^{-\pi p}-\frac{3}{p}e^{-2\pi p}$, $(p^2+3)X=\frac{3}{p}e^{-\pi p}-\frac{3}{p}e^{-2\pi p}$,
$X(p)=\frac{3}{p(p^2+3)}e^{-\pi p}-\frac{3}{p(p^2+3)}e^{-2\pi p}=\big(\frac{1}{p}-\frac{p}{p^2+3}\big)e^{-\pi p}-\big(\frac{1}{p}-\frac{p}{p^2+3}\big)e^{-2\pi p}$.
Thus $x(t)=\big(1-\cos(\sqrt{3}\,t)\big)\big|_{t-\pi}H(t-\pi)-\big(1-\cos(\sqrt{3}\,t)\big)\big|_{t-2\pi}H(t-2\pi)$
$=\big(1-\cos(\sqrt{3}(t-\pi))\big)H(t-\pi)-\big(1-\cos(\sqrt{3}(t-2\pi))\big)H(t-2\pi)$
$=\begin{cases}0, & t\in[0,\pi);\\ 1-\cos(\sqrt{3}(t-\pi)), & t\in[\pi,2\pi);\\ \cos(\sqrt{3}(t-2\pi))-\cos(\sqrt{3}(t-\pi)), & t\ge 2\pi.\end{cases}$

21. It is $x'-2e^t*x(t)=2t\,e^t$, hence $pX-1-2\frac{1}{p-1}X=-\big[\frac{2}{p-1}\big]'=\frac{2}{(p-1)^2}$, also $=\frac{1}{p}\big|_{p-2}=\frac{2}{(p-1)^2}$.
$p(p-1)X-(p-1)-2X=\frac{2}{p-1}$, $(p^2-p-2)X=\frac{p^2-2p+3}{p-1}$, thus $X(p)=\frac{p^2-2p+3}{(p-2)(p+1)(p-1)}=\frac{1}{p-2}+\dfrac{1}{p+1}-\frac{1}{p-1}$.
Hence $x(t)=e^{2t}+e^{-t}-e^t$, $t\ge 0$.

22. $pY-1+\frac{4}{Y}=\frac{1}{p}-e^{-\pi p}\frac{1}{p}$, $Y(p)=\frac{p}{p^2+4}+\frac{1}{p^2+4}-\frac{1}{p^2+4}e^{-\pi p}$, hence
$y(t)=\cos(2t)+\frac{1}{2}\sin(2t)-\frac{1}{2}\sin(2t)\big|_{t-\pi}H(t-\pi)=\cos(2t)+\frac{1}{2}\sin(2t)-\frac{1}{2}\sin(2(t-\pi))H(t-\pi)$
$=\cos(2t)+\frac{1}{2}\sin(2t)-\frac{1}{2}\sin(2t-2\pi)H(t-\pi)=\cos(2t)+\frac{1}{2}\sin(2t)-\frac{1}{2}\sin(2t)H(t-\pi)$
$=\begin{cases}\cos(2t)+\frac{1}{2}\sin(2t), & t\in[0,\pi);\\ \cos(2t), & t\ge\pi.\end{cases}$

23. $p^2Y-0p-1+3(Y-0)+2Y=\frac{2}{p^2}-\mathcal{L}\{2(x+2)\}e^{-2p}=\frac{2}{p^2}-\mathcal{L}\{2x+4\}e^{-2p}=\frac{2}{p^2}-\big(\frac{2}{p^2}+\frac{4}{p}\big)e^{-2p}$,
$Y(p)=\frac{1}{(p+1)(p+2)}+\frac{2}{p^2(p+1)(p+2)}-\big(\frac{2}{p^2(p+1)(p+2)}+\frac{4}{p(p+1)(p+2)}\big)e^{-2p}=\frac{1}{p+1}-\frac{1}{p+2}+\big(-\frac{3/2}{p}+\frac{1}{p^2}+\frac{2}{p+1}-\frac{1/2}{p+2}\big)$
$-\big(-\frac{3/2}{p}+\frac{1}{p^2}+\frac{2}{p+1}-\frac{1/2}{p+2}+\frac{2}{p}-\frac{4}{p+1}+\frac{2}{p+2}\big)e^{-2p}=\frac{3}{p+1}-\frac{3/2}{p+2}-\frac{3/2}{p}+\frac{1}{p^2}-\big(\frac{1/2}{p}+\frac{1}{p^2}-\frac{2}{p+1}+\frac{3/2}{p+2}\big)e^{-2p}$,
thus $y(x)=3e^{-x}-\frac{3}{2}e^{-2x}-\frac{3}{2}+x-\big(\frac{1}{2}+x-2e^{-x}+\frac{3}{2}e^{-2x}\big)\big|_{x-2}H(x-2)$
$=3e^{-x}-\frac{3}{2}e^{-2x}-\frac{3}{2}+x-\big(x-\frac{3}{2}-2e^{2-x}+\frac{3}{2}e^{4-2x}\big)H(x-2)$
$=\begin{cases}3e^{-x}-\frac{3}{2}e^{-2x}-\frac{3}{2}+x, & x\in[0,2);\\ (3+2e^2)e^{-x}-\frac{3}{2}(1+e^4)e^{-2x}, & x\ge 2.\end{cases}$

24. $p^2Y-(-2)p-0-Y=\frac{6}{p-2}-\mathcal{L}\{6e^{2(x+1)}\}e^{-p}=\frac{6}{p-2}-\mathcal{L}\{6e^2e^{2x}\}e^{-p}=\frac{6}{p-2}-\frac{6e^2}{p-2}e^{-p}$,

$Y(p) = -\frac{2p}{(p+1)(p-1)} + \frac{6}{(p+1)(p-1)(p-2)} - \frac{6e^2}{(p+1)(p-1)(p-2)}e^{-p} = \big(-\frac{1}{p-1} - \frac{1}{p+1}\big) + \big(\frac{1}{p+1} - \frac{3}{p-1} + \frac{2}{p-2}\big)$
$\quad -\big(\frac{e^2}{p+1} - \frac{3e^2}{p-1} + \frac{2e^2}{p-2}\big)e^{-p} = \frac{2}{p-2} - \frac{4}{p-1} - \big(\frac{e^2}{p+1} - \frac{3e^2}{p-1} + \frac{2e^2}{p-2}\big)e^{-p}$, hence
$y(x) = 2e^{2x} - 4e^x - \big(e^2e^{-x} - 3e^2e^x + 2e^2e^{2x}\big)\big|_{x-1}H(x-1)$
$\quad = 2e^{2x} - 4e^x - \big(e^2e^{1-x} - 3e^2e^{x-1} + 2e^2e^{2x-2}\big)H(x-1) = 2e^{2x} - 4e^x - \big(e^{3-x} - 3e^{x+1} + 2e^{2x}\big)H(x-1)$
$$= \begin{cases} 2e^{2x} - 4e^x, & x \in [0,1); \\ (3e-4)e^x - e^{3-x}, & x \geq 1. \end{cases}$$

25. $p^2X - 0 - 0 + 2pX - 0 + 5X = \frac{5}{p} - \frac{5}{p}e^{-\pi p}$,
$X(p) = \frac{5}{p(p^2+2p+5)} - \frac{5}{p(p^2+2p+5)}e^{-\pi p} = \frac{1}{p} - \frac{p+2}{p^2+2p+5} - \big(\frac{1}{p} - \frac{p+2}{p^2+2p+5}\big)e^{-\pi p}$.
Thus $x(t) = 1 - \mathcal{L}^{-1}\big\{\frac{(p+1)+1}{(p+1)^2+4}\big\} - \big(1 - \mathcal{L}^{-1}\big\{\frac{(p+1)+1}{(p+1)^2+4}\big\}\big)\big|_{t-\pi}H(t-\pi)$
$\quad = 1 - e^{-t}\mathcal{L}^{-1}\big\{\frac{p+1}{p^2+4}\big\} - \big(1 - e^{-t}\mathcal{L}^{-1}\big\{\frac{p+1}{p^2+4}\big\}\big)\big|_{t-\pi}H(t-\pi)$
$\quad = 1 - e^{-t}\big(\cos(2t) + \frac{1}{2}\sin(2t)\big) - \big(1 - e^{-t}\big(\cos(2t) + \frac{1}{2}\sin(2t)\big)\big)\big|_{t-\pi}H(t-\pi)$
$\quad = 1 - e^{-t}\big(\cos(2t) + \frac{1}{2}\sin(2t)\big) - \big(1 - e^{\pi-t}\big(\cos(2t-2\pi) + \frac{1}{2}\sin(2t-2\pi)\big)\big)H(t-\pi)$
$\quad = 1 - e^{-t}\big(\cos(2t) + \frac{1}{2}\sin(2t)\big) - \big(1 - e^{\pi-t}\big(\cos(2t) + \frac{1}{2}\sin(2t)\big)\big)H(t-\pi)$
$$= \begin{cases} 1 - e^{-t}(\cos(2t) + \frac{1}{2}\sin(2t)), & t \in [0,\pi); \\ (e^\pi - 1)e^{-t}(\cos(2t) + \frac{1}{2}\sin(2t)), & t \geq \pi. \end{cases}$$

26. Choose $y(0^+) = a$ and $y'(0^+) = b$ for $a, b \in \mathbb{R}$.
$p^2Y - ap - b + pY - a - 2Y = \frac{3}{p-1} - \mathcal{L}\{3e^{x+1}\}e^{-p} = \frac{3}{p-1} - \mathcal{L}\{3ee^x\}e^{-p} = \frac{3}{p-1} - \frac{3e}{p-1}e^{-p}$,
$Y(p) = \frac{ap+(a+b)}{(p+2)(p-1)} + \frac{3}{(p+2)(p-1)^2} - \frac{3e}{(p+2)(p-1)^2}e^{-p} = \big(\frac{(a-b)/3}{p+2} + \frac{(2a+b)/3}{p-1}\big) + \big(\frac{1/3}{p+2} - \frac{1/3}{p-1} + \frac{1}{(p-1)^2}\big)$
$\quad -\big(\frac{e/3}{p+2} - \frac{e/3}{p-1} + \frac{e}{(p-1)^2}\big)e^{-p} = \frac{(a-b+1)/3}{p+2} + \frac{(2a+b-1)/3}{p-1} + \frac{1}{(p-1)^2} - \big(\frac{e/3}{p+2} - \frac{e/3}{p-1} + \frac{e}{(p-1)^2}\big)$, hence
$y(x) = \frac{a-b+1}{3}e^{-2x} + \frac{2a+b-1}{3}e^x + e^x\mathcal{L}^{-1}\big\{\frac{1}{p^2}\big\} - \big(\frac{e}{3}e^{-2x} - \frac{e}{3}e^x + e^x\mathcal{L}^{-1}\big\{\frac{e}{p^2}\big\}\big)\big|_{x-1}H(x-1)$
$\quad = \frac{a-b+1}{3}e^{-2x} + \frac{2a+b-1}{3}e^x + x\,e^x - \big(\frac{e}{3}e^{-2x} - \frac{e}{3}e^x + x\,e^x\big)\big|_{x-1}H(x-1)$
$\quad = \frac{a-b+1}{3}e^{-2x} + \frac{2a+b-1}{3}e^x + x\,e^x - \big(\frac{e}{3}e^{2-2x} - \frac{e}{3}e^{x-1} + (x-1)e^{x-1}\big)H(x-1)$
$$= \begin{cases} \frac{a-b+1}{3}e^{-2x} + \frac{2a+b-1}{3}e^x + x\,e^x, & x \in [0,1); \\ \frac{a-b+1-e^3}{3}e^{-2x} + \big(\frac{2a+b}{3} + \frac{1}{e}\big)e^x + \big(1 - \frac{1}{e}\big)x\,e^x, & x \geq 1. \end{cases}$$

27. Choose $x(0^+) = a$ and $\dot{x}(0^+) = b$ for $a, b \in \mathbb{R}$.
$p^2X - ap - b + X = \frac{2p}{p^2+1} - \mathcal{L}\{2\cos(t+\pi)\}e^{-\pi p} = \frac{2p}{p^2+1} + \mathcal{L}\{2\cos(t)\}e^{-\pi p} = \frac{2p}{p^2+1} + \frac{2p}{p^2+1}e^{-\pi p}$,
$X(p) = \frac{ap+b}{p^2+1} + \frac{2p}{(p^2+1)^2} + \frac{2p}{(p^2+1)^2}e^{-\pi p}$.
Note that $\frac{2p}{(p^2+1)^2}$ is already a partial fraction!
Thus $x(t) = a\cos(t) + b\sin(t) + \mathcal{L}^{-1}\big\{-\big[\frac{1}{p^2+1}\big]'\big\} - \mathcal{L}^{-1}\big\{-\big[\frac{1}{p^2+1}\big]'\big\}\big|_{t-\pi}H(t-\pi)$
$\quad = a\cos(t) + b\sin(t) + t\mathcal{L}^{-1}\big\{\frac{1}{p^2+1}\big\} - \big(t\mathcal{L}^{-1}\big\{\frac{1}{p^2+1}\big\}\big)\big|_{t-\pi}H(t-\pi)$
$\quad = a\cos(t) + b\sin(t) + t\sin(t) - \big(t\sin(t)\big)\big|_{t-\pi}H(t-\pi)$
$\quad = a\cos(t) + b\sin(t) + t\sin(t) - (t-\pi)\sin(t-\pi)H(t-\pi)$
$\quad = a\cos(t) + b\sin(t) + t\sin(t) + (t-\pi)\sin(t)H(t-\pi)$
$$= \begin{cases} a\cos(t) + b\sin(t) + t\sin(t), & t \in [0,\pi); \\ a\cos(t) + b\sin(t) + (2t-\pi)\sin(t), & t \geq \pi. \end{cases}$$