Practice problems on Fourier series
Brief solutions

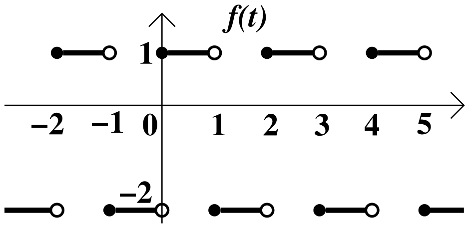
1. Fourier series: $T=2$, $\omega=\pi$. $a_0=\frac{2}{2}\Big(\int_0^1 1dt-2\int_1^2 1dt\Big)=-1.$

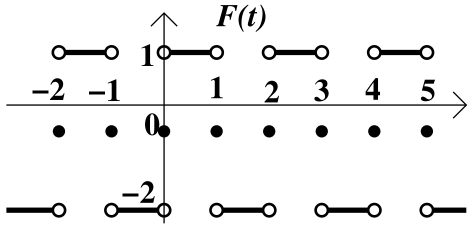
$a_k=\frac{2}{2}\Big(\int_0^1\cos(k\pi t)\,dt-2\int_1^2\cos(k\pi t)\,dt\Big)=0.$

$b_k=\frac{2}{2}\Big(\int_0^1\sin(k\pi t)\,dt-2\int_1^2\sin(k\pi t)\,dt\Big)=\frac{3}{k\pi}[1-(-1)^k]=\begin{cases}0, & k\text{ even},\\ \frac{6}{k\pi}, & k\text{ odd}.\end{cases}$

$f\sim-\frac{1}{2}+\sum\limits_{k=1}^{\infty}\frac{3}{k\pi}[1-(-1)^k]\sin(k\pi t)=-\frac{1}{2}+\sum\limits_{k=0}^{\infty}\frac{6}{(2k+1)\pi}\sin((2k+1)\pi t).$

Sum: Periodic extension of f on the left, sum on the right:





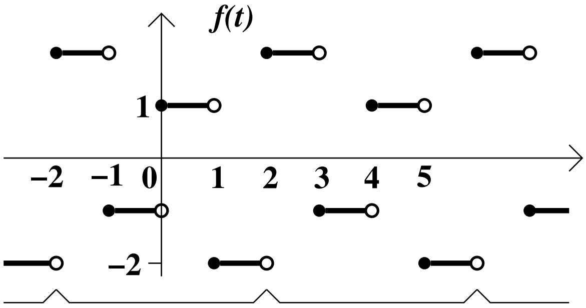
Remark: f is of the form "odd"$-\frac{1}{2}$, hence the series is of the form "sine series"$-\frac{1}{2}$.

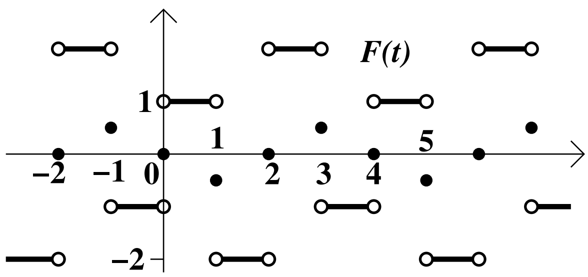
sine Fourier series: $L=2$, $T=4$, $\omega=\frac{\pi}{2}$. $a_k=0$.

$b_k=\frac{2}{2}\Big(\int_0^1\sin\big(k\frac{\pi}{2}t\big)\,dt-2\int_1^2\sin\big(k\frac{\pi}{2}t\big)\,dt\Big)=\frac{2}{k\pi}\big[1+2(-1)^k-3\cos\big(k\frac{\pi}{2}\big)\big].$

$f\sim\sum\limits_{k=1}^{\infty}\frac{2}{k\pi}\big[1+2(-1)^k-3\cos\big(k\frac{\pi}{2}\big)\big]\sin\big(k\frac{\pi}{2}t\big).$

Sum: **Odd** periodic extension of f on the left, sum on the right:



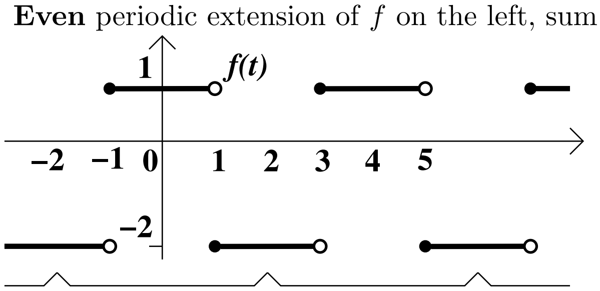


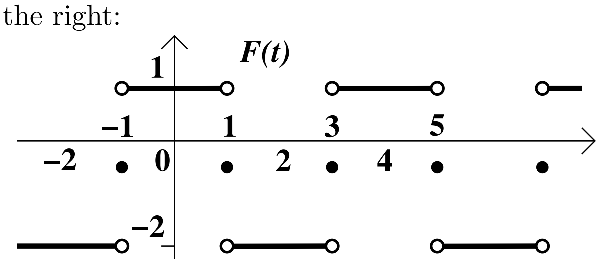
cosine Fourier series: $L=2$, $T=4$, $\omega=\frac{\pi}{2}$. $b_k=0$, $a_0=-1$ (see Fourier series).

$a_k=\frac{2}{2}\Big(\int_0^1\cos\big(k\frac{\pi}{2}t\big)\,dt-2\int_1^2\cos\big(k\frac{\pi}{2}t\big)\,dt\Big)=\frac{6}{k\pi}\sin\big(k\frac{\pi}{2}\big)=\begin{cases}0, & k\text{ even},\\ (-1)^n\frac{6}{k\pi}, & k\text{ odd},\ k=2n+1.\end{cases}$

$f\sim-\frac{1}{2}+\sum\limits_{k=1}^{\infty}\frac{6}{k\pi}\sin\big(k\frac{\pi}{2}\big)\cos\big(k\frac{\pi}{2}t\big)=-\frac{1}{2}+\sum\limits_{k=0}^{\infty}(-1)^k\frac{6}{(2k+1)\pi}\cos\big((2k+1)\frac{\pi}{2}t\big).$

Sum: **Even** periodic extension of f on the left, sum on the right:



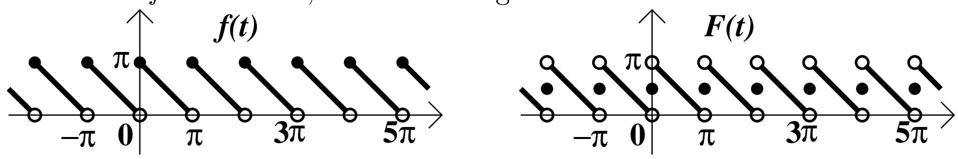


2. Fourier series: $T=\pi$, $\omega=2$. $a_0=\frac{2}{\pi}\int_0^{\pi}(\pi-t)dt=\pi.$

$a_k=\frac{2}{\pi}\int_0^{\pi}(\pi-t)\cos(2kt)\,dt=0$, $b_k=\frac{2}{\pi}\int_0^{\pi}(\pi-t)\sin(2kt)\,dt=\frac{1}{k}$ (using by parts).

$f\sim\frac{\pi}{2}+\sum\limits_{k=1}^{\infty}\frac{1}{k}\sin(2kt).$

Sum: Periodic extension of f on the left, sum on the right:

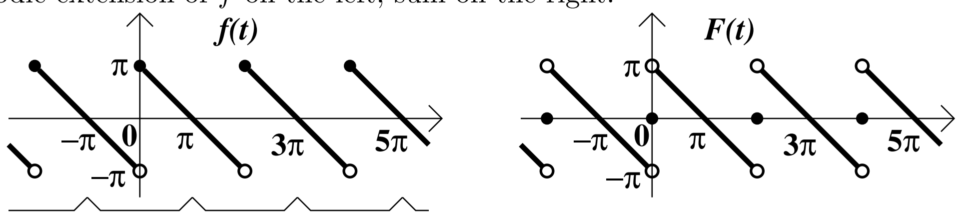


Remark: f is of the form "odd"$+\frac{\pi}{2}$, hence the series is of the form "sine series"$+\frac{\pi}{2}$.

sine Fourier series: $L=\pi$, $T=2\pi$, $\omega=1$. $a_k=0$.

$b_k=\frac{2}{\pi}\int\limits_0^{\pi}(\pi-t)\sin(kt)\,dt=\frac{2}{k}$ (using by parts). $f\sim\sum\limits_{k=1}^{\infty}\frac{2}{k}\sin(kt)$.

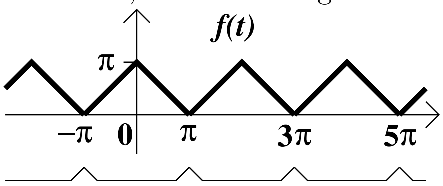
Sum: **Odd** periodic extension of f on the left, sum on the right:



cosine Fourier series: $L=\pi$, $T=2\pi$, $\omega=1$. $b_k=0$, $a_0=\pi$.

$a_k=\frac{2}{\pi}\int\limits_0^{\pi}(\pi-t)\cos(kt)\,dt=\frac{4}{k^2\pi}$ (using by parts). $f\sim\frac{\pi}{2}+\sum\limits_{k=1}^{\infty}\frac{4}{k^2\pi}\cos(kt)$.

Sum: **Even** periodic extension of f on the left, sum on the right:



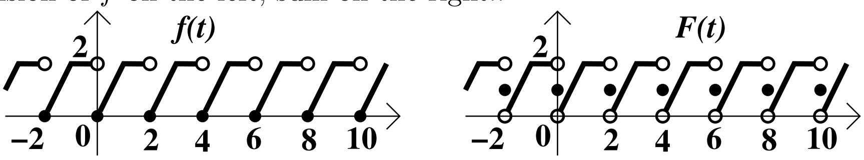
3. Fourier series: $T=2$, $\omega=\pi$. $a_0=\frac{2}{2}\Big(\int\limits_0^1 2t\,dt+2\int\limits_1^2 1\,dt\Big)=3$.

$$a_k=\frac{2}{2}\Big(\int\limits_0^1 2t\cos(k\pi t)\,dt+2\int\limits_1^2\cos(k\pi t)\,dt\Big)=\frac{2}{k\pi}[(-1)^k-1]=\begin{cases}0, & k\text{ even},\\ -\frac{4}{k\pi}, & k\text{ odd}\end{cases}\quad\text{(using by parts).}$$

$$b_k=\frac{2}{2}\Big(\int\limits_0^1 2t\sin(k\pi t)\,dt+2\int\limits_1^2\sin(k\pi t)\,dt\Big)=-\frac{2}{k\pi}\quad\text{(using by parts).}$$

$$f\sim\frac{\pi}{2}+\sum_{k=1}^{\infty}\Big(\frac{2}{k\pi}[(-1)^k-1]\cos(k\pi t)-\frac{2}{k\pi}\sin(k\pi t)\Big)=\frac{\pi}{2}-\sum_{k=0}^{\infty}\frac{-4}{(2k+1)\pi}\cos((2k+1)\pi t)-\sum_{k=1}^{\infty}\frac{2}{k\pi}\sin(k\pi t).$$

Sum: Periodic extension of f on the left, sum on the right::

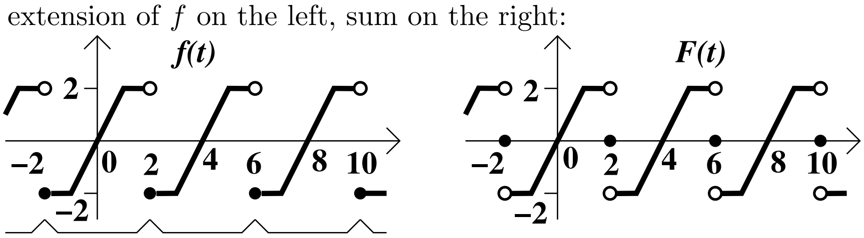


sine Fourier series: $L=2$, $T=4$, $\omega=\frac{\pi}{2}$. $a_k=0$.

$$b_k=\frac{2}{2}\Big(\int\limits_0^1 2t\sin\big(k\tfrac{\pi}{2}t\big)\,dt+2\int\limits_1^2\sin\big(k\tfrac{\pi}{2}t\big)\,dt\Big)=\frac{4}{k^2\pi^2}\sin\big(k\tfrac{\pi}{2}\big)-\frac{4}{k\pi}(-1)^k\quad\text{(using by parts).}$$

$$f\sim\sum_{k=1}^{\infty}\Big[\frac{4}{k^2\pi^2}\sin\big(k\tfrac{\pi}{2}\big)-\frac{4}{k\pi}(-1)^k\Big]\sin\big(k\tfrac{\pi}{2}t\big).$$

Sum: **Odd** periodic extension of f on the left, sum on the right:

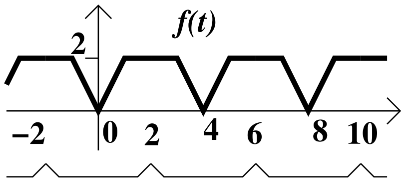


cosine Fourier series: $L=2$, $T=4$, $\omega=\frac{\pi}{2}$. $b_k=0$, $a_0=3$.

$$a_k=\frac{2}{2}\Big(\int\limits_0^1 2t\cos\big(k\tfrac{\pi}{2}t\big)\,dt+2\int\limits_1^2\cos\big(k\tfrac{\pi}{2}t\big)\,dt\Big)=\frac{8}{k^2\pi^2}\Big[\cos\big(k\tfrac{\pi}{2}\big)-1\Big]\quad\text{(using by parts).}$$

$f \sim \frac{3}{2} + \sum\limits_{k=1}^{\infty} \frac{8}{k^2\pi^2}\bigl[\cos\bigl(k\frac{\pi}{2}\bigr) - 1\bigr]\cos\bigl(k\frac{\pi}{2}t\bigr)$.

Sum: **Even** periodic extension of f, it is also the sum of the series:



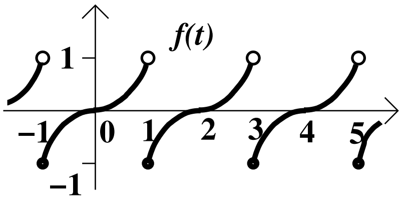
4. Fourier series: $T = 2$, $\omega = \pi$. $a_0 = \frac{2}{2}\int\limits_{-1}^{1} t^3\,dt = 0$.

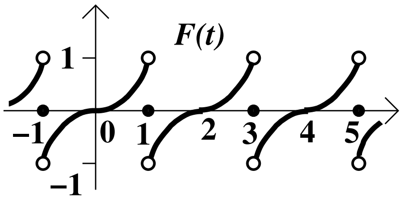
$a_k = \frac{2}{2}\int\limits_{-1}^{1} t^3\cos(k\pi t)\,dt = 0$ (by parts three times).

$b_k = \frac{2}{2}\int\limits_{-1}^{1} t^3\sin(k\pi t)\,dt = -\frac{2}{k\pi}(-1)^k + \frac{12}{k^3\pi^3}$ (by parts three times).

$f \sim \sum\limits_{k=1}^{\infty}\bigl[\frac{12}{k^3\pi^3} - (-1)^k\frac{2}{k\pi}\bigr]\sin(k\pi t)$.

Sum: Periodic extension of f on the left, sum on the right::





sine, cosine Fourier series: The function was defined on an interval that extends "across" the origin, so it includes both positive and negative numbers. Thus we do not have a chance to modify the function to make it symmetric, we can only hope that it already has some symmetry as given. We see that it is odd, therefore also its extension is odd and thus we get a sine series, it is not possible to make a cosine series.

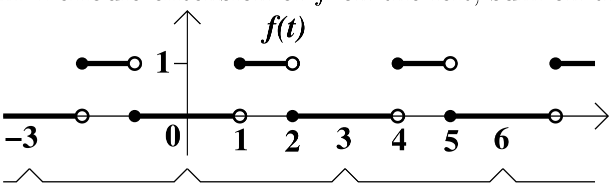
5. Fourier series: $T = 3$, $\omega = \frac{2}{3}\pi$. $a_0 = \frac{2}{3}\int\limits_{1}^{2} 1dt = \frac{2}{3}$.

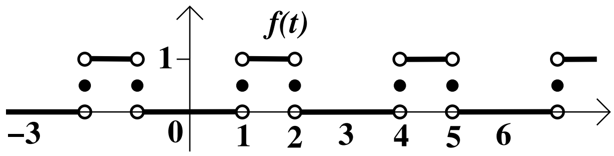
$a_k = \frac{2}{3}\int\limits_{1}^{2}\cos\bigl(\frac{2}{3}k\pi t\bigr)\,dt = \frac{1}{k\pi}\bigl[\sin\bigl(\frac{4\pi}{3}k\bigr) - \sin\bigl(\frac{2\pi}{3}k\bigr)\bigr]$.

$b_k = \frac{2}{3}\int\limits_{1}^{2}\sin\bigl(\frac{2}{3}k\pi t\bigr)\,dt = \frac{1}{k\pi}\bigl[\cos\bigl(\frac{2\pi}{3}k\bigr) - \cos\bigl(\frac{4\pi}{3}k\bigr)\bigr]$.

$f \sim \frac{1}{3} + \sum\limits_{k=1}^{\infty}\bigl[\frac{1}{k\pi}\bigl[\sin\bigl(\frac{4\pi}{3}k\bigr) - \sin\bigl(\frac{2\pi}{3}k\bigr)\bigr]\cos\bigl(\frac{2}{3}k\pi t\bigr) + \frac{1}{k\pi}\bigl[\cos\bigl(\frac{2\pi}{3}k\bigr) - \cos\bigl(\frac{4\pi}{3}k\bigr)\bigr]\sin\bigl(\frac{2}{3}k\pi t\bigr)\bigr]$.

Sum: Periodic extension of f on the left, sum on the right:





Looking at the picture we see that the resulting extension is even, so we should have a cosine series. Is it so? Let's check on the coefficient b_k. Due to periodicity it is enough to investigate it for $k = 1, 2, 3$ and we see that it is always zero. So we have a cosine series indeed:

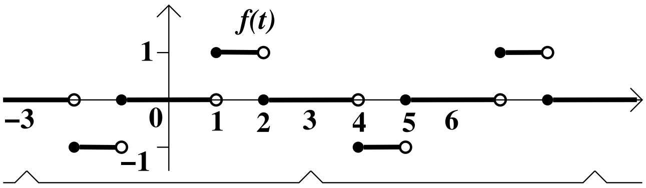
$f \sim \frac{1}{3} + \sum\limits_{k=1}^{\infty}\frac{1}{k\pi}\bigl[\sin\bigl(\frac{4\pi}{3}k\bigr) - \sin\bigl(\frac{2\pi}{3}k\bigr)\bigr]\cos\bigl(\frac{2}{3}k\pi t\bigr)$.

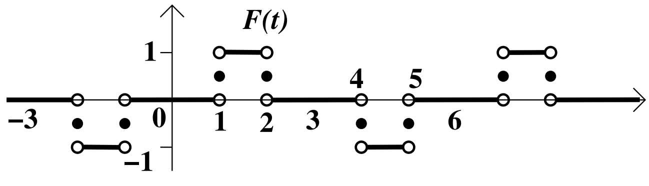
sine Fourier series: $L = 3$, $T = 6$, $\omega = \frac{\pi}{3}$. $a_k = 0$.

$b_k = \frac{2}{3}\int\limits_{1}^{2}\sin\bigl(\frac{\pi}{3}kt\bigr)\,dt = \frac{2}{k\pi}\bigl[\cos\bigl(\frac{\pi}{3}k\bigr) - \cos\bigl(\frac{2\pi}{3}k\bigr)\bigr]$.

$f \sim \sum\limits_{k=1}^{\infty}\frac{2}{k\pi}\bigl[\cos\bigl(\frac{\pi}{3}k\bigr) - \cos\bigl(\frac{2\pi}{3}k\bigr)\bigr]\sin\bigl(\frac{\pi}{3}kt\bigr)$.

Sum: **Odd** periodic extension of f on the left, sum on the right:





cosine Fourier series: We already got one in the first part. Now we try it the standard way. $L = 3$, $T = 6$, $\omega = \frac{\pi}{3}$. $b_k = 0$, $a_0 = \frac{2}{3}$ (see Fourier series).

$a_k = \frac{2}{3}\int_1^2 \cos\left(\frac{\pi}{3}kt\right)dt = \frac{2}{k\pi}\left[\sin\left(\frac{2\pi}{3}k\right) - \sin\left(\frac{\pi}{3}k\right)\right]$.

$f \sim \frac{1}{3} + \sum_{k=1}^{\infty} \frac{2}{k\pi}\left[\sin\left(\frac{2\pi}{3}k\right) - \sin\left(\frac{\pi}{3}k\right)\right]\cos\left(\frac{\pi}{3}kt\right)$.

The sum is as before. How come we have a different formula? Actually, in fact it is not a different formula. It is easy to check that when k is odd, then the coefficients are zero (due to periodicity it is enough to check for $k = 1, 3, 5$). Writing $k = 2n$ we arrive at the same series as before.

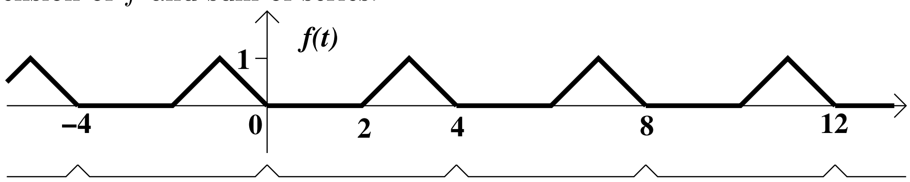
6. Fourier series: $T = 4$, $\omega = \frac{\pi}{2}$. $a_0 = \frac{2}{4}\left(\int_2^3 (t-2)dt + \int_3^4 (4-t)dt\right) = 1$.

$a_k = \frac{2}{4}\left(\int_2^3 (t-2)\cos\left(\frac{\pi}{2}kt\right)dt + \int_3^4 (4-t)\cos\left(\frac{\pi}{2}kt\right)dt\right) = \frac{2}{k^2\pi^2}\left[2\cos\left(\frac{3\pi}{2}k\right) - \cos(\pi k) - \cos(2\pi k)\right]$.

$b_k = \frac{2}{4}\left(\int_2^3 (t-2)\sin\left(\frac{\pi}{2}kt\right)dt + \int_3^4 (4-t)\sin\left(\frac{\pi}{2}kt\right)dt\right) = \frac{4}{k^2\pi^2}\sin\left(\frac{3\pi}{2}k\right)$.

$f \sim \frac{1}{2} + \sum_{k=1}^{\infty}\left[\frac{2}{k^2\pi^2}\left[2\cos\left(\frac{3\pi}{2}k\right) - 1 - (-1)^k\right]\cos\left(\frac{\pi}{2}kt\right) + \frac{4}{k^2\pi^2}\sin\left(\frac{3\pi}{2}k\right)\sin\left(\frac{\pi}{2}kt\right)\right]$.

Sum: Periodic extension of f and sum of series:

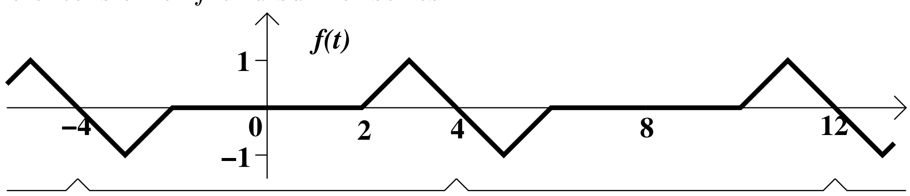


sine Fourier series: $L = 4$, $T = 8$, $\omega = \frac{\pi}{4}$. $a_k = 0$.

$b_k = \frac{2}{4}\left(\int_2^3 (t-2)\sin\left(\frac{\pi}{4}kt\right)dt + \int_3^4 (4-t)\sin\left(\frac{\pi}{4}kt\right)dt\right) = \frac{8}{k^2\pi^2}\left[2\sin\left(\frac{3\pi}{4}k\right) - \sin\left(\frac{\pi}{2}k\right)\right]$

$f \sim \sum_{k=1}^{\infty} \frac{8}{k^2\pi^2}\left[2\sin\left(\frac{3\pi}{4}k\right) - \sin\left(\frac{\pi}{2}k\right)\right]\sin\left(\frac{\pi}{4}kt\right)$.

Sum: **Odd** periodic extension of f and sum of series:



cosine Fourier series: $L = 4$, $T = 8$, $\omega = \frac{\pi}{4}$. $b_k = 0$, $a_0 = 1$ (see Fourier series).

$a_k = \frac{2}{4}\left(\int_2^3 (t-2)\cos\left(\frac{\pi}{4}kt\right)dt + \int_3^4 (4-t)\cos\left(\frac{\pi}{4}kt\right)dt\right) = \frac{8}{k^2\pi^2}\left[2\cos\left(\frac{3\pi}{4}k\right) - \cos\left(\frac{\pi}{2}k\right) - \cos(\pi k)\right]$.

$f \sim \frac{1}{2} + \sum_{k=1}^{\infty} \frac{8}{k^2\pi^2}\left[2\cos\left(\frac{3\pi}{4}k\right) - \cos\left(\frac{\pi}{2}k\right) - (-1)^k\right]\cos\left(\frac{\pi}{4}kt\right)$.

Sum: **Even** periodic extension of f and sum of series: