

Řešené příklady na Laplaceovu transformaci

1. Pomocí Laplaceovy transformace řešte rovnici

$$y'' + 3y' + 2y = 2e^{-3x} \chi_{(1,2)} = \begin{cases} 2e^{-3x}, & x \in \langle 1, 2 \rangle; \\ 0, & \text{jinde,} \end{cases} \quad y(0^+) = 0, \quad y'(0^+) = 1.$$

2. Pomocí Laplaceovy transformace řešte rovnici

$$y' + y = (2t + 3) \chi_{(0,1)} = \begin{cases} 2t + 3, & t \in \langle 0, 1 \rangle; \\ 0, & \text{jinde,} \end{cases} \quad y(0^+) = 1.$$

3. Pomocí Laplaceovy transformace řešte rovnici

$$\dot{x} + 2 \int_0^t x(u) du = 2t + 3, \quad x(0^+) = 0.$$

4. Najděte Laplaceovu transformaci funkce

$$f(t) = t e^{-3t} \sin(2t) + \frac{d}{dt} [1 + \sinh(t)] + \int_0^t (u^2 + \sin(u) e^{3(t-u)}) du.$$

5. Najděte inverzní Laplaceovu transformaci funkce

$$F(p) = \frac{p + 1 - e^{-\pi p}}{p^2 + 4p + 13}.$$

Řešení:

1. Zlapišme s volbou $\mathcal{L}\{y\} = Y$, pravá strana je konečný signál, urazíme to přes Heavisideovy funkce:

$$\begin{aligned} p^2 Y - 0p - 1 + 3(pY - 0) + 2Y &= \mathcal{L}\{2e^{-3x}[H(x-1) - H(x-2)]\} \\ &= e^{-p} \mathcal{L}\{2e^{-3(x+1)}\} - e^{-2p} \mathcal{L}\{2e^{-3(x+2)}\} = e^{-p} 2e^{-3} \mathcal{L}\{e^{-3x}\} - e^{-2p} 2e^{-6} \mathcal{L}\{e^{-3x}\} \\ &= e^{-p} 2e^{-3} \frac{1}{p+3} - e^{-2p} 2e^{-6} \frac{1}{p+3}. \end{aligned}$$

Proto $Y(p^2 + 3p + 2) = \frac{2e^{-3}}{p+3} e^{-p} - \frac{2e^{-6}}{p+3} e^{-2p} + 1$, tedy

$$Y = \frac{2e^{-3}}{(p+1)(p+2)(p+3)} e^{-p} - \frac{2e^{-6}}{(p+1)(p+2)(p+3)} e^{-2p} + \frac{1}{(p+1)(p+2)}.$$

Parciální zlomky udělají

$$Y(p) = \frac{1}{p+1} - \frac{1}{p+2} + \left(\frac{e^{-3}}{p+1} - \frac{2e^{-3}}{p+2} + \frac{e^{-3}}{p+3} \right) e^{-p} - \left(\frac{e^{-6}}{p+1} - \frac{2e^{-6}}{p+2} + \frac{e^{-6}}{p+3} \right) e^{-2p},$$

tedy

$$\begin{aligned} y(x) &= \mathcal{L}^{-1}\{Y(p)\} = e^{-x} - e^{-2x} + \mathcal{L}^{-1}\left\{ \frac{e^{-3}}{p+1} - \frac{2e^{-3}}{p+2} + \frac{e^{-3}}{p+3} \right\} \Big|_{x-1} H(x-1) \\ &\quad - \mathcal{L}^{-1}\left\{ \frac{e^{-6}}{p+1} - \frac{2e^{-6}}{p+2} + \frac{e^{-6}}{p+3} \right\} \Big|_{x-2} H(x-2) \\ &= e^{-x} - e^{-2x} + (e^{-3} e^{-x} - 2e^{-3} e^{-2x} + e^{-3} e^{-3x}) \Big|_{x-1} H(x-1) \end{aligned}$$

$$\begin{aligned}
& - (e^{-6}e^{-x} - 2e^{-6}e^{-2x} + e^{-6}e^{-3x}) \Big|_{x=2} H(x-2) \\
= & e^{-x} - e^{-2x} + (e^{-3}e^{1-x} - 2e^{-3}e^{2-2x} + e^{-3}e^{3-3x})H(x-1) \\
& - (e^{-6}e^{2-x} - 2e^{-6}e^{4-2x} + e^{-6}e^{6-3x})H(x-2) \\
= & \begin{cases} e^{-x} - e^{-2x}, & x \in \langle 0, 1 \rangle, \\ e^{-x} - e^{-2x} + (e^{-2}e^{-x} - 2e^{-1}e^{-2x} + e^{-3x}) \\ \quad = (1 + e^{-2})e^{-x} - (1 + 2e^{-1})e^{-2x} + e^{-3x}, & x \in \langle 1, 2 \rangle, \\ (1 + e^{-2})e^{-x} - (1 + 2e^{-1})e^{-2x} + e^{-3x} - (e^{-4}e^{-x} - 2e^{-2}e^{-2x} + e^{-3x}) \\ \quad = (1 + e^{-2} - e^{-4})e^{-x} - (1 + 2e^{-1} - 2e^{-2})e^{-2x}, & x \geq 2. \end{cases}
\end{aligned}$$

Poznámka: Pravá strana šla docela dobře laplasit i přes definici:

$$\begin{aligned}
\mathcal{L}\{2e^{-3x}\chi_{(1,2)}\} &= \int_1^2 2e^{-3x}e^{-px} dx = 2 \int_1^2 e^{-(p+3)x} dx = 2 \left[\frac{e^{-(p+3)x}}{-(p+3)} \right]_{x=1}^{x=2} \\
&= \frac{2e^{-(p+3)}}{p+3} - \frac{2e^{-2(p+3)}}{p+3} = \frac{2e^{-3}e^{-p}}{p+3} - \frac{2e^{-6}e^{-2p}}{p+3}.
\end{aligned}$$

2. Zlaplasíme s volbou $\mathcal{L}\{y\} = Y$, pravá strana je konečný signál, urazíme to přes Heavisideovy funkce:

$$\begin{aligned}
pY - 1 + Y &= \mathcal{L}\{(2t+3)[H(t) - H(t-1)]\} = \frac{2}{p^2} + \frac{3}{p} - e^{-p}\mathcal{L}\{2(t+1) + 3\} \\
&= \frac{2}{p^2} + \frac{3}{p} - e^{-p}\mathcal{L}\{2t+5\} = \frac{2}{p^2} + \frac{3}{p} - \left(\frac{2}{p^2} + \frac{5}{p}\right)e^{-p}.
\end{aligned}$$

Proto $Y(p+1) = 1 + \frac{2}{p^2} + \frac{3}{p} - \left(\frac{2}{p^2} + \frac{5}{p}\right)e^{-p}$, tedy $Y = \frac{1}{p+1} + \frac{2}{p^2(p+1)} + \frac{3}{p(p+1)} - \left(\frac{2}{p^2(p+1)} + \frac{5}{p(p+1)}\right)e^{-p}$. Parciální zlomky udělají

$$Y(p) = \frac{1}{p+1} + \left(\frac{2}{p+1} - \frac{2}{p} + \frac{2}{p^2}\right) + \left(\frac{3}{p} - \frac{3}{p+1}\right) - \left(\frac{2}{p+1} - \frac{2}{p} + \frac{2}{p^2} + \frac{5}{p} - \frac{5}{p+1}\right)e^{-p} = \frac{1}{p} + \frac{2}{p^2} - \left(\frac{3}{p} + \frac{2}{p^2} - \frac{3}{p+1}\right)e^{-p},$$

tedy

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\{Y(p)\} = 1 + 2t - \mathcal{L}^{-1}\left\{\frac{3}{p} + \frac{2}{p^2} - \frac{3}{p+1}\right\} \Big|_{t-1} H(t-1) \\
&= 1 + 2t - (3 + 2t - 3e^{-t}) \Big|_{t-1} H(t-1) = 1 + 2t - (3 + 2(t-1) - 3e^{1-t})H(t-1) \\
&= 1 + 2t - (1 + 2t - 3e^{1-t})H(t-1) = \begin{cases} 1 + 2t, & t \in \langle 0, 1 \rangle, \\ 3e^{1-t}, & t \geq 1. \end{cases}
\end{aligned}$$

3. Zlaplasíme s volbou $\mathcal{L}\{x\} = X$:

$$pX + \frac{2}{p}X = \frac{2}{p^2} + \frac{3}{p}.$$

Proto

$$p^2X + 2X = \frac{2+3p}{p} \implies (p^2 + 2)X = \frac{2+3p}{p},$$

tedy $X = \frac{2+3p}{(p^2+2)p}$. Parciální zlomky udělají

$$X(p) = \frac{1}{p} - \frac{p}{p^2+2} + \frac{3}{p^2+2},$$

tedy

$$\begin{aligned}
x(t) &= \mathcal{L}^{-1}\{X(p)\} = 1 + \mathcal{L}^{-1}\left\{-\frac{p}{p^2+(\sqrt{2})^2} + \frac{3}{\sqrt{2}} \frac{\sqrt{2}}{p^2+(\sqrt{2})^2}\right\} \\
&= 1 - \cos(\sqrt{2}t) + \frac{3}{\sqrt{2}} \sin(\sqrt{2}t).
\end{aligned}$$

4.

$$\begin{aligned}
\mathcal{L}\{f(t)\} &= \mathcal{L}\{t e^{-3t} \sin(2t)\} + \mathcal{L}\left\{\frac{d}{dt}[1 + \sinh(t)]\right\} + \mathcal{L}\left\{\int_0^t u^2 du\right\} + \mathcal{L}\left\{\int_0^t \sin(u) e^{3(t-u)} du\right\} \\
&= -\frac{d}{dp} \mathcal{L}\{e^{-3t} \sin(2t)\} + p \mathcal{L}\{1 + \sinh(t)\} - [1 + \sinh(t)]|_{t=0+} + \frac{1}{p} \mathcal{L}\{u^2\} + \mathcal{L}\{\sin(t) * e^{3t}\} \\
&= -\frac{d}{dp} (\mathcal{L}\{\sin(2t)\})|_{p+3} + p \mathcal{L}\left\{1 + \frac{1}{2}(e^t - e^{-t})\right\} - [1 + \sinh(0)] + \frac{1}{p} \frac{2}{p^3} + \mathcal{L}\{\sin(t)\} \cdot \mathcal{L}\{e^{3t}\} \\
&= -\frac{d}{dp} \left(\frac{2}{p^2+4}\right)|_{p+3} + p \left(\frac{1}{p} + \frac{1}{2} \frac{1}{p-1} - \frac{1}{2} \frac{1}{p+1}\right) - 1 + \frac{2}{p^4} + \frac{1}{p^2+1} \frac{1}{p-3} \\
&= -\frac{d}{dp} \left(\frac{2}{(p+3)^2+4}\right) + 1 + \frac{p/2}{p-1} - \frac{p/2}{p+1} - 1 + \frac{2}{p^4} + \frac{1}{(p^2+1)(p-3)} \\
&= \frac{4(p+3)}{[(p+3)^2+4]^2} + \frac{\frac{1}{2}p}{p-1} - \frac{\frac{1}{2}p}{p+1} + \frac{2}{p^4} + \frac{1}{(p^2+1)(p-3)}.
\end{aligned}$$

5.

$$\begin{aligned}
\mathcal{L}^{-1}\{F(p)\} &= \mathcal{L}^{-1}\left\{\frac{p+1-e^{-\pi p}}{p^2+4p+13}\right\} = \mathcal{L}^{-1}\left\{\frac{p+1}{(p+2)^2+9}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(p+2)^2+9} e^{-\pi p}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{p+2-1}{(p+2)^2+9}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(p+2)^2+9}\right\}|_{t-\pi} H(t-\pi) \\
&= e^{-2t} \mathcal{L}^{-1}\left\{\frac{p-1}{p^2+9}\right\} - \left(e^{-2t} \mathcal{L}^{-1}\left\{\frac{1}{p^2+9}\right\}\right)|_{t-\pi} H(t-\pi) \\
&= e^{-2t} \mathcal{L}^{-1}\left\{\frac{p}{p^2+9} - \frac{1}{3} \frac{3}{p^2+9}\right\} - \left(e^{-2t} \mathcal{L}^{-1}\left\{\frac{1}{3} \frac{3}{p^2+9}\right\}\right)|_{t-\pi} H(t-\pi) \\
&= e^{-2t} (\cos(3t) - \frac{1}{3} \sin(3t)) - \left(e^{-2t} \frac{1}{3} \sin(3t)\right)|_{t-\pi} H(t-\pi) \\
&= e^{-2t} \cos(3t) - \frac{e^{-2t}}{3} \sin(3t) - \frac{e^{-2(t-\pi)}}{3} \sin(3(t-\pi)) H(t-\pi) \\
&= e^{-2t} \cos(3t) - \frac{e^{-2t}}{3} \sin(3t) - \frac{e^{2\pi-2t}}{3} \sin(3t-3\pi) H(t-\pi) \\
&= e^{-2t} \cos(3t) - \frac{e^{-2t}}{3} \sin(3t) + \frac{e^{2\pi-2t}}{3} \sin(3t) H(t-\pi) \\
&= \begin{cases} e^{-2t} \cos(3t) - \frac{1}{3} e^{-2t} \sin(3t), & t \in \langle 0, \pi \rangle, \\ e^{-2t} \cos(3t) + \frac{1}{3} (e^{2\pi} - 1) e^{-2t} \sin(3t), & t \geq \pi. \end{cases}
\end{aligned}$$

Poznámka: $\cos(3t - 3\pi)$ se dělá pomocí $\cos(\alpha - 3\pi) = -\cos(\alpha)$ s volbou $\alpha = 3t$. Podobně $\cos(\alpha \pm \pi) = -\cos(\alpha)$, $\cos(\alpha \pm \frac{\pi}{2}) = \mp \sin(\alpha)$, $\sin(\alpha \pm \pi) = -\sin(\alpha)$, $\sin(\alpha \pm \frac{\pi}{2}) = \cos(\alpha)$.