

Cvičné příklady na Laplaceovu transformaci
Stručná řešení

$$1. = \mathcal{L}\{t^5\}|_{p-2} = \frac{5!}{p^6}|_{p-2} = \frac{5!}{(p-2)^6} \text{ nebo } -\left[\frac{1}{p-2}\right]''''''.$$

$$2. = -\left[\frac{1}{2}\mathcal{L}\{\sin(\omega t)\}\right]' = -\frac{1}{2}\left[\frac{\omega}{p^2+\omega^2}\right]' = \frac{\omega p}{(p^2+\omega^2)^2}.$$

$$3. = e^{-\frac{\pi}{2}p}\mathcal{L}\{\sin(3(t+\frac{\pi}{2}))\} = e^{-\frac{\pi}{2}p}\mathcal{L}\{\sin(3t+\frac{3\pi}{2})\} = -e^{-\frac{\pi}{2}p}\mathcal{L}\{\cos(3t)\} = -e^{-\frac{\pi}{2}p}\frac{p}{p^2+9}.$$

$$4. = \mathcal{L}\{t[H(t)-H(t-1)]\} + \mathcal{L}\{H(t-1)\} = \frac{1}{p^2} - e^{-p}\mathcal{L}\{t+1\} + e^{-p}\mathcal{L}\{1\} = \frac{1}{p^2} - e^{-p}\frac{1}{p^2}.$$

$$5. = \mathcal{L}\{(1-\frac{t}{2})[H(t)-H(t-2)]\} = (\frac{1}{p} - \frac{1}{2}\frac{1}{p^2}) - e^{-2p}\mathcal{L}\{1-\frac{t+2}{2}\} = (\frac{1}{p} - \frac{1}{2p^2}) + e^{-2p}\frac{1}{2}\mathcal{L}\{t\} = (\frac{1}{p} - \frac{1}{2p^2}) + e^{-2p}\frac{1}{2p^2}.$$

$$6. = \mathcal{L}\{H(t)-2H(t-1)+H(t-2)\} = \frac{1}{p} - 2e^{-p}\mathcal{L}\{1\} + e^{-2p}\mathcal{L}\{1\} = \frac{1}{p}(1-2e^{-p}+e^{-2p}) = \frac{1}{p}(1-e^{-p})^2.$$

$$7. = \mathcal{L}^{-1}\left\{\frac{p-1-1}{(p-1)^2+4}\right\} = e^t\mathcal{L}^{-1}\left\{\frac{p-1}{p^2+4}\right\} = e^t(\cos(2t) - \frac{1}{2}\sin(2t)).$$

$$8. = \frac{1}{2}\sin(2t)|_{t-\frac{\pi}{2}}H(t-\frac{\pi}{2}) = \frac{1}{2}\sin(2(t-\frac{\pi}{2}))H(t-\frac{\pi}{2}) = \frac{1}{2}\sin(2t-\pi)H(t-\frac{\pi}{2}) = -\frac{1}{2}\sin(2t)H(t-\frac{\pi}{2}).$$

$$9. = \mathcal{L}^{-1}\left\{\frac{1}{p-1} - \frac{p-4}{p^2+4}\right\} = e^t - \cos(2t) + 2\sin(2t).$$

$$10. = \mathcal{L}^{-1}\left\{\frac{1}{(p-2)^2}\right\}|_{t-3}H(t-3) = \left(e^{2t}\mathcal{L}^{-1}\left\{\frac{1}{p^2}\right\}\right)|_{t-3}H(t-3) = (e^{2t})|_{t-3}H(t-3) = (t-3)e^{2(t-3)}H(t-3).$$

$$11. p^2Y - 3p - 3 - (pY - 3) - 2Y = \frac{3}{p+2}, Y(p) = \frac{3p}{(p+1)(p-2)} + \frac{3}{(p+1)(p-2)(p+2)} = \frac{9/4}{p-2} + \frac{3/4}{p+2}, \text{ tedy}$$

$$y(x) = \frac{9}{4}e^{2x} + \frac{3}{4}e^{-2x}.$$

Alternativa: odhad pravé strany: $\lambda^2 - \lambda - 2 = 0$, $\lambda = -1, 2$, $y_h(x) = ae^{-x} + be^{2x}$, odhad $y(x) = Ae^{-2x}$, po dosazení $A = \frac{1}{4}$, $y_p(x) = \frac{1}{4}e^{-2x}$, $y(x) = y_h(x) + y_p(x) = ae^{-x} + be^{2x} + \frac{1}{4}e^{-2x}$, počáteční podmínky dají $y(x) = \frac{9}{4}e^{2x} + \frac{3}{4}e^{-2x}$.

Alternativa: variace konstanty: $y(x) = a(x)e^{-x} + b(x)e^{2x}$, rovnice $a'(x)e^{-x} + b'(x)e^{2x} = 0$ a $-a'(x)e^{-x} + 2b'(x)e^{2x} = 3e^{-2x}$. Řešení jsou $a'(x) = -e^{-x}$ a $b'(x) = e^{-4x}$, tedy $a(x) = e^{-x}$ a $b(x) = -\frac{1}{4}e^{-4x}$, proto $y_p(x) = e^{-2x} - \frac{1}{4}e^{-2x} = \frac{3}{4}e^{-2x}$, $y = y_p + y_h$, počáteční podmínky dají $y(x) = \frac{9}{4}e^{2x} + \frac{3}{4}e^{-2x}$.

$$12. p^2Y - 0p - (-1) - (pY - 0) = \mathcal{L}\{4x\}|_{p+1} = \frac{4}{p^2}|_{p+1} = \frac{4}{(p+1)^2},$$

$$Y(p) = \frac{-1}{p(p-1)} + \frac{4}{(p+1)^2p(p-1)} = \left(\frac{1}{p} - \frac{1}{p-1}\right) + \left(\frac{3}{p+1} + \frac{2}{(p+1)^2} - \frac{4}{p} + \frac{1}{p-1}\right) = \frac{3}{p+1} + \frac{2}{(p+1)^2} - \frac{3}{p}, \text{ tedy}$$

$$y(x) = 3e^{-x} + 2e^{-x}\mathcal{L}^{-1}\left\{\frac{2}{p^2}\right\} - 3 = 3e^{-x} + 2xe^{-x} - 3.$$

$$13. p^2X - 0p - 0 + X = \frac{6}{p^2+4} - \frac{1}{p},$$

$$X(p) = \frac{6}{(p^2+1)(p^2+4)} - \frac{1}{p(p^2+1)} = \left(\frac{2}{p^2+1} - \frac{2}{p^2+4}\right) - \left(\frac{1}{p} - \frac{p}{p^2+1}\right) = \frac{2}{p^2+1} + \frac{p}{p^2+1} - \frac{2}{p^2+4} - \frac{1}{p}, \text{ tedy}$$

$$x(t) = 2\sin(t) + \cos(t) - \sin(2t) - 1.$$

$$14. pY - Y = e^{-p}\frac{1}{p} - e^{-2p}\frac{1}{p}, Y(p) = \frac{1}{p(p-1)}e^{-p} - \frac{1}{p(p-1)}e^{-2p} = \left(\frac{1}{p-1} - \frac{1}{p}\right)e^{-p} - \left(\frac{1}{p-1} - \frac{1}{p}\right)e^{-2p}, \text{ tedy}$$

$$y(t) = (e^t - 1)|_{t-1}H(t-1) - (e^t - 1)|_{t-2}H(t-2) = (e^{t-1} - 1)H(t-1) - (e^{t-2} - 1)H(t-2)$$

$$= \begin{cases} 0, & t < 1; \\ e^{t-1} - 1, & t \in \langle 1, 2 \rangle; \\ (e^{-1} - e^{-2})e^t, & t \geq 2. \end{cases}$$

$$15. pX - 1 + 4X + 13\frac{1}{p}X = \frac{1}{p}, p^2X - p + 4pX + 13X = 1, X(p) = \frac{p+1}{p^2+4p+13} = \frac{(p+2)-1}{(p+2)^2+9}, \text{ tedy}$$

$$x(t) = e^{-2t}\mathcal{L}^{-1}\left\{\frac{p-1}{p^2+9}\right\} = e^{-2t}\left[\cos(3t) - \frac{1}{3}\sin(3t)\right].$$

$$16. pY - 0 - 2Y = \frac{4}{p^2} - \mathcal{L}\{4(x+1)\}e^{-p} = \frac{4}{p^2} - \mathcal{L}\{4x+4\}e^{-p} = \frac{4}{p^2} - \left(\frac{4}{p^2} + \frac{4}{p}\right)e^{-p},$$

$$Y(p) = \frac{4}{p^2(p-2)} - \left(\frac{4}{p^2(p-2)} + \frac{4}{p(p-2)}\right)e^{-p} = \left(\frac{1}{p-2} - \frac{2}{p^2} - \frac{1}{p}\right) - \left(\frac{1}{p-2} - \frac{2}{p^2} - \frac{1}{p} - \frac{2}{p} + \frac{2}{p-2}\right)e^{-p}$$

$$= \left(\frac{1}{p-2} - \frac{2}{p^2} - \frac{1}{p}\right) - \left(\frac{3}{p-2} - \frac{2}{p^2} - \frac{3}{p}\right)e^{-p}, \text{ proto}$$

$$y(x) = e^{2x} - 2x - 1 - (3e^{2x} - 2x - 3)|_{x-1}H(x-1)$$

$$= e^{2x} - 2x - 1 - (3e^{2x-2} - 2(x-1) - 3)H(x-1) = \begin{cases} e^{2x} - 2x - 1, & x \in \langle 0, 1 \rangle; \\ e^{2x}(1 - 3/e^2), & x \geq 1. \end{cases}$$

17. $pY - 0 + \frac{1}{p}Y = \frac{2}{p-1} + \frac{2}{p+1}$, $p^2Y + Y = \frac{2p}{p-1} + \frac{2p}{p+1}$,
 $Y(p) = \frac{2p}{(p-1)(p^2+1)} + \frac{2p}{(p+1)(p^2+1)} = \left(\frac{1}{p-1} - \frac{p}{p^2+1} + \frac{1}{p^2+1}\right) + \left(\frac{-1}{p+1} + \frac{p}{p^2+1} + \frac{1}{p^2+1}\right) = \frac{1}{p-1} - \frac{1}{p+1} + \frac{2}{p^2+1}$.
 Proto $y(x) = e^x - e^{-x} + 2 \sin(x) = 2 \sinh(x) + 2 \sin(x)$.

18. $pY + 1 + 3Y = -\frac{13p}{p^2+4} + 13\mathcal{L}\{\cos(2(x + \pi/4))\}e^{-\frac{\pi}{4}p} = -\frac{13p}{p^2+4} + 13\mathcal{L}\{\cos(2x + \pi/2)\}e^{-\frac{\pi}{4}p}$
 $= -\frac{13p}{p^2+4} + 13\mathcal{L}\{-\sin(2x)\}e^{-\frac{\pi}{4}p} = -\frac{13p}{p^2+4} - \frac{26}{p^2+4}e^{-\frac{\pi}{4}p}$, zde se použilo $\cos(\alpha + \pi/2) = -\sin(\alpha)$ pro $\alpha = 2x$,
 proto $Y(p) = \frac{-1}{p+3} - \frac{13p}{(p+3)(p^2+4)} - \frac{26}{(p+3)(p^2+4)}e^{-\frac{\pi}{4}p} = \frac{-1}{p+3} + \frac{3}{p+3} - \frac{3p+4}{p^2+4} - \left(\frac{2}{p+3} + \frac{-2p+6}{p^2+4}\right)e^{-\frac{\pi}{4}p}$
 $= \frac{2}{p+3} - \frac{3p}{p^2+4} - \frac{4}{p^2+4} - \left(\frac{2}{p+3} - \frac{2p}{p^2+4} + \frac{6}{p^2+4}\right)e^{-\frac{\pi}{4}p}$.

Tedy $y(x) = 2e^{-3x} - 3 \cos(2x) - 2 \sin(2x) - (2e^{-3x} - 2 \cos(2x) + 3 \sin(2x))\Big|_{x-\pi/4} H(x - \pi/4)$
 $= 2e^{-3x} - 3 \cos(2x) - 2 \sin(2x) - (2e^{3\pi/4-3x} - 2 \cos(2(x - \pi/4)) + 3 \sin(2(x - \pi/4)))H(x - \pi/4)$
 $= 2e^{-3x} - 3 \cos(2x) - 2 \sin(2x) - (2e^{3\pi/4-3x} - 2 \cos(2x - \pi/2) + 3 \sin(2x - \pi/2))H(x - \pi/4)$
 $= 2e^{-3x} - 3 \cos(2x) - 2 \sin(2x) - (2e^{3\pi/4-3x} - 2 \sin(2x) - 3 \cos(2x))H(x - \pi/4)$
 $= \begin{cases} 2e^{-3x} - 3 \cos(2x) - 2 \sin(2x), & x \in \langle 0, \pi/4 \rangle; \\ 2e^{-3x}(1 - e^{3\pi/4}), & x \geq \pi/4. \end{cases}$

19. $pX + 1 - X = \frac{2}{p} - \frac{1}{p^2} - \mathcal{L}\{2 - (t + 2)\}e^{-2p} = \frac{2}{p} - \frac{1}{p^2} + \mathcal{L}\{t\}e^{-2p} = \frac{2}{p} - \frac{1}{p^2} + \frac{1}{p^2}e^{-2p}$,
 $X(p) = -\frac{1}{p-1} + \frac{2p-1}{p^2(p-1)} + \frac{1}{p^2(p-1)}e^{-2p} = -\frac{1}{p-1} + \frac{1}{p-1} - \frac{1}{p} + \frac{1}{p^2} + \left(\frac{1}{p-1} - \frac{1}{p} - \frac{1}{p^2}\right)e^{-2p}$
 $= \frac{1}{p^2} - \frac{1}{p} + \left(\frac{1}{p-1} - \frac{1}{p} - \frac{1}{p^2}\right)e^{-2p}$.

Tedy $x(t) = t - 1 + (e^t - 1 - t)\Big|_{t-2} H(t - 2) = t - 1 + (e^{t-2} - 1 - (t - 2))H(t - 2)$
 $= t - 1 + (e^{t-2} + 1 - t)H(t - 2) = \begin{cases} t - 1, & t \in \langle 0, 2 \rangle; \\ e^{t-2}, & t \geq 2. \end{cases}$

20. $p^2X - 0 - 0 + 3X = \frac{3}{p}e^{-\pi p} - \frac{3}{p}e^{-2\pi p}$, $(p^2 + 3)X = \frac{3}{p}e^{-\pi p} - \frac{3}{p}e^{-2\pi p}$,
 $X(p) = \frac{3}{p(p^2+3)}e^{-\pi p} - \frac{3}{p(p^2+3)}e^{-2\pi p} = \left(\frac{1}{p} - \frac{p}{p^2+3}\right)e^{-\pi p} - \left(\frac{1}{p} - \frac{p}{p^2+3}\right)e^{-2\pi p}$.

Tedy $x(t) = (1 - \cos(\sqrt{3}t))\Big|_{t-\pi} H(t - \pi) - (1 - \cos(\sqrt{3}t))\Big|_{t-2\pi} H(t - 2\pi)$
 $= (1 - \cos(\sqrt{3}(t - \pi)))H(t - \pi) - (1 - \cos(\sqrt{3}(t - 2\pi)))H(t - 2\pi)$
 $= \begin{cases} 0, & t \in \langle 0, \pi \rangle; \\ 1 - \cos(\sqrt{3}(t - \pi)), & t \in \langle \pi, 2\pi \rangle; \\ \cos(\sqrt{3}(t - 2\pi)) - \cos(\sqrt{3}(t - \pi)), & t \geq 2\pi. \end{cases}$

21. Je to $x' - 2e^t * x(t) = 2te^t$, proto $pX - 1 - 2\frac{1}{p-1}X = -\left[\frac{2}{p-1}\right]' = \frac{2}{(p-1)^2}$, také $= \frac{1}{p}\Big|_{p-2} = \frac{2}{(p-1)^2}$.
 $p(p-1)X - (p-1) - 2X = \frac{2}{p-1}$, $(p^2 - p - 2)X = \frac{p^2 - 2p + 3}{p-1}$, proto $X(p) = \frac{p^2 - 2p + 3}{(p-2)(p+1)(p-1)} = \frac{1}{p-2} + \frac{1}{p+1} - \frac{1}{p-1}$.
 Tedy $x(t) = e^{2t} + e^{-t} - e^t$, $t \geq 0$.

22. $pY - 1 + \frac{4}{Y} = \frac{1}{p} - e^{-\pi p} \frac{1}{p}$, $Y(p) = \frac{p}{p^2+4} + \frac{1}{p^2+4} - \frac{1}{p^2+4}e^{-\pi p}$, tedy
 $y(t) = \cos(2t) + \frac{1}{2} \sin(2t) - \frac{1}{2} \sin(2t)\Big|_{t-\pi} H(t - \pi) = \cos(2t) + \frac{1}{2} \sin(2t) - \frac{1}{2} \sin(2(t - \pi))H(t - \pi)$
 $= \cos(2t) + \frac{1}{2} \sin(2t) - \frac{1}{2} \sin(2t - 2\pi)H(t - \pi) = \cos(2t) + \frac{1}{2} \sin(2t) - \frac{1}{2} \sin(2t)H(t - \pi)$
 $= \begin{cases} \cos(2t) + \frac{1}{2} \sin(2t), & t \in \langle 0, \pi \rangle; \\ \cos(2t), & t \geq \pi. \end{cases}$

23. $p^2Y - 0p - 1 + 3(Y - 0) + 2Y = \frac{2}{p^2} - \mathcal{L}\{2(x + 2)\}e^{-2p} = \frac{2}{p^2} - \mathcal{L}\{2x + 4\}e^{-2p} = \frac{2}{p^2} - \left(\frac{2}{p^2} + \frac{4}{p}\right)e^{-2p}$,
 $Y(p) = \frac{1}{(p+1)(p+2)} + \frac{2}{p^2(p+1)(p+2)} - \left(\frac{2}{p^2(p+1)(p+2)} + \frac{4}{p(p+1)(p+2)}\right)e^{-2p} = \frac{1}{p+1} - \frac{1}{p+2} + \left(-\frac{3/2}{p} + \frac{1}{p^2} + \frac{2}{p+1} - \frac{1/2}{p+2}\right)$
 $- \left(-\frac{3/2}{p} + \frac{1}{p^2} + \frac{2}{p+1} - \frac{1/2}{p+2} + \frac{2}{p} - \frac{4}{p+1} + \frac{2}{p+2}\right)e^{-2p} = \frac{3}{p+1} - \frac{3/2}{p+2} - \frac{3/2}{p} + \frac{1}{p^2} - \left(\frac{1/2}{p} + \frac{1}{p^2} - \frac{2}{p+1} + \frac{3/2}{p+2}\right)e^{-2p}$,
 proto $y(x) = 3e^{-x} - \frac{3}{2}e^{-2x} - \frac{3}{2} + x - \left(\frac{1}{2} + x - 2e^{-x} + \frac{3}{2}e^{-2x}\right)\Big|_{x-2} H(x - 2)$
 $= 3e^{-x} - \frac{3}{2}e^{-2x} - \frac{3}{2} + x - \left(x - \frac{3}{2} - 2e^{-x} + \frac{3}{2}e^{-2x}\right)H(x - 2)$
 $= \begin{cases} 3e^{-x} - \frac{3}{2}e^{-2x} - \frac{3}{2} + x, & x \in \langle 0, 2 \rangle; \\ (3 + 2e^2)e^{-x} - \frac{3}{2}(1 + e^4)e^{-2x}, & x \geq 2. \end{cases}$

24. $p^2Y - (-2)p - 0 - Y = \frac{6}{p-2} - \mathcal{L}\{6e^{2(x+1)}\}e^{-p} = \frac{6}{p-2} - \mathcal{L}\{6e^2e^{2x}\}e^{-p} = \frac{6}{p-2} - \frac{6e^2}{p-2}e^{-p}$,

$$\begin{aligned}
Y(p) &= -\frac{2p}{(p+1)(p-1)} + \frac{6}{(p+1)(p-1)(p-2)} - \frac{6e^2}{(p+1)(p-1)(p-2)}e^{-p} = \left(-\frac{1}{p-1} - \frac{1}{p+1}\right) + \left(\frac{1}{p+1} - \frac{3}{p-1} + \frac{2}{p-2}\right) \\
&\quad - \left(\frac{e^2}{p+1} - \frac{3e^2}{p-1} + \frac{2e^2}{p-2}\right)e^{-p} = \frac{2}{p-2} - \frac{4}{p-1} - \left(\frac{e^2}{p+1} - \frac{3e^2}{p-1} + \frac{2e^2}{p-2}\right)e^{-p}, \text{ proto} \\
y(x) &= 2e^{2x} - 4e^x - \left(e^2e^{-x} - 3e^2e^x + 2e^2e^{2x}\right)\Big|_{x-1} H(x-1) \\
&= 2e^{2x} - 4e^x - \left(e^2e^{1-x} - 3e^2e^{x-1} + 2e^2e^{2x-2}\right)H(x-1) = 2e^{2x} - 4e^x - \left(e^{3-x} - 3e^{x+1} + 2e^{2x}\right)H(x-1) \\
&= \begin{cases} 2e^{2x} - 4e^x, & x \in \langle 0, 1 \rangle; \\ (3e - 4)e^x - e^{3-x}, & x \geq 1. \end{cases}
\end{aligned}$$

$$25. p^2X - 0 - 0 + 2pX - 0 + 5X = \frac{5}{p} - \frac{5}{p}e^{-\pi p},$$

$$X(p) = \frac{5}{p(p^2+2p+5)} - \frac{5}{p(p^2+2p+5)}e^{-\pi p} = \frac{1}{p} - \frac{p+2}{p^2+2p+5} - \left(\frac{1}{p} - \frac{p+2}{p^2+2p+5}\right)e^{-\pi p}.$$

$$\begin{aligned}
\text{Tedy } x(t) &= 1 - \mathcal{L}^{-1}\left\{\frac{(p+1)+1}{(p+1)^2+4}\right\} - \left(1 - \mathcal{L}^{-1}\left\{\frac{(p+1)+1}{(p+1)^2+4}\right\}\right)\Big|_{t-\pi} H(t-\pi) \\
&= 1 - e^{-t}\mathcal{L}^{-1}\left\{\frac{p+1}{p^2+4}\right\} - \left(1 - e^{-t}\mathcal{L}^{-1}\left\{\frac{p+1}{p^2+4}\right\}\right)\Big|_{t-\pi} H(t-\pi) \\
&= 1 - e^{-t}\left(\cos(2t) + \frac{1}{2}\sin(2t)\right) - \left(1 - e^{-t}\left(\cos(2t) + \frac{1}{2}\sin(2t)\right)\right)\Big|_{t-\pi} H(t-\pi) \\
&= 1 - e^{-t}\left(\cos(2t) + \frac{1}{2}\sin(2t)\right) - \left(1 - e^{-\pi-t}\left(\cos(2t-2\pi) + \frac{1}{2}\sin(2t-2\pi)\right)\right)H(t-\pi) \\
&= 1 - e^{-t}\left(\cos(2t) + \frac{1}{2}\sin(2t)\right) - \left(1 - e^{-\pi-t}\left(\cos(2t) + \frac{1}{2}\sin(2t)\right)\right)H(t-\pi) \\
&= \begin{cases} 1 - e^{-t}\left(\cos(2t) + \frac{1}{2}\sin(2t)\right), & t \in \langle 0, \pi \rangle; \\ (e^\pi - 1)e^{-t}\left(\cos(2t) + \frac{1}{2}\sin(2t)\right), & t \geq \pi. \end{cases}
\end{aligned}$$

26. Zvolíme $y(0^+) = a$ a $y'(0^+) = b$ pro $a, b \in \mathbb{R}$.

$$\begin{aligned}
p^2Y - ap - b + pY - a - 2Y &= \frac{3}{p-1} - \mathcal{L}\{3e^{x+1}\}e^{-p} = \frac{3}{p-1} - \mathcal{L}\{3ee^x\}e^{-p} = \frac{3}{p-1} - \frac{3e}{p-1}e^{-p}, \\
Y(p) &= \frac{ap+(a+b)}{(p+2)(p-1)} + \frac{3}{(p+2)(p-1)^2} - \frac{3e}{(p+2)(p-1)^2}e^{-p} = \left(\frac{(a-b)/3}{p+2} + \frac{(2a+b)/3}{p-1}\right) + \left(\frac{1/3}{p+2} - \frac{1/3}{p-1} + \frac{1}{(p-1)^2}\right) \\
&\quad - \left(\frac{e/3}{p+2} - \frac{e/3}{p-1} + \frac{e}{(p-1)^2}\right)e^{-p} = \frac{(a-b+1)/3}{p+2} + \frac{(2a+b-1)/3}{p-1} + \frac{1}{(p-1)^2} - \left(\frac{e/3}{p+2} - \frac{e/3}{p-1} + \frac{e}{(p-1)^2}\right)e^{-p}, \text{ proto} \\
y(x) &= \frac{a-b+1}{3}e^{-2x} + \frac{2a+b-1}{3}e^x + e^x\mathcal{L}^{-1}\left\{\frac{1}{p^2}\right\} - \left(\frac{e}{3}e^{-2x} - \frac{e}{3}e^x + e^x\mathcal{L}^{-1}\left\{\frac{e}{p^2}\right\}\right)\Big|_{x-1} H(x-1) \\
&= \frac{a-b+1}{3}e^{-2x} + \frac{2a+b-1}{3}e^x + xe^x - \left(\frac{e}{3}e^{-2x} - \frac{e}{3}e^x + xe^x\right)\Big|_{x-1} H(x-1) \\
&= \frac{a-b+1}{3}e^{-2x} + \frac{2a+b-1}{3}e^x + xe^x - \left(\frac{e}{3}e^{2-2x} - \frac{e}{3}e^{x-1} + (x-1)e^{x-1}\right)H(x-1) \\
&= \begin{cases} \frac{a-b+1}{3}e^{-2x} + \frac{2a+b-1}{3}e^x + xe^x, & x \in \langle 0, 1 \rangle; \\ \frac{a-b+1-e^3}{3}e^{-2x} + \left(\frac{2a+b}{3} + \frac{1}{e}\right)e^x + \left(1 - \frac{1}{e}\right)xe^x, & x \geq 1. \end{cases}
\end{aligned}$$

27. Zvolíme $x(0^+) = a$ a $\dot{x}(0^+) = b$ pro $a, b \in \mathbb{R}$.

$$p^2X - ap - b + X = \frac{2p}{p^2+1} - \mathcal{L}\{2\cos(t+\pi)\}e^{-\pi p} = \frac{2p}{p^2+1} + \mathcal{L}\{2\cos(t)\}e^{-\pi p} = \frac{2p}{p^2+1} + \frac{2p}{p^2+1}e^{-\pi p},$$

$$X(p) = \frac{ap+b}{p^2+1} + \frac{2p}{(p^2+1)^2} + \frac{2p}{(p^2+1)^2}e^{-\pi p}.$$

Pozor, $\frac{2p}{(p^2+1)^2}$ už je parciální zlomek!

$$\begin{aligned}
\text{Tedy } x(t) &= a\cos(t) + b\sin(t) + \mathcal{L}^{-1}\left\{-\left[\frac{1}{p^2+1}\right]'\right\} - \mathcal{L}^{-1}\left\{-\left[\frac{1}{p^2+1}\right]'\right\}\Big|_{t-\pi} H(t-\pi) \\
&= a\cos(t) + b\sin(t) + t\mathcal{L}^{-1}\left\{\frac{1}{p^2+1}\right\} - \left(t\mathcal{L}^{-1}\left\{\frac{1}{p^2+1}\right\}\right)\Big|_{t-\pi} H(t-\pi) \\
&= a\cos(t) + b\sin(t) + t\sin(t) - \left(t\sin(t)\right)\Big|_{t-\pi} H(t-\pi) \\
&= a\cos(t) + b\sin(t) + t\sin(t) - (t-\pi)\sin(t-\pi)H(t-\pi) \\
&= a\cos(t) + b\sin(t) + t\sin(t) + (t-\pi)\sin(t)H(t-\pi) \\
&= \begin{cases} a\cos(t) + b\sin(t) + t\sin(t), & t \in \langle 0, \pi \rangle; \\ a\cos(t) + b\sin(t) + (2t-\pi)\sin(t), & t \geq \pi. \end{cases}
\end{aligned}$$