Combinatorics

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5. COMBINATORICS

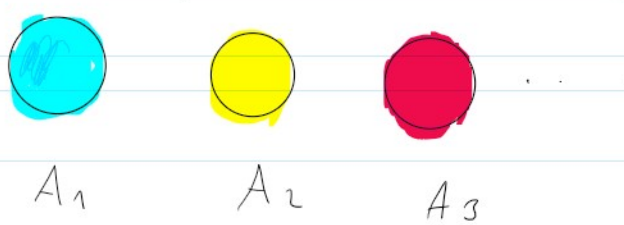
- the art of counting number of sets
- based on two obvious principles

addition principle

(1) $|A_1 \cup A_2 \cup \dots \cup A_n| = \sum_{i=1}^{n} |A_i|$

if $A_i \cap A_j = \emptyset$ for all $i \neq j$

e.g.



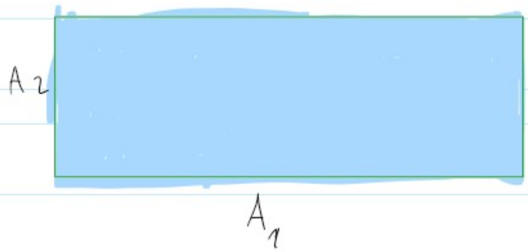
$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3|$

size of disjoint union is the <u>sum</u> of numbers of elements of particular sets

Multiplication principle

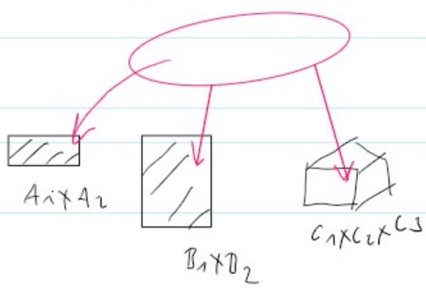
(2) $|A_1 \times A_2 \times \dots \times A_n| = |A_1| \cdot |A_2| \cdots |A_n|$

Size of cartesian product equals to product of particular sets

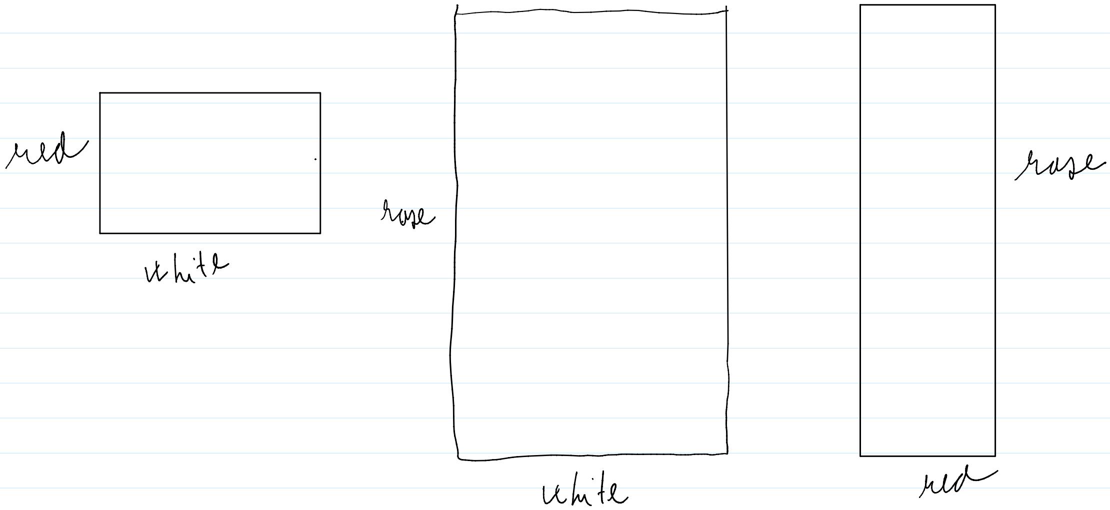


independent choices multiply
"multiplication rule"

Usually we combine both rules



Example: We have five kinds of white wine, three kinds of red wine and ten kinds of rose wine.
What is the number of ways we can prepare two glasses of wine of different colours



Overall number: $5 \times 3 + 5 \times 10 + 3 \times 10 = 15 + 50 + 30 = 95$

Definition Let A be a set with n elements.

a permutation of A is an ordered tuple
$$(a_1, a_2, \dots, a_n) \in A^n$$
where $a_i \neq a_j$ if $i \neq j$.

Suppose $A = \{1, 2, \dots, n\}$

Permutations $\longleftrightarrow$ bijections

$(a_1, a_2, \dots, a_n) \longleftrightarrow$ $\varphi : A \to A$
$\varphi(i) = a_i$

Proposition: Any n-element set has

$$n! = n(n-1)(n-2) \cdots 2 \cdot 1$$

permutations.

Convention $0! = 1$

Proof :

Convention $0! = 1$

Proof:

$A = \{1, 2, \ldots, n\}$

multiplication principle:

1. position : n choices
2. position $n-1$ choices
3. position $n-2$ choices

$\vdots$

n th position 1 choice

multiply

$n(n-1)(n-2)\cdots 2\cdot 1$

Example: How many possible queues consisting of 6 people are possible?

answer: $6! = 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1 = 720$

Definition: Let A be a finite set with n elements.

k-permutation of A is any k-tuple $(a_1, a_2, \ldots, a_k)$ such that $a_i \in A$ and $a_i \neq a_j$ if $i \neq j$

Proposition Let A be n-point set. Then A has

$$\boxed{P(n, k) = n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}}$$

k-permutations

Proof: 1 step : n

2 step $n-1$

multiplication

$n(n-1)\cdots(n-k+1)$

Proof:

1 step: n

2 step $n-1$

$\vdots$

k step $n-k+1$

multiplication $n(n-1)\cdots(n-k+1)$

Example: What is the number of queues with three positions from the set of six people.

answer: $P(6,3) = 6\cdot 5\cdot 4 = \underline{120}$

Definition: Let A be a finite set, $k \in \mathbb{N} \cup \{0\}$,

k-combination of A

is any k-element subset of A.

Proposition: Let $|A| = n$. Then A has

$$C(n,k) = \binom{n}{k} = \frac{1}{k!}P(n,k) = \frac{n(n-1)\cdots(n-k+1)}{k!}$$

$$= \frac{n!}{k!(n-k)!}$$

k-combinations.

$\binom{n}{k}$ is read: "n choose k"

Proof :

$$\underbrace{C(n,k)}_{\substack{k\text{-point}\\ \text{subset}}} \cdot \underbrace{k!}_{\substack{\text{all possible}\\ \text{rearrangements}}} = \underbrace{P(n,k)}_{\substack{\text{all permutations}\\ \text{with repetitions}}}$$

Obviously :

$$\binom{n}{0} = 1$$

$$\binom{n}{n} = 1$$

$$\binom{n}{1} = n$$

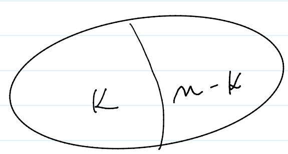
Important identities :

(1) $$\boxed{\binom{n}{k} = \binom{n}{n-k}}$$

(2) $$\boxed{\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}}$$

Proof

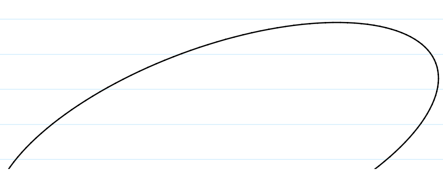
(1)



$X \subseteq A$

$X \to A \setminus X$ is a bijection $\Big\} \Rightarrow$ number of k-point subsets $=$ number of $n-k$ point subset

(2)



n - points

- number of k-point subsets containing a $= \binom{n-1}{k-1}$
- number of k-point subsets not containing a $= \binom{n-1}{k}$

$\Rightarrow$ all k-point subsets $= \binom{n-1}{k-1} + \binom{n-1}{k}$

Pascal triangle

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
 & & & & & \vdots & & & & &
\end{array}
$$

Example: How many groups consisting of 3 people can one form in the class of 30 students

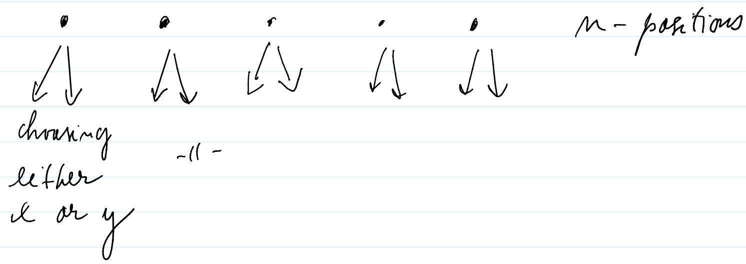
answer: $\binom{30}{3} = \frac{30 \cdot 29 \cdot 28}{\cancel{3} \cdot \cancel{2}} = 10 \cdot 29 \cdot 14 = 4060$

Binomial theorem:

For all $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$ we have

$$\boxed{(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}}$$

Proof: $$(x+y)^n = \underbrace{(x+y)(x+y)\cdots(x+y)}_{n \text{ times}}$$



determined by the subset k- point subset of the set all positions

So we have $x^k y^{n-k}$ $C(n,k)$ times.

Example: $$(x+y)^5 = y^5 + \binom{5}{1} x^4 y + \binom{5}{2} x^3 y^2 + \binom{5}{3} x^2 y^3 + \binom{5}{4} x y^4 + x^4$$

$$= y^5 + 5x^4 y + 10x^3 y^2 + 10 x^2 y^3 + + 5 x y^4 + x^4$$

$$+ 5xy^4 + x^4$$

Two parallel generalizations of $\binom{n}{k}$

$\swarrow$ Combination with repetition

$\downarrow$ multinomial coefficients

Combination with repetition:

Motivating problem: $n \geq 0$ integer

r - natural number

What is a number of decompositions of n into sums of r nonnegative integers (order is relevant).

More precisely, what is the number of all ordered r-tuples $(i_1, i_2, \dots, i_r)$ of nonnegative integers such that

$$i_1 + i_2 + \dots + i_r = n$$

Decompositions for $n = 3$ $r = 3$

$0+0+3$

$0+3+0$

$3+0+0$

$2+0+1$

$0+2+1$

$1+0+2$

$0+1+2$

$1+2+0$

$1+0+2$

$1+1+1$

kind of coding:

3 - boxes (n)

• - 3 balls (k)

$1+1+1 \longrightarrow$ more economically $\bullet|\bullet|\bullet$

$2+0+1 \longrightarrow \bullet\bullet||\bullet$

$0+3+0$ $\quad |\bullet\bullet\bullet|$

So we have m balls •

$k-1$ separators |

So we have $m+n-1$ objects

we are choosing m - point subset

Therefore, we have

$$\binom{m+k-1}{m} = \binom{m+n-1}{k-1}$$

possibilities

This combinatorial category is called combination with repetition

Notation $\left(\binom{n}{k}\right) = \binom{n+k-1}{k}$

In our problem we have

$$\left(\binom{3}{3}\right) = \binom{5}{3} = \binom{5}{2} = \frac{5\cdot 4}{2\cdot 1} = 10$$

Multinomial coefficients

Motivating problem: How many words (also without sense) can we obtain using all letters in the word?

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So we have

$$\left.\begin{array}{l} M - 1X \\ I - 4X \\ S - 4X \\ P - 2X \end{array}\right\} \text{11 letters}$$

n = number of all words:

$$n \cdot \underset{\substack{\text{permutations}\\ \text{of } M}}{1!} \cdot \underset{\text{permutations of } I}{4!} \cdot \underset{\text{permutations of } S}{4!} \cdot \underset{\text{permutations of } S}{2!} = 11!$$

So

$$n = \frac{11!}{4! \cdot 4! \cdot 2!} = \frac{11 \cdot 10 \cdot 9 \cdot \cancel{8} \cdot 7 \cdot \cancel{6} \cdot 5 \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2}}{\cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1 \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1 \cdot \cancel{2}} =$$

$$= 11 \cdot 10 \cdot 9 \cdot 7 = 6930$$

In general we have

$$\binom{n}{k_1, k_2, \ldots, k_m} \text{ - multinomial coefficient}$$

$$k_1 + k_2 + \cdot + k_m = n$$

- If $\begin{array}{l} k_1 = k \\ k_2 = n-k \end{array}$ then $\binom{n}{k_1, n-k} = \binom{n}{k}$

- **Multinomial theorem**

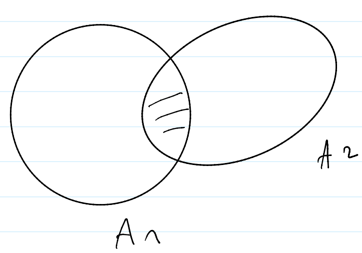
$$(x_1 + x_2 + \dots + x_m)^n =$$

$$= \sum_{k_1+k_2+k_3+ \dots +k_m = n} \binom{n}{k_1, k_2, \dots, k_m} x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m}$$

Inclusion - exclusion principle

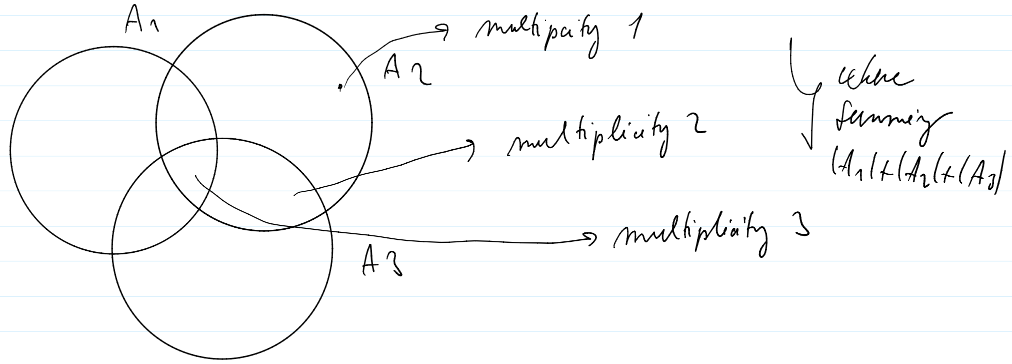
Motivation - compute number of elements in the union of sets $A_1, A_2, \dots, A_n$; knowing the number of elements in all intersections

Case $n = 2$



$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cup A_2|$$

$n = 3$



$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3|$$
$$- |A_1 \cap A_2| - |A_2 \cap A_3| - |A_1 \cap A_3|$$
$$+ |A_1 \cap A_2 \cap A_3|$$

Before formulation of the general case some notation

$$I \subset \{1, 2, \ldots, n\}$$

$$\bigcap_{i \in I} A_i = \{x \mid x \in A_i \ \forall\, i \in I\}$$

e.g. $\quad I = \{1, 3, 4\}$

$$\bigcap_{i \in I} A_i = A_1 \cap A_3 \cap A_4$$

Theorem (inclusion-exclusion)

Let $A_1, A_2, \dots, A_n$ be finite sets.

Then

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{\emptyset \neq I \subset \{1,2,\dots,n\}} (-1)^{|I|-1} \left|\bigcap_{i\in I} A_i\right|$$

summing over all nonempty sets $I \subset \{1,2,\dots,n\}$

Proof Some preparation

- Let $C \subset D$ be finite.

  If f_C is a characteristic function of C

  then $\sum_{x\in D} f_C(x) = |C|$.

- Let $C_1, C_2, \dots, C_K$ be subsets of D

  The $f_{C_1} f_{C_2} \cdots f_{C_K} = f_{\bigcap_{i=1}^{K} C_K}$

- Let f_i be the characteristic function of A_i

  Then, for each $a \in A$,

(*) $(1 - f_1(a))(1 - f_2(a)) \cdots (1 - f_n(a)) = 0$

reason: $\exists\, i$ such that $a \in A_i \Leftarrow 1 - f_i(a) = 0$.

Let us expand the product $(*)$

at each instance we can choose as a factor

$1 - f_1(a)$ $\quad 1 - f_2(a)$ $\quad 1 - f_3(a)$ $\quad \ldots \quad 1 - f_m(a)$

e.g.: all 1's: $1 \cdot 1 \cdot \ldots \cdot 1 = 1$

$$1 \cdot 1 \left(-f_3(a)\right) \cdot 1 \cdots \left(-f_5(a)\right) 1 \cdots 1$$

$$= (-1)^2 f_3(a) f_5(a) = (-1)^2 f_{A_3 \cap A_5}(a)$$

$$= (-1)^{|I|} \cdot f_{A_I}(a)$$

where $I = \{3, 5\}$

general terms:

$$(-1)^{|I|} f_{A_I}(a)$$

where $I \subset \{1, 2, \ldots, m\}$ is nonempty

Summing it up,

$$0 = (1 - f_1(a))(1 - f_2(a)) \cdots (1 - f_m(a)) =$$

$$= 1 + \sum_{\substack{I \subset \{1,2,\ldots,m\} \\ \text{is nonempty}}} (-1)^{|I|} f_{A_I}(a)$$

let us sum all these equations over a and recall that

$$\sum_{a \in A} f_{A_I}(a) = |A_I|.$$

Therefore we get

$$\sum_{a \in A} 1 = |A|$$

$$0 = |A| + \sum_{I \subset \{1,2,\ldots,m\}} (-1)^{|I|} |A_I|$$

$I \neq \phi$

Hence,
$$|A| = \sum_{\substack{I \subset \{1,2,\dots,n\} \\ I \neq \phi}} (-1)^{|I|-1} |A_I|$$

<u>Application</u> :

What is the number of bijections

$$f : \{1,2,\dots,n\} \mapsto \{1,2,\dots,n\}$$

such that $f(i) = i$ (fixed point)

for at least one $i \in \{1,2,\dots,n\}$?

<u>Solution</u>: $X = \{1,2,\dots,n\}$

$Bi(X)$ = all bijections of X

$A = \{ f \in Bi(X) \mid f(j) = j \text{ for some } j,\ 1 \le j \le n\}$

$A_i = \{ f \in Bi(X) \mid f(i) = i \}$.

$|A_i| = (n-1)!$

$|A_i \cap A_j| = (n-2)!$
$i \neq j$

$|A_i \cap A_j \cap A_k| = (n-3)!$
$i \neq j \neq k \neq i$

etc.

Inclusion and exclusion :

$$|A| = n \cdot (n-1)! - \binom{n}{2}(n-2)! + \binom{n}{3}(n-3)!$$

$$+ \binom{n}{4}(n-4)! - \dots$$

$$= \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} (n-k)! =$$

$$= \sum_{k=1}^{n} (-1)^{k-1} \frac{n!}{k!} =$$

$$= n! \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k!}$$

Standard Taylor expansion

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots$$

for $x = -1$

$$\frac{1}{e} = 1 + \underbrace{\frac{-1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots +}_{n}$$

$$= - \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k!}$$

Therefore

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k!} \approx 1 - \frac{1}{e}$$

for big n

Interesting ratios:

$$\frac{\#\ \text{bijections with fixed point}}{\#\ \text{all bijections}} \approx 1 - \frac{1}{e} = 0{,}632$$

$$\frac{\#\ \text{bijections without fixed point}}{\#\ \text{all bijections}} \approx \frac{1}{e} \doteq 0{,}368$$

Famous problem of cloacroom attendant:

n - men are putting there hats to cloackroom. Fangetful cloackroom attendant is giving them back randomly. What is the chance that no gentlemen will get back its hat.

Answer: Approximalely 36,8 % of „total chaos"