Exercises - Functional Analysis

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Hahn-Banach Theorem

1. Let $X = l^\infty$ and define a function $p((\xi_n)) = \limsup_n \xi_n$ on X. Show that p is a sublinear functional. It is a seminorm? Is it a norm?

2. Show that a sublinear functional is continuous whenever it is continuous at 0.

3. Let f be a sublinear functional on a vector space X. Show that the set
$$K = \{x \in X \mid f(x) < 1\}$$
is convex and absorbing.

4. Let K be a convex and absorbing set in a vector space X. Set
$$A = \{x \mid p_K(x) < 1\}, \qquad B = \{x \mid p_K(x) \leq 1\}.$$
Show that $A \subset K \subset B$ and that for the corresponding Minkowski functionals we have
$$p_A = p_K = p_B .$$

5. Using Hahn-Banach theorem show that any normed space can be isometrically embedded into its second dual.

6. Let M be a subspace of a normed space X and N a subspace of its dual X^*. We define
$$M^0 = \{x^* \in X^* \mid x^*(x) = 0 \text{ for all } x \in M\}$$
$$N^0 = \{x \in X \mid x^*(x) = 0 \text{ for all } x^* \in N\}$$
Show that if M is closed, then
$$(M^0)^0 = M$$

7. Let M be a closed subspace of a normed space X. Show that for each $x \in X$ there is $f \in X^*$, $\|f\| = 1$, vanishing on M such that
$$f(x) = dist(x, M)$$
Show that the dual of the quotient space X/M is isometrically isomorphic to M^0.

8. Supose X is a subspace of Hilbert space H. Let f be a bounded functional on X. Show that f has only one norm preserving extension to H.

9. Find example of a bouded functional on a one dimensional subspace of $L^1[0,1]$ which has uncountably many extension to a continuous functional on $L^1[0,1]$.

10. Show that if a normed space has finite-dimensional dual, then it has to be finite-dimensional.

11. Using Hahn-Banach theorem show that for any finite-dimensional subspace M in a normed space X there is a closed subspace of N of X such that
$$M \oplus N = X$$

12. Let B be a convex, absorbing, closed balanced subset of a normed space X. Let $x \in X$ but $x \notin B$. Show that there is $f \in X^*$ such that
$$|f(y)| \le 1$$
for all $y \in B$ and $f(x) > 1$.

13. $X = L^2[0,1]$, $E_\alpha = \{f \in C[0,1] \mid f(0) = \alpha\}$. Show that each E_α is dense in $L^2[0,1]$. Show that E_α's are pairwise disjoint, but cannot be separated by a closed hyperplane.

14. Suppose that (x_n) is a sequence in a normed space X such that there is $x \in X$ such that
$$f(x_n) \to f(x)$$
for all $f \in X^*$. Prove that there is a sequence (y_n) of convex combinations of elements of the sequence (x_n) such that
$$\|y_n - x\| \to 0 \quad \text{as } n \to \infty\,.$$

Uniform Boundedness Principle

1. Using Baire's category theorem show that any infinite-dimensional Banach space has uncontable Hamel basis.

2. Suppose that (α_i) is a sequence such that
$$\sum_i \alpha_i\, \beta_i$$
converges for all (β_i) converging to zero. Show that
$$\sum_i |\alpha_i| < \infty\,.$$

3. Let (x_n) be a sequence in a Banach space X such that $f(x_n)$ is bounded for each $f \in X^*$. Show that (x_n) is bounded in X.

4. Let (T_n) be a sequence in $B(X,Y)$, where X and Y are Banach spaces. Show that (T_n) converges in $B(X,Y)$ if and only if the following two conditions are satisfied: (i) $(T_n x)$ is a cauchy sequence for all x from some total subset of X. (ii) (T_n) is bounded.

Open Mapping Theorem, Closed Graph Theorem

1. Let $(X, \|\cdot\|_1)$, $(X, \|\cdot\|_2)$ be Banach spaces. Suppose that

$$\|\cdot\|_1 \leq K\|\cdot\|_2$$

where $K > 0$. Show that the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent. Show that this statement does not hold if X is not complete with respet to given norms.

2. Let $T \in B(X,Y)$, where X and Y are Banach spaces. Suppose T is injective. Show that T^{-1} is bounded if, and only if, the range $R(T)$ of T is closed.

3. Let M be a finite-dimensional subspace of a Banach space X. Show that there is a bounded projection mapping X onto M.

4. Show that any closed operator has closed kernel.

5. Show that the sum of a closed and bounded operator is closed.

6. Supppose that $T : X \to Y$ is a closed operator between normed space X and Y. Prove that $T(K)$ is closed whenever K is a compact subset of X.

7. Supppose that $T : X \to Y$ is a closed operator between normed spaces X and Y. Prove that $T^{-1}(B)$ is closed whenever B is closed.

Adjoint operator

1. Let $T, S \in B(X,Y)$. Show that

$$(T+S)^* = T^* + S^*$$

$$(TS)^* = S^* T^*$$

2. Let $T \in B(X,Y)$. Show that

$$Ker\, T^* = R(T)^0$$

3. Determine the Hilbert space adjoint of an isometrical linear embedding of one Hilbert space into another.

4. Let H be a Hilbert space. Show that for any $T \in B(H)$

$$H = KerT \oplus \overline{R(T^*)}$$

Formulate and prove the dual statement.

Spectral Theory on Banach Spaces

In the sequel X is always a complex Banach space.

1. Let $T \in B(X)$. Show that the set of eingenvectors of T corresponding to different eigenvalues is linearly independent.
2. Let λ be in the point spectrum of an operator $T \in B(X)$. Show that the space of all eigenvectors corresponding to λ is T-invariant.
3. Let λ be an eigenvalue of $T \in B(X)$ and p be a polynomial. Show that $p(\lambda)$ is an eigenvalue of $p(T)$.
4. Let $T \in B(X)$ be invertible in $B(X)$. Show that $\sigma(T^{-1}) = \{1/\lambda \mid \lambda \in \sigma(T)\}$
5. Let $X = l^\infty$ and $f \in X$. Let T_f be a linear map acting on X by

$$T_f(x_n) = (f_n x_n)$$

Show that this operator is bounded, determine its spectrum and spectral radius. When is the point spectrum of T_f nonempty?

6. Let $T \in B(X)$ be an idempotent, i.e. $T^2 = T$. Show that if T is neither zero nor identity, then $\sigma(T) = \{0, 1\}$.
7. Suppose that $T \in B(X)$ and $\sigma(T) = \{0\}$. Using the spetrum radius formula show that $\lim_n \lambda^n \|T\|^{n+1} = 0$ for all complex λ.
8. Find example of an operator for which the spectral radius is strictly less then its norm. Hint: Consider nilpotent operators.
9. Show that if $S, T \in B(X)$ commute, then $r(ST) \leq r(S)r(T)$.
10. Let $\Phi \in B(X)^*$ and $T \in B(X)$. Show that the sequence of the moments $(\Phi(T^n))_n$ has at most exponential growth.

Classes of operators on a Hilbert space

1. Show that for a two-dimensional real Hilbert space H there are two different operators T_1 and T_2 on H such that $(T_1x, x) = (T_2x, x)$ for all $x \in H$.

2. Show that if $T \in B(H)$ is a normal operator, then $\operatorname{Ker} T = \operatorname{Ker} T^*$.

3. Prove that if $T \in B(H)$ is normal and $F \subset H$ is a subspace of H consisting of eigenvectors of T, then $F^\perp$ is T-invariant.

4. Suppose that $T \in B(H)$ is normal. (i) Show that $\operatorname{Ker} T$ is T^*-invariant and $\operatorname{Ker} T^\perp$ is T-invariant. (ii) Prove that $\operatorname{Ker} T = \operatorname{Ker} T^k$ for any integer k. (iii) Using the previous result show that if a normal operator T is nilpotent (i.e. if $T^k = 0$ for some integer k), then $T = 0$.

5. Show that if $T \in B(H)$, then $T^*\,T$ and $T\,T^*$ are positive. In particular, if T is self-adjoint, then T^2 is positive.

6. It can be proved that for any positive operator $T \in B(H)$ there is a positive operator $S \in B(H)$ such that $S^2 = T$. Taking this for granted determine when the product of two positive operators is positive.

7. Prove that an operator $T \in B(H)$ preserves the inner product if, and only if, $T^*T = I$. Prove that if such an operator is not a surjection, then it cannot be normal.

8. Let U be a unilateral shift on $\ell^2(\mathbb{Z})$, meaning that $U\delta_n = \delta_{n+1}$. Show that the spectrum of U is the unit circle.
   Hint: Given a complex unit λ study the action of U on the vectors $x_n = \frac{1}{\sqrt{2n+1}} \sum_{k=-n}^{n} \lambda^{-k}\delta_k$.

9. Denote by $B(H)^+$ the set of all positive operators on H. Prove that this set forms a positive cone (i.e. that $B(H)^+$ is closed under sums and positive scalar multiples). Show that this cone defines a translation invariant partial order on the set of self-adjoint operators by $S \le T$ if $T - S \ge 0$.

10. Suppose that K is a positive element in $B(H)$. Show that the equation $(x, y)_1 = (Kx, y)$ defines an inner product on H (possibly indefinite). By means of the Cauchy-Schwarz inequality show that

$$\|K\| = \min\{a \in \mathbb{R} \mid K \le aI\}\,.$$

11. Let $P, Q \in B(H)$ be projections. Prove that the following statements are equivalent (i) $Q - P \ge 0$ (ii) $PQ = P$ (iii) $P(H) \subset Q(H)$.

12. Show that the set $E(H)$ of all positive operators on H of norm less than one is a convex set. Prove that any projection is an extreme point of this set. (An extreme point is a point which cannot be written as a proper convex combination of other points).

13. Show that the numerical range of a normal operator may be much larger then its spectrum.
Hint: Consider projection.

14. Let $(G, \cdot)$ be a group. Put $H = \ell^2(G)$. For $s \in G$ let $u_s \in B(H)$ be defined by $f(\cdot) \to f(s^{-1}\cdot)$. Show that u_s is unitary. How does u_s act on standard orthonormal basis $(\delta_g)_{g\in G}$? Prove that $u_s\, u_t = u_{st}$ and $u_{s^{-1}} = u_s^*$.

Compact operators on Hilbert spaces.

1. Prove that the range of any compact operator on a Hilbert space is separable.

2. Suppose that $T \in K(H)$. Let $(e_n)_{n=1}^{\infty}$ be an orthonormal basis of H and P_n a projection onto linear span of $\{e_1, e_2, \cdots, e_n\}$.
Prove that $\|P_n T - T\| \to 0$ as $n \to \infty$.

3. Show that if T is a compact operator on a Hilbert space H, then its adjoint is again compact.

4. Using the spetral theorem for a normal compact operators show that a compact normal operator on a Hilbert space is positive if, and only if, all its eigenvalues are nonnegative.

5. Show that for a positive compat operator T acting on a Hilbert space H there is a positive compact operator S such that $S^2 = T$.