Graphs

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6. GRAPH THEORY

6.1 Basic concepts

- Graph is given by some relation or map on finite sets.

- Graph has the following ingredients

  vertices – basic set (objects)

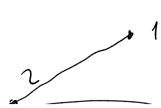
  edges – relation between vertices

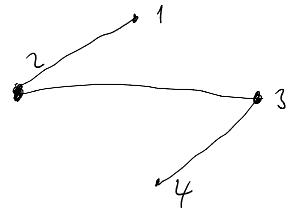
- Notation: $\binom{V}{k}$ ... subsets of V having k elements

Definition: A (simple) graph is a pair (V, E) where V is a set and $E \subseteq \binom{V}{2}$. Elements of V are called vertices and elements of E are called edges.

Example: $V = \{1, 2, 3, 4\}$

$E = \{\{1,2\}, \{3,4\}, \{2,3\}\}$



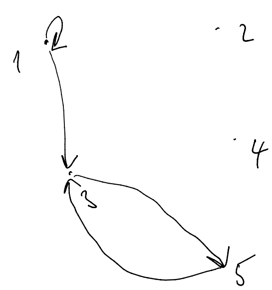


Definition

a directed graph is a pair (V,E), where V is a set of vertices and $E \subseteq V \times V$ is a set of edges.

Example $V = \{1,2,3,4,5\}$

$E = \{(1,1), (3,5), (5,3), (1,3)\}$



Definition In a graph $G=(V,E)$, two vertices $u, w \in G$ are called adjacent if they are connected with an edge $e \in E$, so $\{v, w\} \in E$.

a vertex $v \in V$ is said to be incident with an edge $e \in E$ if $v \in e$.

Definition Let $G = (V, E)$ be a graph.

Let $v \in V$.

We define the degree of v to be a number

$$d_G(v) = |\{w \in V \mid \{v,w\} \in E\}|$$

Handshaking lemma Let $G = (V, E)$ be a graph. Then

(1) $\sum_{v \in V} d_G(v) = 2|E|$

(2) Every graph has even number of vertices with odd degree

(1) Every edge is incident to exactly two vertices

(2) Sum of degrees of odd vertices + Sum of degrees of even vertices = even number

$\downarrow$

must be even

$\downarrow$

So the number of odd vertices has to be even.

Standard examples of graphs

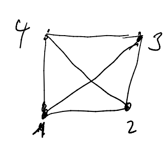
- Complete graph

$K_n = (V, E)$

$V = \{1, \dots, n\}$

$E = \binom{V}{2}$

K_4



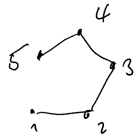
- Path graph

$$P_n = (V, E)$$

$V = \{1, \dots, n\}$

$E = \{\{i, i+1\} \mid i = 1, 2, \dots, n-1\}$

P_5



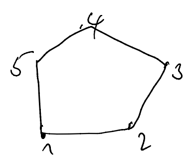
Cycle graph

$$C_n = (V, E)$$

$V = \{1, 2, \dots, n\}$

$E = \{\{i, i+1\} \mid i = 1, \dots, n-1\} \cup \{\{1, n\}\}$

C_5



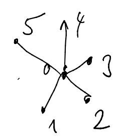
- Star graph

$S_n = (V, E)$

$V = \{0, \dots, n\}$

$E = \{\{0, i\} \mid i = 1, 2, \dots, n\}$

$S_5 =$



Definition: Let $G = (V, E)$ be a graph with $V = \{1, \dots, n\}$

we define its adjacency matrix A_G by

$$[A_G]_{ij} = \begin{cases} 1 & \text{if } \{i,j\} \in E \\ 0 & \text{otherwise} \end{cases}$$

- Adjacency matrix is always symmetric and has zero on diagonal.

Examples:

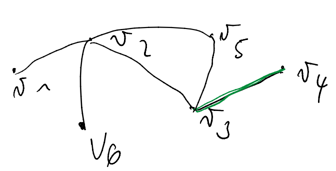
$$A_{P_5} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

6.2. CONNECTIVITY

Definition: Let $G = (V, E)$ be a graph.

- a walk (of length $k-1$) is a sequence of vertices $v_1, \dots, v_k$ s.t.

$$\{v_i, v_{i+1}\} \in E \quad \forall i = 1, \dots, k-1$$



$v_1, v_2, v_3, v_4, v_3, v_5, v_2, v_6$

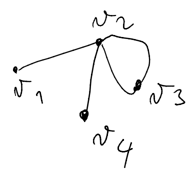
Convention: We shall admit a walk v_1

- a trail is a walk in which all edges are distinct, i.e.

$$\{v_i, v_{i+1}\} \neq \{v_j, v_{j+1}\}$$

whenever $i \neq j$

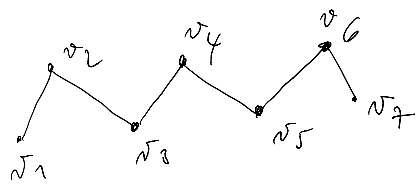
E.g.



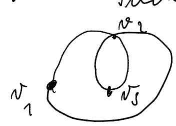
v_1, v_2, v_3, v_2, v_4

- a path is a walk in which all vertices are distinct, i.e.

$v_i \neq v_j$ whenever $i \neq j$.

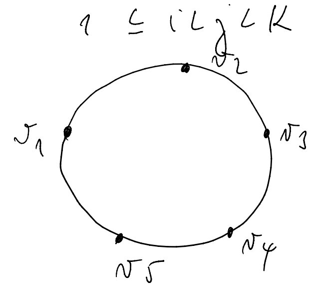


- a circuit is a trail such that $v_1 = v_k$

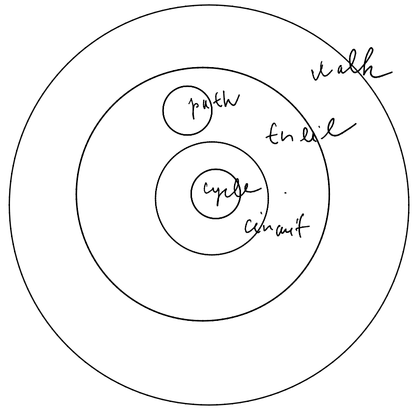


v_1, v_2, v_3, v_2, v_1

- a cycle is a circuit where $v_i \neq v_j$ for any $1 \leq i < j < k$



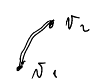
$v_1, v_2, v_3, v_4, v_5, v_1$



<u>Definition</u>: Let $G = (V, E)$ be a graph.
We say that vertices v and w **are connected** if there is a walk starting at v and ending at w.

<u>Theorem</u> Being connected is an equivalence relation on G. (Every vertex is incident with some edge)

<u>Proof</u>: · reflexivity $\quad v_1, v_2, v_k$
we just take a walk v_1 ✓



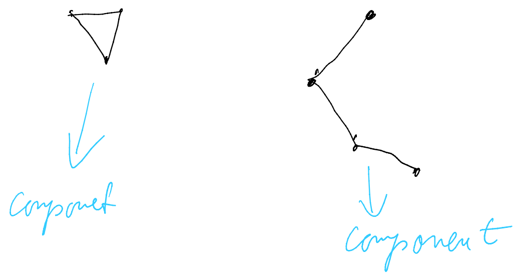
· symmetry ✓

· transitivity ✓

<u>Definition</u> Let $G = (V, E)$ be a graph and $\sim$ the equivalence of being connected.
Equivalence classes of $\sim$ are called **components of $\sim$**.

G is called connected if there is only one component.

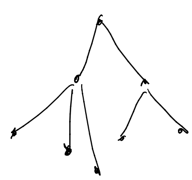
Example:



6.3. Trees

Definition: $G = (V, E)$ that is connected and contains no cycle is called a tree

e.g.



Theorem: Let $G = (V, E)$ be a graph then the following are equivalent

(i) G is a tree

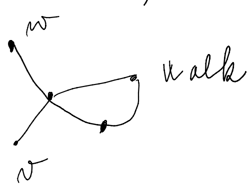
(ii) For every $v, w \in G$, $v \neq w$, there is a unique path from v to w

(iii) G is connected and for every edge $e \in E$, the graph $(V, E \setminus \{e\})$ is not connected.

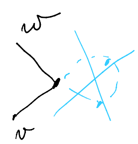
edge $e \in E$, the graph $(V, E \setminus \{e\})$ is not connected.

<u>Proof:</u> (i) $\Rightarrow$ (ii)

Suppose (i) and take $v, w \in G$, $v \neq w$. As G is connected there is a walk from v to w. If there is a walk from $v \to w$ then there must be a path $v \to w$. Indeed,



convert to path $\longmapsto$



We have to prove the path is unique.

Suppose we have two paths

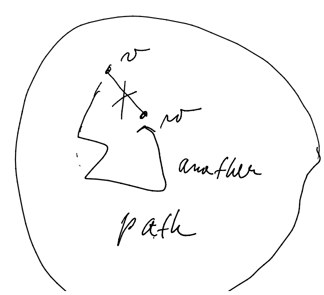
$$v = a_1, a_2, \dots, a_k = w$$
$$v = b_1, b_2, \quad , b_\ell = w.$$

Let i, j be the smallest indices s.t. $a_i = b_j$.
(there exists as $a_k = b_\ell$).

Then $a_1, a_2, \dots, a_i = b_j, b_{j-1}, \dots, b_1$ is a cycle – contradiction.

(ii) $\Rightarrow$ (iii) By (ii) G must be connected.
Take edge $e = \{v, w\}$.
As v and w are connected with just one path by removing the edge e, v and w stop being connected.

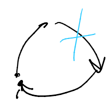
Removing this edge, the points v and w stop being connected.



Removing this edge, the points v and w stop being connected.

path

$(iii) \Rightarrow (i)$ Assume G is connected but contains a cycle. Removing one edge from this cycle G remains connected.



Another characterization of the tree

Theorem Let $G=(V,E)$ be a connected graph. Then G is a tree if and only if

$$|E| = |V| - 1$$

Proof: $(\Rightarrow)$:

By induction number of of vertices $n=1$ $\quad |V|=1 \quad |E|=0$

Hypothesis: Any tree with number of vertices smaller than n satisfies the equation

Take any edge e of G. Removing it we obtain a disconnected graph with two components $G_i=(V_i, E_i)$ $i=1,2$

Components are again trees.

By induction hypotheses

$$\#E_i = \#V_i - 1$$

by induction hyp.

$$\# E_i = \# V_i - 1$$

$E = E_1 \cup E_2 \cup \{e\}$

$V = V_1 \cup V_2$ (disjoint union)

$$|E| = |E_1| + |E_2| + 1 = |V_1| - 1 + |V_2| - 1 + 1 = |V_1| + |V_2| - 1$$
$$= |V| - 1$$

✓

($\Leftarrow$) Suppose $|E| = |V| - 1$ and G is not a tree. It must contain an edge such that by removing it the graph remains connected.

By continuing this way we obtain a tree

$$G' = (V, E')$$

By the previous implication

$$|E'| = |V| - 1$$

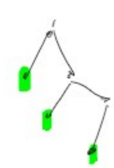
But since $|E| > |E'|$ we have a contradiction.

$\square$

Proposition Any tree must contain a vertex of degree 1.

Proof

$$|E| = |V| - 1$$



$$\sum_{v \in V} \deg(v) = 2|E| = 2|V| - 2$$

If $\deg(v) \geq 2 \quad \forall v \in V$ then

$$2|V| - 2 = \sum_{v \in V} \deg(v) \geq 2|V| \quad \text{- contradiction}$$

NEV v

Applications: programming, data structures, ...

spanning tree:

Definition Let $G=(V,E)$ be a graph. Then graph $G'=(V,E')$, $E'\subseteq E$ which is a tree is called a spanning tree.

Proposition: A graph has a spanning tree if and only if it is connected.

Proof: Having connected graph we can break all cycles. We obtain a tree $G'=(V,E')$ $E'\subseteq E$

Spanning tree is connected so $V=V'$ $E'\subseteq E$
$\Rightarrow$ G is connected

Problem of minimal spanning tree

Given a connected graph $G=(V,E)$ and a cost function

$$c: E \to (0,\infty)$$

find $E'\subseteq E$ such that

$G'=(V,E')$ is connected and

$$c(E')=\sum_{e\in E'} c(e) \quad \text{is minimal}$$

Motivation: connect every village to the electrical grid while minimizing the length of cables needed.

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(OTAKAR BORŮVKA)

KRUSKAL ALGORITHM

Greedy algorithm

INPUT: Connected weighted graph (V, E, c)

OUTPUT: It's minimal spanning tree

1. Sort E

$$E = \{e_1, e_2, \dots, e_m\}$$

s.t.

$$c(e_1) \le c(e_2) \le \dots \le c(e_m)$$

initial step: $E' \leftarrow \{e_1\}$

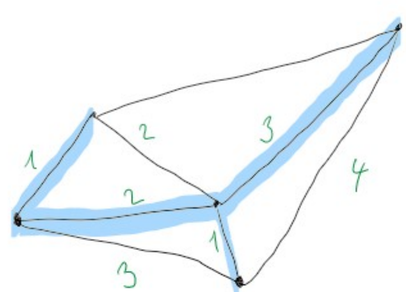
For every $i = 1, 2, \dots, m$

If $E' \cup \{e_i\}$ contains no cycle put

$$E' \leftarrow E' \cup \{e_i\}$$

This algorithm is producing a minimal spanning tree.
(Proof omitted)

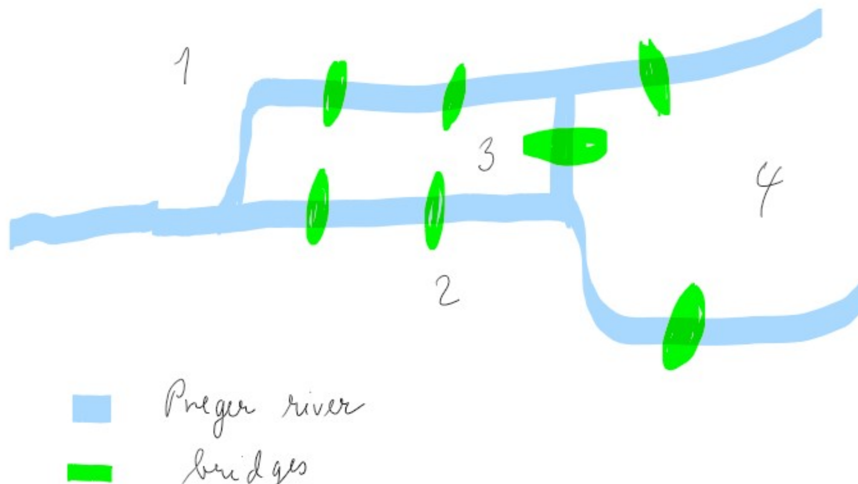
Example



6.4. Euler graphs

Problem : Seven bridges of Königsberg

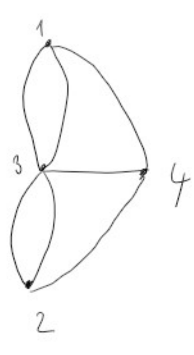
Solved by Leonhard Euler in 1736



Can you cross all the bridges exactly once returning to the original place.

The river separates the city into four parts 1, 2, 3, 4

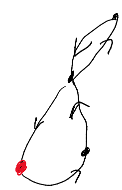
Represent the situation by 1, 2, 3, 4 as vertices and bridges as edges



strictly speaking it is not a graph as we have two edges between same vertex

Is there a circuit going through all vertices?

If there is a circuit going through all vertices then its degree must be twice the number of times the circuit visits the vertex.



$\Rightarrow$ all degrees are even

So the answer to the problem of 7 bridges is NO!

<u>Definition</u> : Let $G=(V,E)$ be a graph.
an <u>Eulerian trail</u> is a trail that uses all edges of the graph exactly once.

<u>Eulerian circuit</u> is a circuit using all edges exactly once

<u>Eulerian graph</u> is a graph admitting Eulerian circuit.

<u>Theorem</u> Let $G=(V,E)$ be a connected graph. Then G is Eulerian $\Leftrightarrow$ each vertex has even degree.

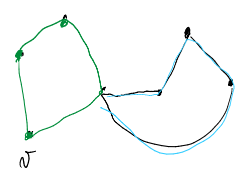
<u>Proof</u>: $\Rightarrow$ see above

$\Leftarrow$

Pick vertex. Start a walk and color all edges you use. Never use the colored edges again. Since the degree of all vertices is even one has always possibility to continue unless one arrives at starting vertex. If it happens we have produced a circle

$$v_1, v_2, \dots, v_k = v_1$$

If we use all edges we are done. Otherwise pick some vertex v_i which is incident to coloured edge as well as uncoloured one. It must exist thanks to connectedness.



Do the same and avoid previous edges. This produces another circle

$$v_i = w_1, w_2, \dots w_e = w_i$$

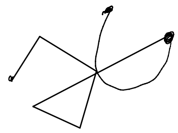
make one circuit

$$v_1, v_2, \dots v_i = w_1 \dots, w_e = v_i, \dots v_k = v_1$$

Then continue this way $\square$

Theorem Let $G = (V, E)$ be a connected graph. Then there is an Eulerian trail on G $\Leftrightarrow$ exactly 2 vertices in G have odd degree

Proof. $\Rightarrow$



Eulerian trail
exactly first and last
vertex has odd degree.

$\Leftarrow$ Repeat the proof for circle.