Midterm 2024:

Requirements and rules

- Separate solutions to particular problems - one sheet of paper should contain solution to only one problem.
- Sign all sheets of paper and print your name.
- Do not forget to clearly write the answer to every question.
- All your computations and derivations should be clear and properly explained.
- Text that is illegible or crossed out will be ignored.

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1. Let us have the following formula

$$[(P \wedge (P \Rightarrow Q)) \Rightarrow Q] \wedge R$$

Write the truth table of this formula. Is it satisfiable?

2. Let $f : X \to Y$ be a function defined on all of X. Define relation R on X by

$$xRy \Leftrightarrow f(x) = f(y)\,.$$

(a) Show that R is equivalence.
(b) For what function f are the equivalence classes of R one-point sets?
(c) For what function f has the partition corresponding to R just one element?

3. For which integer p, $0 \leq p \leq 15$, there is an integer x such that

$$xp \equiv 2 (\mathrm{mod}\, 16)\,.$$

Find all such x if $p = 11$. In solution use Diophantine equation.

4. Let $p > 100$ be a prime number. Determine the remainder of 3^{p+2} when divided by p.

1. Let us have the following formula

$$[(P \wedge (P \to Q)) \to Q] \wedge R$$

Write the truth table of this formula. Is it satisfiable?

Solution:
Denote

$$\alpha \equiv P \wedge (P \Rightarrow Q)$$
$$\beta \equiv \alpha \Rightarrow Q$$
$$\gamma \equiv \beta \wedge R$$

P	Q	R	α	β	γ
0	0	0	0	1	0
0	0	1	0	1	1
0	1	0	0	1	0
0	1	1	0	1	1
1	0	0	0	1	0
1	0	1	0	1	1
1	1	0	1	1	0
1	1	1	1	1	1

The formula is satisfiable since the truth function attains at least one value 1.

2. Let $f : X \to Y$ be a function defined on all of X. Define relation R on X by

$$xRy \Leftrightarrow f(x) = f(y)\,.$$

(a) Show that R is equivalence.
(b) For what function f are the equivalence classes one-point sets?
(c) For what function f is the equivalence class the whole set X?

Solution:
(a)
reflexivity: xRx holds as $f(x) = f(x)$ for all $x \in X$.
symmetry: xRy means $f(x) = f(y)$ and so yRx
transitivity: xRy and yRz means $f(x) = f(y)$ and $f(y) = f(z)$, which gives $f(x) = f(z)$ and so xRz.

(b) Equivalence classes are singletons if and only if f is injective: Indeed, if f is injective, then $f(x) = f(y)$ if only if $x = y$. Therefore $[x] = \{x\}$ for all $x \in X$.
Reciprocal implication: Suppose that $[x] = \{x\}$ for all $x \in X$. Then if $x \neq y$ then $x \notin [y]$ and so $f(x) \neq f(y)$.

(c) $[x] = X$ for all $x \in X$ if and only if $f(x) = f(y)$ for all $x, y \in X$, if and only if f is constant.

3. For which integer p, $0 \leq p \leq 15$, there is an integer x such that

$$xp \equiv 2 (\mathrm{mod}\, 16)\,.$$

Find all such x if $p = 11$. Use Diophantine equation.

Solution:

Equivalent Diophantine equation

$$xp - 2 = k \cdot 16\,,$$

where k and p are integers. Equivalently,

$$xp - 16k = 2$$

In case of $p = 0$ we have $xp = 0$ for any integer x and so $xp \equiv 0(mod 16)$. Therefore there is no solution to given equation.

Suppose $p \neq 0$. By Bezout theorem the solution exists for nonzero p if and only if $gcd(p, 16)$ divides 2. Therefore $gcd(p, 16)$ is equal either to 1 or to 2.

All possibilities:

$$p = 1, 2, 3, 5, 6, 7, 9, 10, 11, 13, 14, 15$$

For $p = 11$ we have

$$11x - 16k = 2$$

Euclid algorithm:

$$16 = 1 \cdot 11 + 5$$
$$11 = 2 \cdot 5 + 1\,.$$

Therefore, $gcd(11, 16) = 1\,.$

Reverse Euclid:

$$1 = 11 - 2 \cdot 5 = 11 - 2 \cdot (16 - 11) = 3 \cdot 11 - 2 \cdot 16.$$

Therefore

$$2 = 6 \cdot 11 - 4 \cdot 16$$

Particular solution

$$x = 6, k = 4$$

Solutions of homogeneous equation:

$$(16t, 11t),\ t \in \mathbb{Z}\,.$$

Therefore $x = 6 + 16t$, $t \in \mathbb{Z}$.

3. Let $p > 100$ be a prime number. Determine the remainder of 3^{p+1} when divided by p.

Solution:

Fermat Little Theorem says:

Let p be a prime number and a a natural number such that $a \perp p$. Then

$$a^{p-1} \equiv 1 (\mathrm{mod} p) \,.$$

We have to justify that $a = 3 \perp p$. But it is obvious as $p \neq 3$ and p is prime.

So as $p \perp 3$, by Little Fermat theorem we have

$$3^{p-1} \equiv 1 (\mathrm{mod} p)$$

Multiplying both sides by 3^3 we obtain

$$3^{p-1+3} = 3^{p+2} \equiv 27 \, (\mathrm{mod} p) \,.$$

Answer: the remainder is 27.