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★**Quantum measure theory.**

Fundamental Theories of Physics, 134.
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This is an important book which deals with a branch of functional analysis whose roots go back to von Neumann's work on the foundations of quantum mechanics and which is going through vigorous growth and development today. Experts on operator algebras will need it on their bookshelves but the target audience is much wider. Because the standard of exposition is so high, this book is accessible to anyone with a foundation in classical measure theory and a basic knowledge of operators on Hilbert space.

Let H be any Hilbert space and let $L(H)$ be the algebra of all bounded operators on H, equipped with the usual operator norm and with the usual adjoint operation, $*$. Let us recall that $L(H)$ is a von Neumann algebra. More generally, let M be a $*$-subalgebra of $L(H)$ which is closed in the weak operator topology of $L(H)$. Then M is also a von Neumann algebra and, essentially, all von Neumann algebras look like this. In particular, when H is of finite dimension n, $L(H)$ can be identified with $M_n(\mathbf{C})$, the algebra of n by n complex matrices.

Let (Ω,μ) be a probability space and let $L(\Omega,\mu)$ be the algebra of all bounded (complex valued) μ-integrable functions modulo the ideal of μ-null functions. Then $L(\Omega,\mu)$ is a commutative von Neumann algebra, with a natural representation as operators on the Hilbert space $L^2(\Omega,\mu)$. All "small" commutative von Neumann algebras look like this. Hence general von Neumann algebra theory is sometimes referred to as "non-commutative measure theory". This is analogous to the description of C^*-algebra theory as "non-commutative topology". Going from the commutative to the non-commutative situation usually increases difficulties enormously.

Let M be any von Neumann algebra. Let us recall that an element p of M is a projection if it is idempotent and self-adjoint, that is, $p^2=p$ and $p=p^*$. Let $P(M)$ be the set of all projections in M.

When $M = L(\Omega, \mu)$ the projections correspond to the measurable subsets of Ω. So, for a general von Neumann algebra, the projections may be thought of as the non-commutative generalisation of the notion of measurable set.

A quantum measure on a von Neumann algebra M, with values in a Banach space V, is a function m mapping $P(M)$ into V such that, whenever p and q are orthogonal projections then $m(p+q) = m(p) + m(q)$. The Mackey-Gleason problem asks: when does a bounded function m have an extension to a bounded linear operator T from M into V? When M is commutative this is equivalent to obtaining an integral for a finitely additive vector valued measure, which is a straightforward exercise. When M is not commutative this becomes a subtle and difficult question.

Does the Mackey-Gleason problem always have a positive answer? No, for if $M = M_2(\mathbf{C})$ then it is easy to give examples of m for which no linear extension exists, even with m taking its values in the positive real numbers. The remarkable fact is that, essentially, this is the only obstruction to obtaining a positive answer. To formulate it, we need to recall a definition. A von Neumann algebra is said to be of type I_2, if it can be represented as the algebra of all two-by-two matrices over a commutative von Neumann algebra. A von Neumann algebra M has a type I_2 direct summand if there exists a nonzero central projection c such that cM is of type I_2. In its most general form, the generalised Gleason theorem states that the Mackey-Gleason theorem has a positive answer precisely when M does not have a type I_2 direct summand.

The generalised Gleason theorem is the culmination of the work of many hands over many years, starting of course with the initial breakthrough by Gleason. The proofs in the literature are difficult and complex. Hamhalter has used his wide knowledge and deep understanding to give an exceptionally lucid and readable account of this work.

Of course there is much more to this book than the generalised Gleason theorem, although this result and its ramifications (e.g. the multiform generalized Gleason theorem and its connections with decoherence functions in quantum physics) forms a substantial part of the book. The chapter headings are: I. Introduction; II. Operator algebras; III. Gleason theorem; IV. Completeness criteria; V. Generalized Gleason theorem; VI. Basic principles of quantum measure theory; VII. Applications of Gleason theorem; VIII. Orthomorphisms of projections; IX. Restrictions and extensions of states; X. Jauch-Piron states; XI. Independence of quantum systems.

This is a marvellous book. My advice is: rush out and buy it.

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