# WAVELETS.
# A TUTORIAL AND A BIBLIOGRAPHY (*)

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## Contents



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## Foreword

This is a working-out of the material on wavelets, presented at the Second and Third Conference on Measure Theory and Real Analysis, held at Grado, May 10-22, 1992, and September 10 - October 1, 1993, respectively.

The author would like to thank the organizer of these meetings, professor Aljoša Volčič, for the invitation to deliver a tutorial on wavelets. The author also expresses his gratitude to his student, Elke Wilczok, for her work on the final form of these notes, for putting together the bibliography, and for type-writing the whole treatise. Many thanks also to Dr. Peter Singer for preparing the preceding seminars.

Erlangen, May 1994 DIETRICH KÖLZOW

## A Historical Once-Over - First Papers

A list of authors in wavelet theory and related topics, with date and title of their first paper on this subject.

1910 A. Haar [Ha10]:
Über eine Klasse von orthogonalen Funktionensystemen.

1928 P. Franklin [Frank28]:
A set of continuous orthogonal functions.

1931 J. Littlewood, R. Paley [LiP31]:
Theorems on Fourier series and power series.

1946 D. Gabor [Gabo46]:
Theory of communication.

1952 R.J. Duffin, A.C. Schaeffer [DufS52]:
A class of nonharmonic Fourier series.

1961 V. Bargmann [Barg61]:
On a Hilbert space of analytic functions and an associated integral transform.

1963 Z. Ciesielski [Ci63]:
Properties of the orthonormal Franklin system.
1964 A.P. Calderón [Cal64]:
Intermediate spaces and interpolation, the complex method.
1968 E.W. Aslaksen, J.R. Klauder [AslK68]:
Unitary representations of the affine group.
1977 D. Esteban, C. Galand [EstG77]:
Applications of quadrature mirror filters to split band voice coding schemes.
1981 R. Balian [Bal81]:
Un principe d'incertitude fort en théorie du signal ou en mécanique quantique.
1982 J.O. Stromberg [Strom82]:
A modified Franklin system and higher order spline systems on $\mathbf{R}^n$ as unconditional bases for Hardy spaces.
1983 P.J. Burt, E.H. Adelson [BurA83a]:
The Laplacian pyramid as a compact image code.
1984 J. Bertrand, P. Bertrand [BertB84]:
Répresentations temps fréquences des signaux.
P. Goupillaud, A. Grossmann, J. Morlet [GouG84]:
Cycle octave and related transforms in seismic signal analysis.
T. Paul [Paul84]:
Functions analytic on the half plane as quantum mechanical states.
1985 H.G. Feichtinger, P. Gröbner [FeiGb85]:
Banach spaces of distributions defined by decomposition methods.
F. Low [Low85]:
Complete sets of wave packets.
1986 Y. Meyer [Mey86a]:
Principe d'incertitude, bases Hilbertiennes et algèbres d'operateurs.
1987 G. Battle [Bat87]:
A block spin construction of ondelettes. Part I: Lemarié functions.
I. Daubechies [Dau89]:
Orthonormal bases of wavelets with finite support - connection with discrete filters.

S. Mallat [Mal87]:
Multiresolution approximation and wavelets.

1988 M. Farge, G. Rabreau [FarR88]:
Transformée en ondelettes pour detecter et analyser les structures cohérentes dans les écoulements turbulents bidimensionnels.
M. Holschneider [Hol88]:
On the wavelet transform of fractal objects.
P.G. Lemarié [Lem88]:
Ondelettes à localisation exponentielle.

1989 F. Argoul, A. Arnéodo, J. Elezgaray, G. Grasseau, R. Murenzi [ArgAE89]:
Wavelet transforms of fractal aggregates.
J.J. Benedetto [Ben89a]:
Uncertainty principle inequalities and spectrum estimation.
S. Jaffard [Jaf89a]:
Exposants de Hölder en des points donnés et coefficients d'ondelettes.

1990 A. Cohen [Co90a]:
Ondelettes, analyses multiresolutions et filtres miroirs en quadrature.
W. Dahmen, C. Micchelli [DahmM90a]:
On stationary subdivision and the construction of compactly supported wavelets.
W. Lawton [Law90]:
Tight frames of compactly supported wavelets.

1991 J.-P. Antoine, R: Murenzi, B. Piette, M. Duval-Destin [AntMP91]:
Image analysis with 2D continuous wavelet transform: Detection of position, orientation and visual contrast of simple objects.
Y. Maday, V. Perrier, J.-C. Ravel [MadP91]:
Adaptivité dynamique sur bases d'ondelettes pour l'approximation d'équations aux dérivées partielles.

1992 B.K. Alpert [Alp92a]:
Wavelets and other bases for fast numerical linear algebra.
C.K. Chui [Chui92a]:

Wavelets and spline interpolation.
K. Gröchenig, W. Madych [GröM92]:
Multiresolution analysis, Haar bases and self-similiar tilings of $\mathbf{R}^n$.
Ph. Tchamitchian, B. Torrésani [TcT92]:
Ridge and skeleton extraction from the wavelet transform.
M.V. Wickerhauser [Wi92]:
Acoustic signal compression with wavelet packets.

1993 L. Cohen [Coh93]:
The scale representation.
J. Lewalle [Lew93]:
Energy dissipation in the wavelet transformed Navier-Stokes equation.
R.S. Strichartz [Stri93]:
Wavelets and self-affine tilings.

1994 C. Houdré [Hou94]:
Wavelets, probabilities and statistics: some bridges.
M. Mitrea [Mitr94]:
Clifford wavelets, singular integrals, and Hardy spaces.

## General Literature on Wavelets

### Introductions.

[AkH92], [Berg91a], [Chui92b], [Dau92], [FrK91], [Grosk89], [HeilW89], [JafM89], [Koo93], [Lem90a], [Mal87], [Mey88], [Mey89a], [Mey89b], [MeyJ87],
[Stra89], [Stri94]

### Surveys.

[Chui91], [Chui92b], [Coi90], [Dau89], [Dau92], [DeVL92], [Far92], [Fei90], [JawS93],
[Mey86c], [Mey90b], [Mey93], [MeyJ87], [RioV91], [Stra89 ], [Yo93]

### Monographs.

[AkH92], [Chui92b], [Dau92], [Dav91], [Fo89], [Mey90c], [Mey90d], [Mey91a], [Mey93],
[Mitr94], [Va92]

**Special Issues.**

[DauMW92], [DeVM93], [DuhFN93]

**Proceedings.**

[AskF90], [BenF94], [BeyCD92], [Chui92c], [ComGT89], [FarHV92], [KaiR90], [Lem90],
[Lig92], [SCW93]

The literature, listed in the last two sections, can be found, in detail, in the bibliography, at the end of the article.

## List of Symbols and Abbreviations

| Symbol | Meaning |
|---|---|
| $\diamondsuit$ | end of proof |
| $\mathbf{N}$ | positive integers, without zero |
| $\mathbf{N_0}$ | positive integers, including zero |
| $\mathbf{Z}$ | integers |
| $\mathbf{Q}$ | rational numbers |
| $\mathbf{R}$ | real numbers |
| $\mathbf{R}^*$ | real numbers, without zero |
| $\mathbf{R}^+$ | strictly positive real numbers |
| $\mathbf{C}$ | complex numbers |
| $\mathbf{T}$ | complex numbers $z$ such that $\lvert z\rvert = 1$ |
| $A^B$ | set of mappings from $B$ to $A$ |
| $RgA$ | range of the operator $A$ |
| $A^*$ | Hilbert adjoint of the operator $A$ |
| $\overline{M}$ | closure of $M$ |
| $spanM$ | linear span of $M$ |
| $A \otimes B$ | tensor product of $A$ and $B$ |
| $(\cdot,\cdot)_{\mathcal{H}},\ \ \lVert\cdot\rVert_{\mathcal{H}}$ | scalar product, resp. norm, of the Hilbert space $\mathcal{H}$ |
| $(\cdot,\cdot),\ \lVert\cdot\rVert$ | scalar product, resp. norm, of the Hilbert space $L^2(\mathbf{R})$ |
| $f \perp g$ | $f$ orthogonal to $g$ |
| $V^{\perp}$ | orthogonal complement of the Hilbert subspace $V \subseteq \mathcal{H}$ |
| $V \oplus W$ | orthogonal sum of the Hilbert subspaces $V, W \subseteq \mathcal{H}$ |
| $\lambda^n,\ d^n x$ | n-dimensional Lebesgue measure |
| $L^p(X,\mu), 1 \leq p < \infty$ | equivalence classes $[f]$, modulo Lebesgue zero sets, of functions $f$ such that $x \mapsto \lvert f\rvert^p$ $(x \in X)$ is $\mu$-integrable on $X$; convention: $[f] \equiv f$ |
| $\lVert f\rVert_{L^p(X,\mu)}$ | $:= (\int_X \lvert f(x)\rvert^p d\mu)^{\frac{1}{p}}\ \forall f \in L^p(X,\mu)$ |
| $L^\infty(\mathbf{R})$ | equivalence classes $[f]$, modulo Lebesgue zero sets, of functions $f$, bounded a.e.; convention: $[f] = f$ |

| | |
|---|---|
| $\|f\|_{L^\infty}$ | $:= \inf\{c \in \mathbf{R} : \exists A \subseteq \mathbf{R} : \|f\| \leq c$ on $\mathbf{R} \setminus A, \lambda(A) = 0\}$ |
| $C^n(\mathbf{R})$ | $n$-times continuously differentiable functions |
| $supp f$ | support of a function $f$: closure of the set of points in which $f$ does not vanish |
| $\mathbf{1}_A$ | characteristic function of $A$: $\mathbf{1}_A(x) = 1$, if $x \in A$, 0 otherwise $\delta_{ij}$ Kronecker delta: $\delta_{ij} = 1$, if $i = j$, 0 otherwise |
| $\hat{f}(\omega)$ | $L^2$-Fourier transform of $f$, defined by $\hat{f}(\omega) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$. |
| $f = O(g),\ as\ x \to x_0$ | $: \overline{\lim}_{x \to x_0} (\frac{\|f(x)\|}{\|g(x)\|}) < +\infty$. |
| $f = o(g),\ as\ x \to x_0$ | $: \lim_{x \to x_0} (\frac{\|f(x)\|}{\|g(x)\|}) = 0$. |

| **Abbreviation** | **Meaning** |
|---|---|
| a.e. | almost everywhere (i.e., up to a Lebesgue zero set) |
| CWFT | continuous windowed Fourier transform |
| CWT | continuous wavelet transform |
| DWFT | discrete windowed Fourier transform |
| DWT | discrete wavelet transform |
| iff | if and only if |
| LCG | locally compact group |
| MRA | multiresolution analysis |
| ONB | orthonormal basis |
| QMF | quadrature mirror filter |
| WONB | wavelet orthonormal basis |

# I. The Continuous Wavelet Transform (CWT)

## I.0. Definition and Basic Properties

DEFINITION.

i) An *analyzing wavelet*, or *mother wavelet*, is a function $\psi \in L^2(\mathbf{R})$ with $\|\psi\| > 0$.

ii) For an analyzing wavelet $\psi$ and $(a, b) \in \mathbf{R}^* \times \mathbf{R}$, define:

$$\psi_{ab}(x) := \frac{1}{\sqrt{|a|}} \psi\left(\frac{x-b}{a}\right). \qquad (I.1)$$

The functions $\psi_{ab}$, $(a, b) \in \mathbf{R}^* \times \mathbf{R}$, are called the *daughter wavelets* of $\psi$.

iii) The *continuous wavelet transform (CWT)* of a function $f \in L^2(\mathbf{R})$ with respect to the analyzing wavelet $\psi$ is defined as the following function:

$$T_\psi f : \ \mathbf{R}^* \times \mathbf{R} \to \mathbf{C}$$

$$(a, b) \mapsto T_\psi f(a, b) := \int_{-\infty}^{\infty} f(x) \overline{\psi_{ab}(x)} dx. \qquad (I.2)$$

For a fixed value $(a, b) \in \mathbf{R}^* \times \mathbf{R}$, the complex number $T_\psi f(a, b)$ is called the *wavelet coefficient* of f with respect to the analyzing wavelet $\psi$ at the point $(a, b)$. The *continuous wavelet transform operator* with respect to the analyzing wavelet $\psi$ is given by the following integral operator:

$$T_\psi : \quad L^2(\mathbf{R}) \to \mathbf{C}^{\mathbf{R}^* \times \mathbf{R}}$$

$$f \mapsto T_\psi f.$$

If there is no danger of confusion the attribute "*with respect to the analyzing wavelet* $\psi$" will be dropped in the following.

Lemma 1. (*Norm conservation and continuity*)

i) $\forall (a,b) \in \mathbf{R}^* \times \mathbf{R}$
$$\|\psi_{ab}\| = \|\psi\|; \tag{I.3}$$

ii) $\forall (a_0,b_0) \in \mathbf{R}^* \times \mathbf{R}$
$$\lim_{(a,b)\to(a_0,b_0)} \|\psi_{ab} - \psi_{a_0 b_0}\| = 0.$$

*Proof.*

i) Substituting $y := \frac{x-b}{a}$ results in:
$$\|\psi_{ab}\|^2 = \int_{-\infty}^{\infty} \frac{1}{|a|}|\psi(\frac{x-b}{a})|^2 dx = \int_{-\infty}^{\infty} |\psi(y)|^2 dy = \|\psi\|^2.$$

ii) First, assume $\psi$ to be continuous with compact support. Then, $\psi_{ab}(x) \to \psi_{a_0 b_0}(x)$ uniformly in $x$ as $(a,b) \to (a_0,b_0)$, so
$$\lim_{(a,b)\to(a_0,b_0)} \|\psi_{ab} - \psi_{a_0 b_0}\|^2 = \int_{-\infty}^{\infty} \lim_{(a,b)\to(a_0 b_0)} |\frac{1}{\sqrt{a}}\psi(\frac{x-b}{a}) -$$
$$- \frac{1}{\sqrt{a_0}}\psi(\frac{x-b_0}{a_0})|^2 dx = 0.$$
Next, consider $\psi \in L^2(\mathbf{R})$ arbitrary. Since the continuous functions with compact support are dense in $L^2(\mathbf{R})$, for given $\epsilon > 0$, there exists a continuous function $\tilde{\psi}$ with compact support, such that $\|\psi - \tilde{\psi}\| < \frac{\epsilon}{3}$. By i), this implies that $\|\psi_{ab} - \tilde{\psi}_{ab}\| < \frac{\epsilon}{3}$. By the first part of the proof, we can choose $(a,b)$ such that $\|\tilde{\psi}_{ab} - \tilde{\psi}_{a_0 b_0}\| < \frac{\epsilon}{3}$. So,
$$\|\psi_{ab} - \psi_{a_0 b_0}\| = \|\psi_{ab} - \tilde{\psi}_{ab} + \tilde{\psi}_{ab} - \psi_{a_0 b_0} + \tilde{\psi}_{a_0 b_0} - \tilde{\psi}_{a_0 b_0}\| \le$$
$$\le \|\psi_{ab} - \tilde{\psi}_{ab}\| + \|\tilde{\psi}_{ab} - \tilde{\psi}_{a_0 b_0}\| + \|\psi_{a_0 b_0} - \tilde{\psi}_{a_0 b_0}\| < \epsilon.$$
Since $\epsilon$ was arbitrary, the assertion follows. $\Diamond$

Corollary. *$T_\psi f$ is continuous as a function in $(a,b)$.*

*Proof.* By Cauchy-Schwarz's inequality,

$$|T_\psi f(a,b) - T_\psi f(a_0,b_0)| \leq \|\psi_{ab} - \psi_{a_0 b_0}\| \|f\| \ \forall (a,b),\ (a_0,b_0) \in \mathbf{R}^* \times \mathbf{R}.$$

The assertion now follows by ii) in the previous lemma. ♢

Lemma 2. *(Fourier representation of $T_\psi$)*

$$T_\psi f(a,b) = \int_{-\infty}^{\infty} \hat{f}(\omega) \frac{a}{\sqrt{|a|}} \overline{\hat{\psi}(a\omega)} e^{ib\omega} d\omega. \qquad (I.4)$$

*Proof.* By Plancharel's identity, one has:

$$T_\psi f(a,b) = (f, \psi_{ab}) = (\hat{f}, \widehat{\psi_{ab}}) = \int_{-\infty}^{\infty} \hat{f}(\omega) \overline{\frac{1}{\sqrt{|a|}} a \hat{\psi}(a\omega) e^{-ib\omega}} d\omega,$$

since $\widehat{\psi_{ab}}(\omega) = \frac{1}{\sqrt{|a|}} a \hat{\psi}(a\omega) e^{-ib\omega}$ by the trans- and dilation rule for Fourier transform. ♢

Theorem. (*Elementary properties of $T_\psi$*).

*$T_\psi$ is an injective, bounded, linear operator from $L^2(\mathbf{R})$ to $L^\infty(\mathbf{R}^* \times \mathbf{R})$ possessing the following invariance properties:*

i) $[T_\psi f(\,\cdot - x_0)](\cdot,\cdot) = T_\psi f(\cdot,\ \cdot - x_0)$ *(translation invariance)*;

ii) $[T_\psi f(c\,\cdot\,)](\cdot,\cdot) = \frac{\sqrt{|c|}}{c} T_\psi f(c\,\cdot\,, c\,\cdot\,)$ *(dilation invariance).*

*Proof.* Injectivity:

It will be shown indirectly: $T_\psi f(a,b) \equiv 0$ implies $\|f\| = 0$.

Assume: $\exists\ f \in L^2(\mathbf{R})$ such that $\|f\| > 0$ and $T_\psi f(a,b) = 0\ \ \forall (a,b) \in \mathbf{R}^* \times \mathbf{R}$.

Integrating the assumption with respect to the measure $\frac{da\,db}{a^2}$ and applying $(I.4)$ yields:

$$0 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |T_\psi f(a,b)|^2 \frac{da\,db}{a^2} =$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{a^2}{|a|}|\int_{-\infty}^{\infty}\hat{f}(\omega)\overline{\hat{\psi}(a\omega)}e^{ib\omega}d\omega|^2\frac{dadb}{a^2}.$$

Since the existence of the integral is ensured, one can apply Fubini's theorem twice, which gives, together with Plancharel's identity:

$$0 = \int_{-\infty}^{\infty}\frac{a^2}{|a|}[2\pi\int_{-\infty}^{\infty}|\hat{f}(\omega)\overline{\hat{\psi}(a\omega)}|^2 d\omega]\frac{da}{a^2} =$$

$$= \int_{-\infty}^{\infty}|\hat{f}(\omega)|^2[2\pi\int_{-\infty}^{\infty}|\hat{\psi}(a\omega)|^2\frac{da}{|a|}]d\omega.$$

Now $c_\psi := 2\pi\int_{-\infty}^{\infty}|\hat{\psi}(a\omega)|^2\frac{da}{|a|} = 2\pi\int_{-\infty}^{\infty}|\psi(u)|^2\frac{du}{|u|} > 0$, for analyzing wavelets $\psi$.

Otherwise $\frac{|\hat{\psi}(u)|}{\sqrt{|u|}}$ would vanish a.e., and hence $\|\psi\| = 0$. Using Plancharel's identity again results therefore in $c_\psi\|f\|^2 = 0$, i.e $\|f\| = 0$, in contradiction to the assumption.

Boundedness:

The Cauchy-Schwarz-inequality gives:

$$|T_\psi f(a,b)| \leq \|f\|\|\psi_{ab}\| = \|f\|\|\psi\|$$

by $(I.3)$, hence, $\|T_\psi f\|_\infty \leq C\|f\|$, where $C = \|\psi\|$.

Linearity:

$$[T_\psi(\lambda f + \mu g)](a,b) = \int_{-\infty}^{\infty}(\lambda f + \mu g)\overline{\psi_{ab}(x)}dx =$$

$$= \lambda\int_{-\infty}^{\infty}f(x)\overline{\psi_{ab}(x)}dx + \mu\int_{-\infty}^{\infty}g(x)\overline{\psi_{ab}(x)}dx =$$

$$= \lambda T_\psi f(a,b) + \mu T_\psi g(a,b).$$

Translation invariance:

$$[T_\psi f(\cdot - x_0)](a,b) = \int_{-\infty}^{\infty}f(x - x_0)\overline{\psi_{ab}(x)}dx =$$

$$= \int_{-\infty}^{\infty}f(y)\frac{1}{\sqrt{|a|}}\overline{\psi(\frac{y + x_0 - b}{a})}dy = T_\psi f(a, b - x_0).$$

Dilation invariance:

$$[T_\psi f(c\cdot)](a,b) = \int_{-\infty}^{\infty} f(cx)\frac{1}{\sqrt{|a|}}\overline{\psi(\frac{x-b}{a})}dx =$$

$$= \int_{-\infty}^{\infty} f(y)\frac{1}{\sqrt{|a|}}\overline{\psi(\frac{\frac{y}{c}-b}{a})}\frac{1}{c}dy =$$

$$= \frac{\sqrt{|c|}}{c}\int_{-\infty}^{\infty} f(x)\frac{1}{\sqrt{|ac|}}\overline{\psi(\frac{y-bc}{ac})}dy = \frac{\sqrt{|c|}}{c}T_\psi f(ca,cb). \qquad \diamondsuit$$

## I.1. Time and Frequency Localization

### General remarks.

In this paragraph, the CWT is restricted to values $(a,b) \in \mathbf{R}^+ \times \mathbf{R}$.

The variable $x$ is interpreted as a time-, the variable $\omega$ as a frequency-parameter.

### I.1.1. Time Localization

**General assumption.**

$$\psi \ \textit{analyzing wavelet}, \quad x\psi(x) \in L^2(\mathbf{R}). \tag{I.5}$$

(In particular, this is satisfied by compactly supported analyzing wavelets.)

The function

$$p_\psi : \mathbf{R} \to \mathbf{R}^+, \quad x \mapsto \frac{|\psi(x)|^2}{\|\psi\|^2}$$

can be considered as density of a (Borel) probability measure $\mu_\psi$ on $\mathbf{R}$, the function

$$X : \mathbf{R} \to \mathbf{R}, \quad x \mapsto x$$

as a random variable for $\mu_\psi$ with mean *(center)*

$$m_\psi := \int_{-\infty}^{\infty} X(x)p_\psi(x)dx = \frac{1}{\|\psi\|^2}\int_{-\infty}^{\infty} x|\psi(x)|^2dx, \qquad (I.6)$$

and standard deviation *(radius)*

$$\Delta_\psi := (\int_{-\infty}^{\infty}(X(x)-m_\psi)^2 p_\psi(x)dx)^{\frac{1}{2}} =$$

$$= (\frac{1}{\|\psi\|^2}\int_{-\infty}^{\infty}(x-m_\psi)^2|\psi(x)|^2dx)^{\frac{1}{2}}. \qquad (I.7)$$

By the general assumption, $t_\psi$ and $\Delta_\psi$ exist.

For $c \in [1,\infty[$ let $I(\psi,c) := [m_\psi - c\Delta_\psi, m_\psi + c\Delta_\psi]$.

Lemma 1. (*Center and radius of* $\psi_{ab}$)

i) *$\psi_{ab}$ fulfills the general assumption.*

ii) $m_{\psi_{ab}} = am_\psi + b$.

iii) $\Delta_{\psi_{ab}} = a\Delta_\psi$.

iv) $I(\psi_{ab},c) = [am_\psi + b - ca\Delta_\psi, am_\psi + b + ca\Delta_\psi]$.

*Proof.*

i) $\psi_{ab} \in L^2(\mathbf{R})$ by $(I.3)$.
$\int_{-\infty}^{\infty} x^2|\psi_{ab}(x)|^2dx = \int_{-\infty}^{\infty}(ay+b)^2|\psi(y)|^2dy < \infty$ by assumption.
The next two points make use of identity $(I.3)$.

ii) $m_{\psi_{ab}} = \frac{1}{\|\psi_{ab}\|^2}\int_{-\infty}^{\infty} x|\psi_{ab}(x)|^2dx = \frac{1}{\|\psi\|^2}\int_{-\infty}^{\infty}(ay+b)|\psi(y)|^2dy = am_\psi + b$.

iii) $\Delta^2_{\psi_{ab}} = \frac{1}{\|\psi_{ab}\|^2}\int_{-\infty}^{\infty}(x-m_{\psi_{ab}})^2|\psi_{ab}(x)|^2dx = \frac{1}{\|\psi\|^2}\int_{-\infty}^{\infty}(ay+b-am_\psi-b)^2|\psi(y)|^2dy = a^2\Delta^2_\psi$.

iv) Follows from ii) and iii), by definition of $I(\psi_{ab},c)$. ◇

To prove the next lemma, we need

**Tchebychev's inequality.**

*Let $\mu$ be a probability measure on* $\mathbf{R}$, *$Y$ an arbitrary random variable for $\mu$ such that the mean $EY$ and the standard deviation $\sigma_Y$ exist. Then holds*

$$\forall \epsilon > 0: \quad \mu(|Y - EY| \geq \epsilon) \leq \frac{{\sigma_Y}^2}{\epsilon^2}.$$

LEMMA 2. (*Tchebychev estimation*)

$$\|\psi - \psi \cdot \mathbf{1}_{I(\psi,c)}\| \leq \frac{\|\psi\|}{c}.$$

*Proof.* $\|\psi - \psi \cdot \mathbf{1}_{I(\psi,c)}\|^2 = \int_{\mathbf{R}\setminus I(\psi,c)} |\psi(x)|^2 dx = \|\psi\|^2 \mu_\psi(x \in \mathbf{R} : |x - m_\psi| \geq c\Delta_\psi) \leq \frac{\|\psi\|^2}{c^2}$, where the last estimation follows from Tchebychev's inequality with $\mu = \mu_\psi, \quad Y = X, \quad EY = m_\psi, \quad \sigma_Y = \Delta_\psi, \quad \epsilon = c\Delta_\psi$. $\diamondsuit$

COROLLARY.

$$\|\psi \cdot \mathbf{1}_{I(\psi,c)}\| \geq \|\psi\|(1 - \frac{1}{c}).$$

THEOREM 1. (*Time localization*). *Let $c \in ]1, \infty[$. Assume*

$$|T_\psi f(a,b)| > \frac{\|\psi\|\|f\|}{c}. \tag{I.8}$$

*Then, $f$ does not vanish on some set of positive measure in $I(\psi_{ab}, c)$.*

*Proof.* (indirectly). Assume, $f = 0$ a.e. on $I(\psi_{ab}, c)$. Then:

$$|T_\psi f(a,b)| = |\int_{-\infty}^{\infty} f(x)\overline{\psi_{ab}(x)}dx| = |\int_{\mathbf{R}\setminus I(\psi,c)} f(x)\overline{\psi_{ab}(x)}dx| =$$

$$= |\int_{\mathbf{R}} f(x)[\overline{\psi_{ab}(x)} - \overline{\psi_{ab}(x)} \cdot \mathbf{1}_{I(\psi_{ab},c)}(x)]dx|,$$

what is by Cauchy-Schwarz's inequality less or equal to

$$\|f\|\|\psi_{ab} - \psi_{ab} \cdot \mathbf{1}_{I(\psi_{ab},c)}\|.$$

By lemma 2 and $(I.3)$, this is smaller or equal to

$$\frac{\|f\| \ \|\psi_{ab}\|}{c} = \frac{\|f\| \ \|\psi\|}{c},$$

in contradiction to $(I.8)$. $\diamondsuit$

Note that, for $c = 1$, the above theorem makes no sense, since $|T_\psi f(a,b)| \leq \|\psi\|\|f\| \ \forall \ (a,b) \in \mathbf{R}^+ \times \mathbf{R}$, by Cauchy-Schwarz's inequality.

Corollary 1. *If $T_\psi f(a,b) \neq 0$, then there exists a number $c > 1$ such that $f$ does not vanish on some set of positive measure in $I(\psi_{ab}, c)$.*

*Proof.* Choose $c > \frac{\|\psi\| \ \|f\|}{|T_\psi f(a,b)|}$, in theorem 1. $\diamondsuit$

Note that $T_\psi f(a,b) = 0$ does *not* imply that $f(x) = 0$, for $x$ in a neighbourhood of $am_\psi + b$. Choose, for example, $f = r \cdot \mathbf{1}_{[-s,s]}$, where $r > 0, \ s > 1$ arbitrary,

$$\psi(x) = \begin{cases} -x-1, & x \in [-1, -\frac{1}{2}[ \\ x, & x \in [-\frac{1}{2}, \frac{1}{2}[ \\ 1-x, & x \in [\frac{1}{2}, 1] \\ 0, & otherwise \end{cases} .$$

Then, $m_\psi = 0, \ 0.36 < \Delta_\psi < 0.37$, $T_\psi f(1,0) = \int_{-\infty}^{\infty} \overline{\psi(x)} dx = 0$, but, $f(x) \equiv r > 0$ on $I(\psi_{10}, \frac{s}{\Delta_\psi})$. The essential property of $\psi$, responsible for the described phenomenon, is that $\psi$ fulfills the *vanishing moment condition*

$$\int_{-\infty}^{\infty} \psi(x) dx = 0. \qquad (I.9)$$

Corollary 2. *(Time zooming). Let $b \in \mathbf{R}$. If there exists a constant $c \in ]1, \infty[$ such that $(I.8)$ holds, for $a > 0$ arbitrarily small, then $b \in supp f$.*

*Proof.* Lemma 1 iv) implies: $\forall c > 1$, $\forall \epsilon > 0$ $\exists a_0 > 0$ such that $I(\psi_{ab}, c) \subseteq ]b-\epsilon, b+\epsilon[$ $\forall a \leq a_0$. The assertion now follows by theorem 1. $\diamondsuit$

QUESTION. Do stronger assumptions on $T_\psi f(a,b)$ allow more precise information about the behaviour of $f$ on $I(\psi_{ab}, c)$, $c \in ]1, \infty[$, than just non-vanishing?

THEOREM 2. *(Time modulus estimation). Let $c \in ]1, \infty[$, $\gamma > 0$. If*

$$|T_\psi f(a,b)| > \frac{\|\psi\|}{c}(\|f\| + \gamma), \tag{$I.10$}$$

*then, on some set of positive measure in $I(\psi_{ab}, c)$,*

$$|f(x)| > \frac{\gamma}{c^{\frac{3}{2}}(2a\Delta_\psi)^{\frac{1}{2}}}. \tag{$I.11$}$$

*Proof* (indirectly): Assume,

$$|f(x)| \leq \frac{\gamma}{c^{\frac{3}{2}}(2a\Delta_\psi)^{\frac{1}{2}}} \quad a.e. \;\; on \;\; I(\psi_{ab}, c). \tag{$I.12$}$$

$$|T_\psi f(a,b)| = |\int_{-\infty}^{\infty} f(x)\overline{\psi_{ab}(x)}dx| \leq$$

$$\leq |\int_{-\infty}^{\infty} f(x)[\overline{\psi_{ab}(x)} - \overline{\psi_{ab}(x)} \cdot \mathbf{1}_{I(\psi_{ab},c)}]dx| +$$

$$+|\int_{-\infty}^{\infty} f(x)\overline{\psi_{ab}(x)} \cdot \mathbf{1}_{I(\psi_{ab},c)}(x)dx|.$$

The Cauchy-Schwarz-inequality, together with lemma 2, yields that the first term is smaller or equal to $\frac{\|\psi\| \, \|f\|}{c}$. The second term can be estimated by $(I.12)$ as less or equal to $\frac{\gamma}{c^{\frac{3}{2}}(2a\Delta_\psi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} |\psi_{ab}(x)| \cdot \mathbf{1}_{I(\psi_{ab},c)}(x)dx$. By the Cauchy-Schwarz-inequality, the integral is less or equal to $\|\psi\| \cdot \lambda(I(\psi_{ab}, c))^{\frac{1}{2}} = \|\psi\| \cdot (2ca\Delta_\psi)^{\frac{1}{2}}$ (lemma 1 iv)). Hence, the second term is majorized by $\frac{\gamma\|\psi\|}{c}$. Altogether we get: $|T_\psi f(a,b)| \leq \frac{\|\psi\|}{c}(\|f\| + \gamma)$, in contradiction to $(I.10)$. $\diamondsuit$

Corollary 1. *(Time modulus estimation by the norm). Let* $c \in ]1, \infty[$. If

$$|T_\psi f(a,b)| > \frac{2\|\psi\|\|f\|}{c}, \qquad (I.13)$$

*then, on some set of positive measure in* $I(\psi_{ab}, c)$,

$$|f(x)| > \frac{\|f\|}{c^{\frac{3}{2}}(2a\Delta_\psi)^{\frac{1}{2}}}.$$

*Proof.* Choose $\gamma = \|f\|$ in theorem 2. ♢

Corollary 2. (*Localization of singularities in time, by zooming*). *If there exists a constant* $c \in ]1, \infty[$ *such that* $(I.8)$ *holds, for* $a > 0$ *arbitrarily small, then*

$$\limsup_{x \to b} |f(x)| = \infty, \quad \textit{uniformly in } b.$$

*Proof.* Choose $\delta > 0$ arbitrary. Define $\tilde{c} := c(1+\delta) > 1$, $\gamma := \|f\|\delta > 0$. By assumption,

$$|T_\psi f(a,b)| > \frac{\|\psi\|\|f\|(1+\delta)}{c(1+\delta)} = \frac{\|\psi\|}{\tilde{c}}(\|f\| + \gamma).$$

Since $I(\psi_{ab}, \tilde{c})$ shrinks to $b$, as $a$ tends to zero, theorem 2 yields the assertion. ♢

Note that the last corollary is a strengthening of corollary 2 to theorem 1.

### I.1.2 Frequency Localization

**General assumption.**

$$\psi \ \textit{analyzing wavelet}, \ \omega\hat{\psi}(\omega) \in L^2(\mathbf{R}). \qquad (I.14)$$

(In particular, this is satisfied by analyzing wavelets with compactly supported Fourier transform.)

The general assumption guarantees that $m_{\hat{\psi}}$, $\Delta_{\hat{\psi}}$ and $I(\hat{\psi}, c)$ exist (cf. I.1.1).

LEMMA. *(Center and radius of $\widehat{\psi_{ab}}$)*

i) $\psi_{ab}$ *fulfills the general assumption.*

ii) $m_{\widehat{\psi_{ab}}} = \frac{m_{\hat{\psi}}}{a}$.

iii) $\Delta_{\widehat{\psi_{ab}}} = \frac{\Delta_{\hat{\psi}}}{a}$.

iv) $I(\widehat{\psi_{ab}}, c) = [\frac{m_{\hat{\psi}}}{a} - c\frac{\Delta_{\hat{\psi}}}{a}, \frac{m_{\hat{\psi}}}{a} + c\frac{\Delta_{\hat{\psi}}}{a}]$.

*Proof.*

i) $\psi_{ab} \in L^2(\mathbf{R})$ by $(I.3)$.
$\int_{-\infty}^{\infty} \omega^2 |\widehat{\psi_{ab}}|^2 d\omega = \int_{-\infty}^{\infty} (\frac{u}{a})^2 |\hat{\psi}(u)|^2 du = \frac{1}{a^2} \int_{-\infty}^{\infty} u^2 |\hat{\psi}(u)|^2 du$, which is finite by assumption.

The next two points make use of identity $(I.3)$.

ii) $m_{\widehat{\psi_{ab}}} = \frac{1}{||\widehat{\psi_{ab}}||^2} \int_{-\infty}^{\infty} \omega |\widehat{\psi_{ab}}|^2 d\omega = \frac{1}{||\psi||^2} \int_{-\infty}^{\infty} \frac{u}{a} |\hat{\psi}(u)|^2 du = \frac{m_{\hat{\psi}}}{a}$.

iii) $\Delta^2_{\widehat{\psi_{ab}}} = \frac{1}{||\widehat{\psi_{ab}}||^2} \int_{-\infty}^{\infty} (\omega - \frac{m_{\hat{\psi}}}{a})^2 |\widehat{\psi_{ab}}|^2 d\omega$
$= \frac{1}{||\hat{\psi}||^2} \int_{-\infty}^{\infty} (\frac{u}{a} - \frac{m_{\hat{\psi}}}{a})^2 |\hat{\psi}(u)|^2 du = \frac{1}{a^2} \Delta^2_{\hat{\psi}}$.

iv) Follows from ii) and iii) by definition of $I(\widehat{\psi_{ab}}, c)$. $\diamondsuit$

Note that the lemma implies:

$$Q(a, b) := \frac{m_{\widehat{\psi_{ab}}}}{\Delta_{\widehat{\psi_{ab}}}} = \frac{m_{\hat{\psi}}}{\Delta_{\hat{\psi}}} = const. \quad \forall\ (a, b) \in \mathbf{R}^+ \times \mathbf{R}.$$

This is called *Constant-Q*-property, in signal analysis.

From iv) follows: The interval $I(\widehat{\psi_{ab}}, c)$ increases, as $a$ tends to 0, and is independent of $b$. In contrary, $I(\psi_{ab}, c)$ contracts to the single point $b$, in the same limit. Hence, if there exist frequency-analogues of the time-zooming-statements in I.1.1, they must have a different form.

THEOREM 1. *(Frequency localization). If* $(I.8)$ *holds, then* $\hat{f}$ *does not vanish on some set of positive measure in* $I(\widehat{\psi_{ab}}, c)$.

*Proof* (indirectly). Assume, $f = 0$ a.e. on $I(\widehat{\psi_{ab}}, c)$. By Plancharel's theorem follows:

$$|T_\psi(a,b)| = |\int_{-\infty}^{\infty} \hat{f}(\omega)\overline{\widehat{\psi_{ab}}(\omega)}d\omega| = |\int_{\mathbf{R}\setminus I(\widehat{\psi_{ab}},c)} \hat{f}(\omega)\overline{\widehat{\psi_{ab}}(\omega)}d\omega| =$$

$$= |\int_{-\infty}^{\infty} \hat{f}(\omega)[\overline{\widehat{\psi_{ab}}(\omega)} - \overline{\widehat{\psi_{ab}}(\omega)} \cdot \mathbf{1}_{I(\widehat{\psi_{ab}},c)}]d\omega|,$$

what is, by Cauchy-Schwarz's inequality, smaller or equal than

$$\|f\|\|\widehat{\psi_{ab}} - \widehat{\psi_{ab}} \cdot \mathbf{1}_{I(\widehat{\psi_{ab}},c)}\|,$$

for which $\frac{\|f\|\|\psi\|}{c}$ is a majorante, by the Tchebychev estimation, Plancharel's theorem and $(I.3)$. But, this contradicts $(I.8)$. ◊

COROLLARY 1. *If* $T_\psi f(a,b) \neq 0$, *then there exists a number* $c > 1$ *such that* $\hat{f}$ *does not vanish, on some set of positive measure in* $I(\widehat{\psi_{ab}}, c)$.

*Proof.* Choose $c > \frac{\|\psi\|\|f\|}{|T_\psi f(a,b)|}$, in theorem 1. ◊

Note that, analogously to time localization, $T_\psi f(a,b) = 0$ does *not* imply that $\hat{f}(\omega) = 0$, for $\omega$ in a neighbourhood of $\frac{m_{\hat{\psi}}}{a}$. To show this, it suffices to consider functions, $\psi$ and $f$ having as Fourier transforms the functions, defined in the corresponding example in I.1.1.

COROLLARY 2. *(Frequency zooming). If there exist constants* $b \in \mathbf{R}$, $c \in ]1, \infty[$ *such that* $(I.8)$ *holds, for a arbitrarily large, then* $0 \in supp \hat{f}$.

*Proof.* Lemma 1 iv) implies: $\forall\ c > 1\ \forall\ \epsilon > 0\ \ \exists\ a_0 > 0$ such that $I(\widehat{\psi_{ab}}, c) \subseteq ] - \epsilon, \epsilon[\ \forall a > a_0$. The assertion now follows by theorem 1. $\diamondsuit$

Frequency zooming is less important, for applications, than time zooming. The reason is that the only point in frequency-space, one can zoom in, is zero, while, in time-space, one can zoom in arbitrary points $b \in \mathbf{R}$, in time-space.

THEOREM 2. *(Frequency modulus estimation). If* $(I.10)$ *holds, then*

$$|\hat{f}(\omega)| > \frac{\gamma}{c(\frac{2c}{a}\Delta_{\hat{\psi}})^{\frac{1}{2}}},$$

*on some set of positive measure* in $I(\widehat{\psi_{ab}}, c)$.

*Proof* (indirectly): Assume,

$$|\hat{\psi}(\omega)| \leq \frac{\gamma}{c(\frac{2c}{a}\Delta_{\hat{\psi}})^{\frac{1}{2}}} \quad a.e. \ \ on \ \ I(\widehat{\psi_{ab}}, c). \tag{$I.15$}$$

By the triangle inequality: $|T_\psi f(a,b)| \leq$

$$|\int_{-\infty}^{\infty} \hat{f}(\omega)[\overline{\widehat{\psi_{ab}}(\omega)} - \overline{\widehat{\psi_{ab}}(\omega)} \cdot \mathbf{1}_{I(\widehat{\psi_{ab}},c)}]d\omega| +$$

$$+ |\int_{-\infty}^{\infty} \hat{f}(\omega)\overline{\widehat{\psi_{ab}}(\omega)} \cdot \mathbf{1}_{I(\widehat{\psi_{ab}},c)}(\omega)d\omega|.$$

The Tchebychev estimation, together with the Cauchy-Schwarz-inequality and $(I.15)$, yields:

$$|T_\psi f(a,b)| \leq \frac{\|f\|\|\psi\|}{c} + \frac{\gamma}{c(\frac{2c}{a}\Delta_{\hat{\psi}})^{\frac{1}{2}}}\|\psi\|\lambda(I(\widehat{\psi_{ab}}, c))^{\frac{1}{2}},$$

where $\lambda(I(\widehat{\psi_{ab}}, c)) = \frac{2c}{a}\Delta_{\hat{\psi}}$ by lemma 1 iv), hence $|T_\psi f(a,b)| \leq \frac{\|\psi\|}{c}(\|f\| + \gamma)$, in contradiction to assumption $(I.10)$. $\diamondsuit$

Corollary 1. *(Frequency modulus estimation by the norm)* *If* $(I.13)$ *holds, then, on some set of positive measure in* $I(\widehat{\psi_{ab}}, c)$,

$$|\hat{f}(\omega)| > \frac{\|f\|}{c(\frac{2c}{a}\Delta_{\hat{\psi}})^{\frac{1}{2}}}.$$

*Proof.* Choose $\gamma = \|f\|$, in theorem 2. ♢

Corollary 2. *(Localization of singularities in frequency, by zooming). If there exist constants* $b \in \mathbf{R}$, $c \in ]1, \infty[$ *such that* $(I.8)$ *holds, for a arbitrarily large, then*

$$\limsup_{\omega \to 0} |\hat{f}(\omega)| = \infty.$$

*Proof.* Analogously to the corresponding corollary in I.1.1. ♢

This corollary is a strengthening of corollary 2 to theorem 1.

### I.1.3 Time-frequency Localization

**General assumption.**

$$\psi \ \textit{analyzing wavelet}, \ x\psi(x) \in L^2(\mathbf{R}), \ \ \omega\hat{\psi}(\omega) \in L^2(\mathbf{R}). \quad (I.16)$$

One may ask, whether there exist functions satisfying the general assumption. Previously it was mentioned that the condition $x\psi(x) \in L^2(\mathbf{R})$ is in particular satisfied by compactly supported $\psi$, the condition $\omega\hat{\psi}(\omega) \in L^2(\mathbf{R})$ by $\psi$ with compactly supported Fourier transform $\hat{\psi}$. So, the most simple function, satisfying the general assumption, would be one that is compactly supported in time as well as in frequency space. However, by an immediate consequence of the *Paley-Wiener-Theorem*[1], the only function with this

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1 See e.g. Y. Katznelson, *An introduction in harmonic analysis*, New York (1969), p. 173.

property is $\psi \equiv 0$, which is no analyzing wavelet, by definition. But, there are functions satisfying the general assumption, e.g. the

$$Gaussian \quad functions: \quad g_\alpha(x) := \frac{1}{2\sqrt{\pi\alpha}} e^{-\frac{x^2}{4\alpha}} \quad (\alpha > 0), \qquad (I.17)$$

as well as their derivatives and linear combinations of these functions.

The general assumption being valid, one can combine the results of sections I.1.1 and I.1.2 and extract simultaneously time- and frequency-information about $f$, from the single wavelet coefficient $T_\psi f(a,b)$. This will become more transparent by using the following notation:

For $f \in L^2(\mathbf{R})$, let

$$F : \mathbf{R} \times \mathbf{R} \to \mathbf{C} \times \mathbf{C} \qquad (x,\omega) \mapsto (f(x), \hat{f}(\omega)).$$

$\mathbf{R} \times \mathbf{R}$ is interpreted as *time-frequency-plane*. Accordingly, for $c \in [1,\infty[$, the rectangle $R(\psi,c) := I(\psi,c) \times I(\hat{\psi},c)$ is called the *time-frequency-window* of $\psi$ and $c$.

THEOREM 1. *(Time-frequency localization)*
*Under assumption* $(I.8)$ *holds*

$$F(x,\omega) \neq (0,0),$$

*on some set of positive (2-dim.) measure in* $R(\psi_{ab}, c)$.

*Proof.* Combination of theorems 1 in I.1.1 and I.1.2 (time localization and frequency localization). ◊

Strictly speaking, the theorems on time localization and frequency localization contain more than the last theorem tells.

COROLLARY. *If* $T_\psi f(a,b) \neq 0$, *then there exists a number* $c > 1$ *such that* $F$ *does not vanish on some set of positive measure in* $R(\psi_{ab}, c)$.

*Proof.* Combination of corollaries 1 to theorems 1 in I.1.1 and I.1.2. ◊

THEOREM 2.

$$\lambda^2(R(\psi_{ab}, c)) = 4c^2 \Delta_\psi \Delta_{\hat{\psi}}. \qquad (I.18)$$

*Proof.* $\lambda^2(R(\psi_{ab}, c)) = 2ca\Delta_\psi \cdot 2c\frac{\Delta_{\hat{\psi}}}{a}$. ◊

Using the area of $R(\psi_{ab}, c)$, for $c$ fixed, as a measure for the uncertainty of time-frequency localization by $T_\psi$ at $(am_\psi + b, \frac{m_{\hat{\psi}}}{a})$, one gets:

COROLLARY. *(Area invariance of the time-frequency window). The uncertainty of time-frequency localization by* $T_\psi$ at $(am_\psi + b, \frac{m_{\hat{\psi}}}{a})$ *is independent of* $a$ *and* $b$, *bounded below by* $2c^2$, *uniformly in* $\psi$, *and minimal (equal to* $2c^2$*) iff* $\psi$ *is a Gaussian function.*

*Proof.* Follows from the last theorem, in combination with

**Heisenberg's Uncertainty Principle**.

$$\Delta_\psi \Delta_{\hat{\psi}} \geq \frac{1}{2},$$

*where equality holds iff* $\psi$ *is a Gaussian function.* ◊

Illustration of the $(a, b)$-dependence of time-frequency-windows, for $c = 1$:

[Chui92b]

## I.2. Orthogonality Relation and Applications

### I.2.1. The Admissibility Condition

Definition. An analyzing wavelet $\psi$ is called *admissible*, if it satisfies the following
*admissibility condition*:

$$c_\psi := 2\pi \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega < \infty, \tag{$I$.19}$$

what means that the function $x \mapsto \frac{|\hat{\psi}(\omega)|}{\sqrt{|\omega|}}$ is an element of $L^2(\mathbf{R})$.

Remarks.

i) For $\psi \in L^2(\mathbf{R}) \cap L^1(\mathbf{R})$, the Fourier transform $\hat{\psi}$ is continuous. If for such a $\psi$ ($I$.19) holds, this implies the vanishing moment condition

$$\hat{\psi}(0) = \int_{-\infty}^{\infty} \psi(x) dx = 0$$

(cf. ($I$.9)). So a real-valued $\psi$ changes its sign at least once, i.e. *$\psi$ oscillates*. On the other hand, due to $\psi \in L^2(\mathbf{R})$, *$\psi$ decays at infinity*. Both observations together explain the naming *wave-let* (resp. *onde-lette* in French).

ii) For all analyzing wavelets $\psi$, $\quad c_\psi > 0$ holds, since $\|\psi\| > 0$, by definition.

iii) By restricting to *admissible* analyzing wavelets, one can prove explicit inversion formulas for CWT. This will be done in the following sections. But, the admissibility condition will also be important in the discrete wavelet transform theory (cf.II.2). A deeper insight into the meaning of $c_\psi$, one can get from the group theoretical analysis of wavelet transform, treated in chapter III.

iv) Since the Gaussian functions are elements of $L^2(\mathbf{R}) \cap L^1(\mathbf{R})$, and their Fourier transforms are again Gaussian functions (therefore possess no zeros), by remark i), *no* Gaussian function is an admissible analyzing wavelet. However, from a Gaussian wavelet, one can construct admissible, analyzing wavelets, which approximate a time-frequency-localization of minimal uncertainty. (See the following examples i), iii) and iv).)

EXAMPLES.

i) *Morlet wavelets* (complex-valued):
$\psi(x) = \pi^{\frac{1}{4}}(e^{-i\gamma x} - e^{-\frac{\gamma^2}{2}})e^{-\frac{x^2}{2}} \quad (\gamma \in \mathbf{R})$.
This corresponds to a modulated Gaussian.

ii) *Paul wavelets* (complex-valued):
$\psi(x) = \frac{\Gamma(\beta+1)}{(1-ix)^{\beta+1}} \quad (\beta \in \mathbf{R}^+ \setminus \{0\})$.

iii) *Derivations of the Gaussian* (real-valued):
$\psi(x) = (-1)^m \frac{d^m}{dx^m} e^{-\frac{x^2}{2}} \quad (m \in \mathbf{N} \setminus \{1\})$.
Especially for m=2 the function $\psi$ is called *Marr wavelet* or *Mexican hat.*

iv) *Difference of Gaussians* (DOG wavelet, real-valued):

$\psi(x) = e^{-\frac{x^2}{2}} - \frac{1}{2}e^{-\frac{x^2}{8}}$.

v) *Top hat.* (real-valued)

$$\psi(x) = \begin{cases} 1 & |x| < 1 \\ -\frac{1}{2} & 1 \leq |x| \leq 3 \\ 0 & |x| > 3. \end{cases}$$

This yields an approximation for the Mexican hat.

Illustrations.

Morlet wavelet [Far92]

Paul wavelet [Far92]

Mexican hat [Far92]

DOG wavelet [Far92]

Top hat

THEOREM 1. *(Abundance of admissible wavelets)*

*The set of admissible analyzing wavelets, together with the zero-element of $L^2(\mathbf{R})$, constitutes a dense vector subspace of $L^2(\mathbf{R})$.*

*Proof. Step 1.* Subspace property. Let $\psi_1$, $\psi_2$ be admissible analyzing wavelets, $\lambda$, $\mu \in \mathbf{C}$. Define: $\psi_3 := \lambda\psi_1 + \mu\psi_2$. Then $\psi_3 \in L^2(\mathbf{R})$ and

$$c_{\psi_3} = 2\pi \int_{-\infty}^{\infty} \frac{|\lambda\psi_1(\omega) + \mu\psi_2(\omega)|^2}{|\omega|} d\omega \leq$$

$$\leq 2\pi|\lambda|^2 \int_{-\infty}^{\infty} \frac{|\psi_1(\omega)|^2}{|\omega|} d\omega + 2\pi|\mu|^2 \int_{-\infty}^{\infty} \frac{|\psi_2(\omega)|^2}{|\omega|} d\omega =$$

$$= |\lambda|^2 c_{\psi_1} + |\mu|^2 c_{\psi_2} < \infty,$$

i.e. $\psi_3$ is an admissible analyzing wavelet, too.

*Step 2.* Density. Even more will be shown:

a) If $\psi$ is an admissible wavelet, then, $\psi_{ab}$ is an admissible wavelet $\forall\ (a,b) \in \mathbf{R}^* \times \mathbf{R}$.

b) If $\psi$ is analyzing wavelet, then, $span\{\psi_{ab},\ (a,b) \in \mathbf{R}^* \times \mathbf{R}\}$ is dense in $L^2(\mathbf{R})$.
a) and b), together with step 1, prove the assertion.

*Proof.*
a) Let $\psi$ be an admissible wavelet. $\psi_{ab} \in L^2(\mathbf{R})$ by $(I.3)$.

$$c_{\psi_{ab}} = 2\pi \int_{-\infty}^{\infty} |a| \frac{|\hat{\psi}(a\omega)|^2}{|\omega|} d\omega = |a| c_\psi < \infty,$$

i.e. $\psi_{ab}$ is an admissible wavelet,too.
b) (indirectly). Assume $\overline{span}\{\psi_{ab}, (a,b) \in \mathbf{R}^* \times \mathbf{R}\} \neq L^2(\mathbf{R})$. Then there exists a function $f \in L^2(\mathbf{R})$ such that $\|f\| > 0$, and

$$f \perp \overline{span}\{\psi_{ab}, (a,b) \in \mathbf{R}^* \times \mathbf{R}\}.$$

In particular, $0 = (f, \psi_{ab}) = T_\psi f(a,b)\ \forall\ (a,b) \in \mathbf{R}^* \times \mathbf{R}$. Since $T_\psi$ is injective, by the theorem in I.0, this leads to the desired contradiction. $\diamondsuit$

THEOREM 2. *(Square integrability). Let $\psi$ be an admissible analyzing wavelet. Then,*

$$T_\psi f \in L^2(\mathbf{R}^* \times \mathbf{R}, \frac{dadb}{a^2}) \qquad \forall f \in L^2(\mathbf{R}) \qquad (I.20).$$

*Proof.* One has to show: $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |T_\psi f(a,b)|^2 \frac{dadb}{a^2} < \infty$. Since the considered integrand is positive, one can change the order of integration, by Tonelli's theorem. So one gets, by the same calculation as in the injectivity proof in I.0,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |T_\psi f(a,b)|^2 \frac{dadb}{a^2} = c_\psi \|f\|^2,$$

using the assumption $c_\psi < \infty$.

### I.2.2. Orthogonality Relation.

**General assumption**. $\psi$ admissible analyzing wavelet.

THEOREM. *(Orthogonality)*

$$\forall\ f,\ g \in L^2(\mathbf{R}):$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [T_\psi f(a,b)\overline{T_\psi g(a,b)}] \frac{dadb}{a^2} = c_\psi \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx. \qquad (I.21)$$

*Proof.* [Dau92,p.24], [HeilW89,p.640]. The existence of the integral is ensured, by theorem 2 of the last section. So, in the sequel, the order of integration can be interchanged, by Fubini's theorem. The following calculations are an extension of those, used in the injectivity proof in I.0.

Define $F_a(\omega) := \hat{f}(\omega)\overline{\hat{\psi}(a\omega)}$, $G_a(\omega) := \hat{g}(\omega)\overline{\hat{\psi}(a\omega)}$. Applying $(I.4)$ results in:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} T_\psi f(a,b)\overline{T_\psi g(a,b)}\frac{dadb}{a^2} =$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[\int_{-\infty}^{\infty} F_a(\omega)e^{ib\omega}d\omega\overline{\int_{-\infty}^{\infty} G_a(\tilde{\omega})e^{ib\tilde{\omega}}d\tilde{\omega}}]\frac{dadb}{|a|}$$

$$= \int_{-\infty}^{\infty}\{\int_{-\infty}^{\infty}[\int_{-\infty}^{\infty}\overline{G_a(\tilde{\omega})}e^{-ib\tilde{\omega}}d\tilde{\omega}\overline{\int_{-\infty}^{\infty}\overline{F_a(\omega)}e^{-ib\omega}d\omega}]db\}\frac{da}{|a|},$$

which is by Plancharel's formula equal to

$$\int_{-\infty}^{\infty} 2\pi[\int_{-\infty}^{\infty}\overline{G_a(\xi)}F_a(\xi)d\xi]\frac{da}{|a|} =$$

$$= \int_{-\infty}^{\infty} 2\pi[\int_{-\infty}^{\infty}\hat{f}(\xi)\overline{\hat{\psi}(a\xi)}\overline{\hat{g}(\xi)}\hat{\psi}(a\xi)d\xi]\frac{da}{|a|} =$$

$$= \int_{-\infty}^{\infty}\hat{f}(\omega)\overline{\hat{g}(\omega)}[2\pi\int_{-\infty}^{\infty}|\hat{\psi}(a\xi)|^2\frac{da}{|a|}]d\xi = c_\psi\int_{-\infty}^{\infty} f(x)\overline{g}(x)dx$$

by Plancharel's formula. ♢

$(I.21)$ is often called *resolution of unity*, in accordance with the language, used in the theory of *coherent states* (see Appendix B). Wavelets constitute a special example for those.

COROLLARY. *(Isometry). The map* $\frac{1}{\sqrt{c_\psi}}T_\psi : L^2(\mathbf{R}) \to L^2(\mathbf{R}^* \times \mathbf{R}, \frac{da\,db}{a^2})$ *is an isometry.*

*Proof.* Choose $f = g$, in above theorem. ♢

The next two sections are dedicated to further consequences of $(I.21)$.

### I.2.3. Inversion Formulas.

**General assumption**. $\psi$ admissible analyzing wavelet, $f \in L^2(\mathbf{R})$ arbitrary.

An alternative interpretation of $(I.21)$ is that the following inversion formula holds, in the weak sense

$$f = \frac{1}{c_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} T_\psi f(a,b) \psi_{ab} \frac{dadb}{a^2}. \qquad (I.22)$$

This is the starting point to prove some stronger inversion formulas for CWT.

$L^2$**-inversion.**
*For* $0 < A_1 < A_2 < \infty,\ 0 < B < \infty$ *define* $f_{A_1A_2B}$ *by*

$$(f_{A_1A_2B}, g) := \frac{1}{c_\psi} \int_{-B}^{B} \int_{A_1 \le |a| \le A_2} T_\psi f(a,b)(\psi_{ab}, g) \frac{dadb}{a^2} \quad \forall g \in L^2(\mathbf{R}).$$

*Then,*

$$\lim_{A_1 \to 0,\ A_2, B \to \infty} \|f - f_{A_1A_2B}\| \to 0.$$

*Proof.* [Dau92,p.25]. $f_{A_1A_2B}$ is a uniquely defined element of $L^2(\mathbf{R})$, by Riesz' representation theorem, since

$$|(f_{A_1A_2B}, g)| \le \frac{1}{c_\psi} 4B(\frac{1}{A_1} - \frac{1}{A_2})\|f\|\|\psi\|^2\|g\| \quad \forall g \in L^2(\mathbf{R}),$$

by Cauchy-Schwarz's inequality. I.e. $g \mapsto (f_{A_1A_2B}, g)$ defines a bounded, linear functional on $L^2(\mathbf{R})$.

$$\|f - f_{A_1A_2B}\| = \sup_{\|g\|=1} |(f - f_{A_1A_2B}, g)| \le$$

$$\le \sup_{\|g\|=1} |\frac{1}{c_\psi} \int_{|b| \ge B} \int_{|a| \ge A_2 \ or \ |a| \le A_1} T_\psi f(a,b) \overline{T_\psi g(a,b)} \frac{dadb}{a^2}|,$$

and the last term is, by the Cauchy-Schwarz inequality, smaller or equal to

$$\sup_{\|g\|=1} [\frac{1}{c_\psi} \int_{|b| \ge B} \int_{|a| \ge A_2 \ or \ |a| \le A_1}$$

$$|T_\psi f(a,b)|^2 \frac{dadb}{a^2}]^{\frac{1}{2}} [\frac{1}{c_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |T_\psi g(a,b)|^2 \frac{dadb}{a^2}]^{\frac{1}{2}}.$$

By the orthogonality relation, the second factor is equal to 1, while the first one converges to zero, since the integral exists. $\diamondsuit$

LEMMA. *(Gaussian functions as approximate identity). Define* $g_\alpha$ *as in* $(I.17)$. *Let* $u \in L^1(\mathbf{R})$. *Then, in every point* $x \in \mathbf{R}$, *in which* $u$ *is continuous, holds.*

$$\lim_{\alpha \to 0+} \; u * g_\alpha(x) \to u(x). \qquad (I.23)$$

*Proof.* [Chui92b,p.29]

Gaussian functions [Chui92b]

Note first that $\forall\ \alpha > 0 \quad \int_{-\infty}^{\infty} g_\alpha(x)dx = 1, \quad \lim_{|x| \to \infty} g_\alpha(x) \to 0$. Let $u$ be continuous at $x$, i.e. $\forall \epsilon > 0 \ \ \exists \delta > 0 : |u(x-h) - u(x)| < \epsilon$ $\forall |h| < \delta$.

From these remarks follows:

$$|u * g_\alpha(x) - u(x)| = |\int_{-\infty}^{\infty} u(x-h) g_\alpha(h) dh - u(x) \cdot 1| =$$

$$= |\int_{-\infty}^{\infty} (u(x-h) - u(x)) g_\alpha(h) dh| \leq$$

$$\leq \int_{-\delta}^{\delta} |u(x-h) - u(x)| g_\alpha(h) dh + \int_{|h| \geq \delta} (|u(x-h)| + |u(x)|) g_\alpha(h) dh \leq$$

$$\leq \epsilon \int_{-\delta}^{\delta} g_\alpha(h) dh + \|u\|_1 max_{|h| \geq \delta} g_\alpha(h) + |u(x)| \int_{|h| \geq \delta} g_\alpha(h) dh \leq$$

$$\leq \epsilon \cdot 1 + \|u\|_1 g_\alpha(\delta) + |u(x)| \int_{|h| \geq \frac{\delta}{\sqrt{\alpha}}} g_1(h)dh.$$

Now, $g_\alpha(x) \to 0, \ \forall\, x$, as $\alpha \to 0+$, and $\frac{\delta}{\sqrt{\alpha}} \to \infty$ , as $\alpha \to 0+$, so the last two summands vanish, as $\alpha$ tends to $0+$. Since $\epsilon$ was arbitrary, the assertion is proved. $\diamondsuit$

**Pointwise inversion**.

*In addition to the general assumption, let $f, \psi \in L^1(\mathbf{R})$* and $\psi$ be continuous.

*Then, in every point $x$ in which $f$ is continuous, holds.*

$$f(x) = \frac{1}{c_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} T_\psi f(a,b) \psi_{ab}(x) \frac{dadb}{a^2}. \qquad (I.24)$$

*Proof.* [Chui92b,p.62]. Assume $f$ is continuous in $x$. Choose $g(y) = \overline{g_\alpha(x-y)}$, in the orthogonality relation $(I.21)$. By this, one gets:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} T_\psi f(a,b) \overline{T_\psi \overline{g_\alpha(x-\cdot)}(a,b)} \frac{dadb}{a^2} =$$

$$= c_\psi \int_{-\infty}^{\infty} f(y) g_\alpha(x-y) dy = c_\psi (f * g_\alpha)(x),$$

which goes to $c_\psi f(x)$ as $\alpha$ tends to $0+$ by the last lemma.

On the other hand:

$$\overline{T_\psi \overline{g_\alpha(x-y)}(a,b)} = \int_{-\infty}^{\infty} g_\alpha(x-y) \psi_{ab}(y) dy,$$

which goes to $\psi_{ab}(x)$, as $\alpha \to 0+$, by the same lemma. Therefore:

$$\lim_{\alpha \to 0+} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} T_\psi f(a,b) \overline{T_\psi \overline{g_\alpha(x-\cdot)}(a,b)} \frac{dadb}{a^2} =$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} T_\psi f(a,b) \psi_{ab}(x) \frac{dadb}{a^2}. \qquad \diamondsuit$$

### I.2.4. The Range of the CWT.

By theorem 2 in I.2.1 and the corollary in I.2.2, one knows already that the operator $T_\psi$ is a multiple of an isometry of $L^2(\mathbf{R})$ into $L^2(\mathbf{R}^* \times \mathbf{R}, \frac{da\,db}{a^2})$. But, the orthogonality relation shows even more:

THEOREM. *(Characterization of the Range)*

*Let $\psi$ be an admissible analyzing wavelet. Then, $Rg\ T_\psi$ is a closed subspace of $L^2(\mathbf{R}^* \times \mathbf{R}, \frac{da\,db}{a^2})$, and additionally a reproducing kernel Hilbert space with reproducing kernel*

$$K_\psi((a,b),(a',b')) := \frac{1}{c_\psi} T_\psi \psi_{a'b'}(a,b). \qquad (I.25)$$

*Proof.* $Rg\ T_\psi$ is closed:

Let $(F_n)_{n\in\mathbf{N}}$ be a sequence in $Rg\ T_\psi$ with limit $F \in L^2(\mathbf{R})^* \times \mathbf{R}, \frac{da\,db}{a^2})$. Define $f_n \in L^2(\mathbf{R})$ by $F_n = T_\psi f_n$. Since $(F_n)_{n\in\mathbf{N}}$ is a Cauchy-sequence in $Rg\ T_\psi$, one has:

$$\forall\ \epsilon > 0 \quad \exists\ N \in \mathbf{N} : \quad \|F_n - F_m\|_{L_2(\mathbf{R}^*\times\mathbf{R}, \frac{da\,db}{a^2})} < \epsilon \quad \forall\ n, m > N.$$

So, $\epsilon > c_\psi \|f_n - f_m\|$ by isometry of $\frac{1}{\sqrt{c_\psi}} T_\psi$, i.e. $(f_n)_{n\in\mathbf{N}}$ is a Cauchy-sequence in $L^2(\mathbf{R})$. The completeness of $L^2(\mathbf{R})$ ensures the existence of a limit $f \in L^2(\mathbf{R})$ for this sequence. Since $\frac{1}{\sqrt{c_\psi}} T_\psi$ is an isometry, $T_\psi$ is in particular continuous, so, $T_\psi f = F$. Hence, $Rg\ T_\psi$ is closed.

$Rg\ T_\psi$ reproducing kernel Hilbert space:

Let $F \in Rg\ T_\psi$ be arbitrary, $f \in L^2(\mathbf{R})$ such that $F = T_\psi f$. Applying the orthogonality relation $(I.21)$, yields:

$$F(a,b) = \int_{-\infty}^{\infty} f(x)\overline{\psi_{ab}(x)}dx =$$

$$= \frac{1}{c_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} T_\psi f(a',b') \overline{T_\psi \psi_{ab}}(a',b') \frac{da'db'}{a'^2} =$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(a',b') K_\psi((a,b),(a',b')) \frac{da'db'}{a'^2},$$

with $K_\psi$ as given by $(I.25)$. $\diamondsuit$

**I.2.5. Generalized Orthogonality Relation and Generalized Inversion.**

THEOREM. *(Generalized Orthogonality)*
*Let $\psi$, $\Psi$ be two admissible analyzing wavelets, $f$, $g \in L^2(\mathbf{R})$. Then, the following formula holds.*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} T_\psi f(a,b)\overline{T_\Psi g(a,b)}\frac{dadb}{a^2} = c_{\psi\Psi}\int_{-\infty}^{\infty} f(x)\overline{g(x)}dx, \quad (I.26)$$

where

$$c_{\psi\Psi} := 2\pi\int_{-\infty}^{\infty}\frac{\overline{\hat{\psi}(\omega)}\hat{\Psi}(\omega)}{|\omega|}d\omega. \quad (I.27)$$

*Proof.* The existence of the left hand side is justified by theorem 1, in I.2.1, the right hand side is finite, because of

$$|c_{\psi\Psi}| \le 2\pi\|\frac{\hat{\psi}(\omega)}{\sqrt{|\omega|}}\|\|\frac{\hat{\Psi}(\omega)}{\sqrt{|\omega|}}\| = \frac{1}{2\pi}c_\psi c_\Psi < \infty,$$

since $\psi$, $\Psi$ are admissible. The assertion now follows by the same calculation, as in the proof of $(I.21)$. (Use $G_\alpha(\omega) := \hat{g}(\omega)\overline{\hat{\varphi}(a\omega)}$.) $\diamondsuit$

If additionally

$$c_{\psi\Psi} \neq 0, \quad (I.28)$$

$(I.26)$ can be interpreted as an inversion formula, in the weak sense, again. Analogously to section I.2.3, one can prove the following stronger versions:

**Generalized $L^2$-inversion.**
*For $0 < A_1 < A_2 < \infty$, $0 < B < \infty$, define $f_{A_1A_2B}$ by*

$$(f_{A_1A_2B}, g) :=$$

$$= \frac{1}{c_{\psi\Psi}}\int_{-B}^{B}\int_{A_1|a|<A_2} T_\psi f(a,b)(\Psi_{ab}, g)\frac{dadb}{a^2} \qquad \forall g \in L^2(\mathbf{R}). \quad (I.29)$$

*Then,*

$$\lim_{A_1\to 0,\ A_2,B\to\infty}\|f - f_{A_1A_2B}\| \to 0.$$

**Generalized pointwise inversion.**

*Let $\psi$, $\Psi$ be two admissible analyzing wavelets such that (I.28) holds,* $\Psi$ in $L^1(\mathbf{R})$ *and continuous. Let $f \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$. Then, in every point $x$ in which $f$ is continuous, holds*

$$f(x) = \frac{1}{c_{\psi\varphi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} T_\psi f(a,b) \Psi_{ab}(x) \frac{dadb}{a^2}. \qquad (I.30)$$

Note that the admissibility of both $\psi$ and $\Psi$ is a sufficient, but not a necessary condition for $(I.26)$. Actually, the following assertion already implies $(I.26)$:

i) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} T_\psi f(a,b) \overline{T_\Psi g(a,b)} \frac{dadb}{a^2} < \infty \qquad \forall\ f,\ g \in L^2(\mathbf{R})$,

ii) $c_{\psi\Psi} < \infty$.

If, in addition to i) and ii), the condition $(I.28)$ is satisfied, $\Psi$ is called a *reconstruction wavelet*, for the analyzing wavelet $\psi$, since then, the above inversion formulas are still valid.

Another consequence of $(I.26)$ is the following characterization of the reproducing kernel $K_\psi$ defined in $(I.25)$:

COROLLARY. *(General description of $K_\psi$)*
*Let $\psi$ be an admissible analyzing wavelet, $K_\psi$ the reproducing kernel for the range of $T_\psi$. Then, one has, for all admissible analyzing wavelets* $\Psi$ :

$$K_\psi((a,b),(a',b')) = \frac{1}{c_{\psi\Psi}} T_\Psi \psi_{a'b'}(a,b).$$

*Proof.* Analogously to the proof in I.2.4, using the uniqueness of reproducing kernels. ◊

### I.2.6. Restriction of the CWT to $a > 0$.

The orthogonality relations $(I.21)$ and $(I.26)$ and the resulting inversion formulas require the full knowledge of all wavelet coefficients $T_\psi f(a,b),\ \ (a,b) \in \mathbf{R}^* \times \mathbf{R}$. For analytic purposes (see I.1 and I.3), it often suffices to know the CWT of $f$, for positive values of the scale parameter $a$, only.

QUESTION. Under which conditions on $\psi$, an arbitrary function $f \in L^2(\mathbf{R})$ is determined, uniquely, by its wavelet coefficients $T_\psi f(a,b)$, for $(a,b) \in \mathbf{R}^+ \times \mathbf{R}$ ? Equivalently: For which analyzing wavelets $\psi$ is the operator $T_\psi^+$, defined on $L^2(\mathbf{R})$ via

$$T_\psi^+ : \quad f \mapsto T_\psi f|_{\mathbf{R}^+ \times \mathbf{R}},$$

injective ?

REMARK.
*The admissiblity of the analyzing wavelet $\psi$ does not imply the injectivity of $T_\psi^+$.*

*Proof by counterexample.* Choose $\psi$ such that $\hat{\psi}$ is supported in $[r, \infty[$, where $r > 0$. This choice is clearly compatible with the admissibility condition: On the other hand choose $f$ with $\|f\| > 0$ such that $\hat{f}$ is supported in $]-\infty, s]$, where $s < 0$. Then one gets by $(I.4)$:

$$T_\psi f(a,b) = \int_{-\infty}^{\infty} \hat{f}(\omega)\overline{\sqrt{a}\hat{\psi}(a\omega)}e^{ib\omega}d\omega = 0 \ \ \forall\ (a,b) \in \mathbf{R}^+ \times \mathbf{R},$$

since the supports of both factors overlap for no scale parameter $a > 0$. Hence $T_\psi^+$ is not injective for such $\psi$. $\diamondsuit$

To avoid such cases, one has to pose a harder admissibility condition on $\psi$.

DEFINITION. An analyzing wavelet $\psi$ is called *strongly admissible*, if it satisfies the following *strong admissibility condition*:

$$2\pi \int_0^\infty \frac{|\hat{\psi}(\omega)|^2}{\omega} d\omega = 2\pi \int_{-\infty}^0 \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega =: c_\psi^{st} < \infty. \qquad (I.31)$$

Note that the wavelet, used in the above counterexample, is admissible, but not strongly admissible.

LEMMA. *(Connection between "admissible" and "strongly admissible")*

i) *Every strongly admissible analyzing wavelet is admissible, and* $c_\psi = 2c_\psi^{st}$ *holds.*

ii) *Every real-valued admissible analyzing wavelet is strongly admissible, and* $c_\psi^{st} = \frac{c_\psi}{2}$ *holds.*

*Proof.*
i): Follows by adding both integrals in $(I.31)$.
ii) For $\psi$ real-valued one has $\overline{\hat{\psi}(\omega)} = \hat{\psi}(-\omega)$, i.e.

$$\int_{-\infty}^{0} \frac{|\hat{\psi}(\omega)|^2}{|\hat{\omega}|} d\omega = \int_0^\infty \frac{|\hat{\psi}(-\xi)|^2}{\xi} d\xi = \int_0^\infty \frac{|\overline{\hat{\psi}(\xi)}|^2}{\xi} d\xi =$$

$$= \int_0^\infty \frac{|\psi(\xi)|^2}{\xi} d\xi < \infty,$$

since $\psi$ was assumed to be admissible. $\diamondsuit$

THEOREM. *(Positive version of the orthogonality relation). Let* $\psi$ *be a strongly admissible analyzing wavelet,* $f,\ g \in L^2(\mathbf{R})$ *arbitrary. Then*

$$\int_{-\infty}^{\infty} \int_0^\infty T_\psi f(a,b) \overline{T_\psi g(a,b)} \frac{dadb}{a^2} = \frac{c_\psi}{2} \int_{-\infty}^{\infty} f(x) \overline{g(x)} dx. \qquad (I.32)$$

*Proof* follows by repeating the calculations in the proof of $(I.21)$ with "$\int_{-\infty}^{\infty} da$" replaced by "$\int_0^\infty da$". $\diamondsuit$

COROLLARY. *(Positive version of the inversion formulas). Let* $\psi$ *be a strongly admissible analyzing wavelet. Then* $T_\psi^+$ *is injective and the inversion formulas in I.2.3 hold with "*$c_\psi$*", replaced by "*$\frac{c_\psi}{2}$*", and "*$\int_{-\infty}^{\infty} da$*", replaced by "*$\int_0^\infty da$*".*

*Proof* completely analogous to the proofs in I.2.3. $\diamondsuit$

The symmetry of $(I.31)$ allows to prove *negative* versions of the orthogonality relation and the inversion formulas, under the same conditions: The above theorem and its corollary still hold, if one replaces "$\int_0^\infty da$" by "$\int_{-\infty}^0 da$".

Analogously, one can prove positive/negative versions of the *generalized* orthogonality relation and the *generalized* inversion formulas, posing similiar conditions on $c_{\psi\Psi}$.

Supplementary remark. The counterexample, given above, is generic. Define the *Hardy space*

$$H^2(\mathbf{R}) := \{f \in L^2(\mathbf{R}) : \ \hat{f}(\omega) = 0 \ \ for \ \ \omega \leq 0\}.$$

Analogously as in I.0, one can show: For arbitrary analyzing wavelets $\psi \in H^2(\mathbf{R})$, the transform $T_\psi^+$ is injective on $H^2(\mathbf{R})$. (See [GrosM84].) Similiar results hold for functions with Fourier transforms vanishing on $\mathbf{R}^+$, and for the restriction of $T_\psi$ to *negative* values.

This fact will find a natural explanation in the group theoretical setting of ch.III.

### I.2.7. Extension of the CWT to More than One Dimension.

The CWT has been extended from $L^2(\mathbf{R})$ to $L^2(\mathbf{R}^n)$ $(n \in \mathbf{N}$ arbitrary) by R. Murenzi, see [AntMP91], [Mur89].

In the one dimensional setting, daughter wavelets $\psi_{ab}$ were built from the mother wavelet $\psi$ by using the following symmetry operations on the real line:

*Dilation,* marked by a parameter $a \in \mathbf{R}^*$, and *translation* marked by $b \in \mathbf{R}$.

Trying to imitate this construction in $n$ dimensions, Murenzi [Mur89] studied the symmetries of the Euclidean space $\mathbf{R}^n$. This lead him to introduce an additional *rotation parameter* $\mathcal{R}$ (an element of the orthogonal group $SO(n)$), while maintaining the dilation and translation parameters $a$, $b$ (where $b \in \mathbf{R}^n$ now). So the definitions of I.0 and I.2.1 can be rewritten as follows.

DEFINITION.

i) A *n-dimensional analyzing wavelet* is a function $\psi \in L^2(\mathbf{R}^n)$ with $\|\psi\|_{L^2(\mathbf{R}^n)} > 0$.

ii) A n-dimensional analyzing wavelet $\psi$ is called *admissible*, if it satisfies the following *admissibility condition.*

$$c_\psi^n := (2\pi)^n \int_{\mathbf{R}^n} \frac{|\hat{\psi}(\omega)|^2}{|\omega|^n} d^n\omega < \infty.$$

iii) For a $n$-dimensional analyzing wavelet $\psi$ and $(a, b, \mathcal{R}) \in \mathbf{R}^* \times \mathbf{R}^n \times SO(n)$, define *daughter wavelets* as follows:

$$\psi_{ab\mathcal{R}}(x) := \frac{1}{\sqrt{|a|^n}} \psi(\mathcal{R}^{-1}(\frac{x-b}{a})).$$

iv) The *n-dimensional continuous wavelet transform* of a function $f \in L^2(\mathbf{R}^n)$, *with respect to the n-dimensional analyzing wavelet* $\psi$, is defined as follows.

$$T_\psi^n f: \quad \mathbf{R} \times \mathbf{R}^n \times SO(n) \to \mathbf{C}$$

$$(a, b, \mathcal{R}) \mapsto T_\psi^n f(a, b, \mathcal{R}) = \int_{\mathbf{R}^n} f(x) \psi_{ab\mathcal{R}}(x) d^n x.$$

For this transform, the orthogonality relations $(I.21)$ and $(I.26)$ and their corollaries can be generalized. In particular, $(I.21)$ now looks as follows.

THEOREM. *(n-dimensional orthogonality)*

*Let $\psi$ be an admissible n-dimensional analyzing wavelet, $f,\ g \in L^2(\mathbf{R}^n)$. Then.*

$$\int_{SO(n)} \int_{\mathbf{R}^n} \int_{\mathbf{R}} T^n_\psi f(a,b,\mathcal{R}) \overline{T^n_\psi g(a,b,\mathcal{R})} \frac{da d^n b d\mathcal{R}}{a^{n+1}} =$$

$$= c^n_\psi \int_{\mathbf{R}^n} f(x)\overline{g(x)} d^n x,$$

*where $\int_{SO(n)} d\mathcal{R}$ denotes the integral over $SO(n)$ with respect to its left Haar measure*[2].

The case $n = 2$ is of special importance for applications of CWT to image analysis, see [AntMP91], [ArnAF92]. In this case, the integral $\int_{SO(n)} d\mathcal{R}$ possesses the simple form $\int_0^{2\pi} d\theta$, where $\theta$ is the angle characterizing the 2-dimensional rotation matrix $\mathcal{R} = \mathcal{R}_\theta$.

For $n = 1$, one gets $T^1_\psi = T_\psi$.

## I.3. Pointwise Wavelet Analysis of Differentiability. Application to the Riemann Function.

**Motivation**. Until 1970, it was supposed that the so-called *Riemann function*

$$R(x) := \sum_{n=1}^{\infty} \frac{1}{n^2} sin(\pi n^2 x) \qquad (I.33)$$

is not differentiable, at any point $x \in \mathbf{R}$. But in 1970, J. Gerver proved the differentiability of $R$ in exactly the points $x_0$ of the type

$$x_0 = \frac{2p+1}{2q+1}, \quad where \;\; p, q \in \mathbf{Z}. \qquad (I.34)$$

His proof was elementary, but complicated. So, in the sequel, several mathematicians reproved Gerver's result, more directly, using different techniques[3]. One special method, discovered by M. Holschneider

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2 See Chapter III.1, for a definition of *Haar measure*.

3 E.g.: M. Queffelec, C.R. Séan, Acad. Sci. **273**, Ser. A (1971) 291-293 or S. Itatsu, Proc. Japan Acad., Ser. A, Math. Sci. **57** (1981) 492-495.

and Ph. Tchamitchian in 1989, [HolT89], is founded on the CWT and takes advantage of its localization properties. This will be studied in the following.

THEOREM. *(Pointwise analysis of differentiability)*
*Assume* $G : ]0,1] \times \mathbf{R} \to \mathbf{C}$ *is a measurable function satisfying the following two conditions.*

i) $\exists\ C_1 > 0,\ \alpha > 0,$ *such that*

$$|G(a,b)| \leq C_1 a^{\alpha+\frac{1}{2}} \quad \forall\ a \in ]0,1]$$

*uniformly in* $b$.

ii) $\exists\ C_2 > 0,\ \rho : \mathbf{R}_0^+ \to \mathbf{R}$ *monotone, continuous with* $\int_0^1 \rho(x)\frac{dx}{x} < \infty$ *such that*

$$|G(a, b + x_0)| \leq C_2 a^{\frac{1}{2}}(a\rho(a) + |b|\rho(|b|))$$

*for some fixed* $x_0 \in \mathbf{R}$ *and* $\forall\ (a,b) \in ]0,1] \times \mathbf{R}$.
*Let* $\Psi \in C^2(\mathbf{R})$ *be compactly supported. For* $x \in \mathbf{R}$, *define*

$$g(x) := \int_{-\infty}^{\infty}\int_0^1 G(a,b)\frac{1}{\sqrt{a}}\Psi(\frac{x-b}{a})\frac{dadb}{a^2}. \qquad (I.35)$$

*Then,* $g$ *is differentiable at* $x_0$.

*Sketch of Proof.* [HolT89], [HolT91], [Mey93,p.115]. By a global translation and dilation which does not affect the differentiability properties of $g$, one can assume that $x_0 = 0$ and $supp\Psi \subseteq [-\frac{1}{2}, \frac{1}{2}]$.

Aim of the proof is to show the existence of

$$\lim_{h\to 0}\frac{g(h) - g(0)}{h},$$

or equivalently

$$\lim_{h,h'\to 0}\{\frac{g(h) - g(0)}{h} - \frac{g(h') - g(0)}{h'}\} = 0, \qquad (I.36)$$

where in the following $0 < h' < h < \infty$ is assumed. The case $h < 0$ can be treated analogously. Now

$$\frac{g(h)-g(0)}{h} - \frac{g(h')-g(0)}{h'} =$$

$$= \int_{-\infty}^{\infty}\int_0^1 G(a,b)\{\frac{1}{h}[\frac{1}{\sqrt{a}}\Psi(\frac{h-b}{a}) - \frac{1}{\sqrt{a}}\Psi(-\frac{b}{a})] -$$

$$-\frac{1}{h'}[\frac{1}{\sqrt{a}}\Psi(\frac{h'-b}{a}) - \frac{1}{\sqrt{a}}\Psi(-\frac{b}{a})]\}\frac{dadb}{a^2} =$$

$$= \int_{-\infty}^{\infty}\int_0^{h^{\frac{2}{\alpha}}} G(a,b)\frac{1}{h}\frac{1}{\sqrt{a}}\Psi(\frac{h-b}{a})\frac{dadb}{a^2} + \tag{1}$$

$$+\int_{-\infty}^{\infty}\int_{h^{\frac{2}{\alpha}}}^{h} G(a,b)\frac{1}{h}\frac{1}{\sqrt{a}}\Psi(\frac{h-b}{a})\frac{dadb}{a^2} - \tag{2}$$

$$-\int_{-\infty}^{\infty}\int_0^{h} G(a,b)\frac{1}{h}\frac{1}{\sqrt{a}}\Psi(-\frac{b}{a})\frac{dadb}{a^2} - \tag{3}$$

$$-\int_{-\infty}^{\infty}\int_0^{h'^{\frac{2}{\alpha}}} G(a,b)\frac{1}{h'}\frac{1}{\sqrt{a}}\Psi(\frac{h'-b}{a})\frac{dadb}{a^2} - \tag{4}$$

$$-\int_{-\infty}^{\infty}\int_{h'^{\frac{2}{\alpha}}}^{h} G(a,b)\frac{1}{h'}\frac{1}{\sqrt{a}}\Psi(\frac{h'-b}{a})\frac{dadb}{a^2} + \tag{5}$$

$$+\int_{-\infty}^{\infty}\int_0^{h} G(a,b)\frac{1}{h'}\frac{1}{\sqrt{a}}\Psi(-\frac{b}{a})\frac{dadb}{a^2} + \tag{6}$$

$$-\int_{-\infty}^{\infty}\int_{h'}^{h} G(a,b)\frac{1}{h'}\frac{1}{\sqrt{a}}(\Psi(\frac{h'-b}{a}) - \Psi(-\frac{b}{a}))\frac{dadb}{a^2} + \tag{7}$$

$$+\int_{-\infty}^{\infty}\int_h^1 G(a,b)\{\frac{1}{h}[\frac{1}{\sqrt{a}}\Psi(\frac{h-b}{a}) - \frac{1}{\sqrt{a}}\Psi(-\frac{b}{a})] -$$

$$-\frac{1}{h'}[\frac{1}{\sqrt{a}}\Psi(\frac{h'-b}{a}) - \frac{1}{\sqrt{a}}\Psi(-\frac{b}{a})]\}\frac{dadb}{a^2}. \tag{8}$$

Terms (1) and (4), as well as (3) and (6), can be estimated to be of the order $o(h)$, respectively $o(h')$. In the first case, this follows from condition i); in the second case, with help of condition ii), together with the fact that $\Psi$ is compactly supported and continuous, therefore, it attains its maximum on the finite interval of integration.

Similiary to the second case, one can show that (2), (5) and (7) are $o(1)$.

To estimate (8), one needs the differentiability properties of $\Psi$: the limit of the integrand of (8) can be interpreted as a second derivative of $\Psi$, as $h, h'$ tend to zero. From this it follows that (8) is $o(1)$, too. This proves $(I.36)$. $\diamondsuit$

The next corollary describes the connection between the last theorem and CWT.

COROLLARY. *(CWT-reformulation of the theorem)*

*Let $\Psi \in C^2(\mathbf{R})$ be compactly supported. Assume $\psi$ and $f$ are such that $T_\psi f(a,b)$ is well-defined, and that*

$$f(x) = \frac{1}{c_{\psi\Psi}} \int_{-\infty}^{\infty} \int_0^{\infty} T_\psi f(a,b) \frac{1}{\sqrt{a}} \Psi(\frac{x-b}{a}) \frac{dadb}{a^2} \qquad (I.37)$$

*is valid*[4]*, for all $x$ in a neighbourhood $U_{x_0}$ of $x_0$, where $c_{\psi\Psi}$ is defined as in $(I.27)$. Assume further, conditions i) and ii) of the theorem hold and that*

$$\tilde{g}(x) := \frac{1}{c_{\psi\Psi}} \int_{-\infty}^{\infty} \int_1^{\infty} T_\psi f(a,b) \frac{1}{\sqrt{a}} \Psi(\frac{x-b}{a}) \frac{dadb}{a^2} \qquad (I.38)$$

*is bounded and differentiable in $U_{x_0}$.*

*Then, $f$ is differentiable at $x_0$.*

*Proof.* Choose $G = T_\psi f|_{a\in]0,1]}$, $g = f - \tilde{g}$, in the theorem, and note that the sum of functions differentiable at $x_0$ is again differentiable at $x_0$. $\diamondsuit$

**Application to the Riemann function**. In the following, Gerver's results shall be reproved, using the last corollary. Instead of $R(x)$, consider the following complex-valued function

$$\tilde{R}(x) := \sum_{n=1}^{\infty} \frac{1}{n^2} e^{i\pi n^2 x}.$$

4 This is an extension, of the results in I.2.5 and I.2.6, beyond $L^2$-theory.

($R(x)$ is the imaginary part of $\tilde{R}(x)$.) Choose

$$\psi(x) := \frac{1}{(1-ix)^2}$$

(Paul wavelet with $\beta = 1$, cf. I.2.1, expl. ii)).

The wavelet transform of $\tilde{R}$ with respect to $\psi$ is given by

$$T_\psi \tilde{R}(a,b) = \frac{\pi}{2} a^{\frac{3}{2}} (\theta(b+ia) - 1) \quad for \quad a \in \mathbf{R}^+,\ b \in \mathbf{R}, \qquad (I.39)$$

where $\theta$ denotes the Jacobian $\theta$-function

$$\theta(b+ia) := \sum_{n=-\infty}^{\infty} e^{i\pi n^2 (b+ia)}.$$

(This can be checked, using the Fourier representation $(I.4)$ of CWT, and noting that

$$\hat{\psi}(\omega) = \begin{cases} \omega e^{-\omega}, & \omega \geq 0 \\ 0, & otherwise, \end{cases}$$

while the Fourier transform of $\tilde{R}$ is a sum of $\delta$-functions.)

Assume, $\Psi \in C^2(\mathbf{R})$ is a compactly supported, admissible analyzing wavelet, satisfying [5]

$$0 < c_{\psi\Psi} = 2\pi \int_0^\infty \overline{\hat{\psi}(\omega)} \hat{\Psi}(\omega) \frac{d\omega}{\omega} = 2\pi \int_0^\infty \hat{\Psi}(\omega) \omega e^{-\omega} \frac{d\omega}{\omega} < \infty. \qquad (I.40)$$

Then, one has $\forall\ \lambda > 0$

$$\frac{1}{c_{\psi\Psi}} \int_{-\infty}^\infty \int_0^\infty a^{\frac{3}{2}} e^{i\lambda(b+ia)} \frac{1}{\sqrt{a}} \Psi(\frac{x-b}{a}) \frac{dadb}{a^2} =$$

$$= \frac{1}{c_{\psi\Psi}} \int_0^\infty e^{-\lambda a} \int_{-\infty}^\infty e^{ib\lambda} \Psi(\frac{x-b}{a}) db \frac{da}{a} =$$

$$= \frac{1}{c_{\psi\Psi}} \int_0^\infty e^{-\lambda a} a \hat{\Psi}(a\lambda) e^{i\lambda x} \frac{da}{a} =$$

5 This corresponds to a generalized admissibility condition, cf. I.2.5.

$$= \frac{e^{i\lambda x}}{c_{\psi\Psi}} \int_0^\infty e^{-u} \hat{\Psi}(u) \frac{1}{\lambda} du = \frac{e^{i\lambda x}}{\lambda},$$

hence,

$$\frac{1}{c_{\psi\Psi}} \int_{-\infty}^{\infty} \frac{\pi}{2} a^{\frac{3}{2}} (\theta(b+ia) - 1) \frac{dadb}{a^2} =$$

$$= \frac{1}{c_{\psi\Psi}} \int_{-\infty}^{\infty} \int_0^\infty \frac{\pi}{2} a^{\frac{3}{2}} 2 \sum_{n=1}^{\infty} e^{i\pi n^2 (b+ia)} \frac{1}{\sqrt{a}} \Psi(\frac{x-b}{a}) \frac{dadb}{a^2} =$$

$$= \sum_{n=1}^{\infty} \frac{e^{ix\pi n^2}}{n^2} = \tilde{R}(x).$$

Therefore, $(I.37)$ holds. By $(I.39)$ follows:

$$\exists C > 0: \qquad |T_\psi \tilde{R}(a,b)| \leq C a^{\frac{3}{2}} e^{-a} \quad \forall\ a \geq 1.$$

Consequently, the function $\tilde{g}$ in $(I.38)$ is bounded and has the same regularity as $\Psi$, so $\tilde{g} \in C^2(\mathbf{R})$.

All this allows us to transfer the differentiability analysis of $\tilde{R}$ to a decay analysis of $T_\psi \tilde{R}(a,b) = \frac{\pi}{2} a^{\frac{3}{2}} (\theta(b+ia) - 1)$.

It is well known[6] that

$$|\theta(b+ia)| \leq C_1 |a|^{-\frac{1}{2}},$$

for a suitable constant $C_1$. Therefore, the global condition i) of the theorem is satisfied with $\alpha = \frac{1}{2}$. To check the local condition ii), Holschneider and Tchamitchian referred to the *modular group* $G_M$ [7].

The $\theta$-function possesses the following invariance properties:

$$\theta(K^2(b+ia)) = \theta(b+ia),$$

$$\theta(U(b+ia)) = \sqrt{-i(b+ia)}\,\theta(b+ia),$$

where $K:\ b+ia \mapsto b+1+ia$ and $U:\ b+ia \mapsto -\frac{1}{b+ia}$ are just the generators of the modular group.

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6 D. Mumford, *Tata Lectures on Theta I*, Boston (1983)

7 R.C. Gunning, *Lectures on Modular Forms*, Princeton (1962)

The subgroup of $G_M$, generated by $K^2$ and $U$, is called the *$\theta$-group* $G_\theta$. For any $G \in G_\theta$, one has:

$$Gx \in \mathbf{R},\ if\ x \in \mathbf{R} \quad and \quad Gx \in \mathbf{Q},\ if\ x \in \mathbf{Q}.$$

Furthermore, $\mathbf{Q}$ splits into two orbits under the action of $G_\theta$, namely the orbit of 1, consisting of all rational numbers $\frac{2p+1}{2q+1}$, $p, q \in \mathbf{Z}$, and the orbit of 0, containing the rest of $\mathbf{Q}$. For $x_0 \in G_\theta 1$, Holschneider and Tchamitchian [HolT89], [HolT91] showed:

$$|\theta(b + x_0 + ia)| \leq C_3 a^{-1}(a^{\frac{3}{2}} + |b|^{\frac{3}{2}}).$$

Choosing $C_2 := \frac{\pi}{2} C_3$, one gets:

$$|T_\psi \tilde{R}(a,b)| \leq C_2 a^{\frac{1}{2}}(a^{\frac{3}{2}} + |b|^{\frac{3}{2}}) =$$

$$= C_2 a^{\frac{1}{2}}(a\rho(a) + |b|\rho(|b|)),$$

where $\rho(x) = \sqrt{|x|}$ is a monotone, continuous function, satisfying

$$\int_0^1 \rho(x)\frac{dx}{x} = \int_0^1 x^{-\frac{1}{2}} dx = 2 < \infty.$$

So, condition ii) of the theorem is satisfied as well, and the corollary establishes the differentiability of $\tilde{R}$ (and therefore of $R = \frac{1}{2i}(\tilde{R} - \overline{\tilde{R}})$) in the points $x_0 \in G_\theta 1$, confirming Gerver's result $(I.34)$.

Remarks. In addition, Holschneider and Tchamitchian proved a theorem concerning the pointwise analysis of *Hölder-continuity*, with help of the CWT. (Its proof is quite similiar to that of the foregoing theorem.) This allowed a more accurate description of the Riemann function on $G_\theta 0$.

Finally, the same authors showed the *non-differentiability* of $R$, at the *irrationals*, using (for an indirect proof) an inverse statement to the above corollary, i.e. a theorem of the following type.

*If $f$ is differentiable at $x_0$, then its CWT with respect to an admissible analyzing wavelet $\psi$ possesses certain decay properties* [HolT89], [HolT91].

The proof of this fact rests on the *vanishing moment property* $(I.9)$ of the admissible analyzing wavelet $\psi$, which allows conclusions like

$$T_\psi f(a,b) = \int_{-\infty}^{\infty} f(x)\frac{1}{\sqrt{a}}\psi(\frac{x-b}{a})dx =$$
$$= \int_{-\infty}^{\infty} (f(x) - f(x_0))\frac{1}{\sqrt{a}}\psi(\frac{x-b}{a})dx.$$

All the foregoing theorems are consequences of the zooming property of CWT (cf. I.1.1, I.1.2).

Note that an analogous pointwise analysis can be performed with the *discrete* wavelet transform, which will be introduced in the next chapter. See [Jaf89a], [Jaf92a-c].

The crucial point in the local analysis of a function $f$ by CWT, described above, is the validity of a generalized *inversion formula of type* $(I.37)$. More information on this can be found in [BerW93], [Dau92], [Hol91], [Hol93b], [HolT89], [HolT91].

Since their publication in 1989, the ideas of Holschneider and Tchamitchian have been picked up by several scientists who were interested in the local analysis of other fractal objects than the Riemann function [BacMA93], [Far92], [MalH92].

## II. The Discrete Wavelet Transform (DWT).

### II.0. Motivation.

In applications, $T_\psi f(a,b)$ can only be computed for values $(a,b)$ in a *discrete*[8] set

$$S \subseteq \mathbf{R}^* \times \mathbf{R}. \qquad (II.1)$$

8 I.e.: $S$ possesses no accumulation point.

In other words:

It is not the CWT-operator $T_\psi$, which is used in praxis, but the *discrete wavelet transform (DWT) operator* $T_{\psi S}$, which is defined as follows:

$$T_{\psi S}: \quad L^2(\mathbf{R}) \to \mathbf{C}^S$$
$$f \mapsto T_{\psi S} f := (T_\psi f(a,b))_{(a,b)\in S} = ((f, \psi_{(a,b)}))_{(a,b)\in S}. \qquad (II.2)$$

An interesting question to ask is the following:

*Is it possible to reconstruct an arbitrary function $f \in L^2(\mathbf{R})$ from the discrete values $T_{\psi S} f$, in a stable manner?*

Or equivalently:

*Under which conditions on $\psi$ and $S$ is $T_{\psi S}$ a continuous, injective operator with a continuous inverse?*

This question leads to the concept of a *frame*, which shall be treated in general Hilbert spaces, in the following section. For the special case of *wavelet* frames, see II.2. Among the wavelet frames, the class of *wavelet orthonormal bases (WONBs)* is of particular interest. It will be considered in the remainder of chapter II.

## II.1. Frames in Hilbert Spaces.

Most of the results in this section can be found in [Dau90], [Dau92] and [HeilW89].

**General assumption.**

*$\mathcal{H}$ separable Hilbert space with scalar product $(\cdot,\cdot)_\mathcal{H}$ and norm $\|\cdot\|_\mathcal{H}$,*
*$\mathcal{N}$ countable index set.*

Definition. A subset $\Phi := (\varphi_n)_{n\in\mathcal{N}}$ of $\mathcal{H}$ is called a *frame for $\mathcal{H}$*, if there exist some constants $A, B \in \mathbf{R}$, such that $0 < A \le B < \infty$, and $\forall\ f \in \mathcal{H}$

$$A\|f\|^2_{\mathcal{H}} \leq \sum_{n\in\mathcal{N}} |(f,\varphi_n)_{\mathcal{H}}|^2 \leq B\|f\|^2_{\mathcal{H}}. \qquad (II.3)$$

More specifically, $\Phi$, as above, is called an *$(A,B)$-frame.* $A$ and $B$ are called the *frame bounds.* An $(A,B)$-frame $\Phi$ is called a *tight $A$-frame*, if for the frame bounds holds $A = B$.

THEOREM 1. *(Elementary properties of a frame)*

a) *Every frame for $\mathcal{H}$ is complete in $\mathcal{H}$.*

b) *Every orthonormal basis of $\mathcal{H}$ is a frame for $\mathcal{H}$*, with frame bounds $A = B = 1$.

c) *If $\Phi$ is a tight $A$-frame with frame bound $A = 1$ and $\|\varphi_n\|_{\mathcal{H}} = 1\ \ \forall\ n \in \mathcal{N}$, then $\Phi$ is an orthonormal basis of $\mathcal{H}$.*

d) *A Schauder basis of $\mathcal{H}$ is not necessarily a frame for $\mathcal{H}$.*

e) *A tight frame for $\mathcal{H}$ is not necessarily a Schauder basis of $\mathcal{H}$.*

*Proof.*

a) Let $\Phi$ be a frame for $\mathcal{H}$. Assume, there exists a function $f \in \mathcal{H}$ with $f \perp \varphi_n\ \ \forall\ n \in \mathcal{N}$. Then

$$\sum_{n\in\mathcal{N}} |(f,\varphi_n)_{\mathcal{H}}|^2 = 0,$$

and so $A\|f\|^2_{\mathcal{H}} = 0$, which means that $f = 0$ since $A \neq 0$ by definition. Hence, $\Phi$ is complete in $\mathcal{H}$.

b) If $\Phi$ is an orthonormal basis of $\mathcal{H}$, one has by Parseval's relation

$$\sum_{n\in\mathcal{N}} |(f,\varphi_n)_{\mathcal{H}}|^2 = \|f\|^2_{\mathcal{H}},$$

i.e. $(II.3)$ with $A = B = 1$.

c) Because of a), any frame $\Phi$ is complete in $\mathcal{H}$. By definition, one has $\|\varphi_n\|_{\mathcal{H}} = 1 \ \ \forall\, n \in \mathcal{N}$. It remains to show $(\varphi_m, \varphi_n)_{\mathcal{H}} = 0$, for $m \neq n$. Since $A = B = 1$, one has $\forall\ m \in \mathcal{N}$

$$1 = \|\varphi_m\|^2_{\mathcal{H}} = \sum_{n\in\mathcal{N}} |(\varphi_m, \varphi_n)_{\mathcal{H}}|^2 =$$

$$= \|\varphi_m\|^2_{\mathcal{H}} + \sum_{n\in\mathcal{N},\ n\neq m} |(\varphi_m, \varphi_n)_{\mathcal{H}}|^2 = 1 + \sum_{n\in\mathcal{N},\ n\neq m} |(\varphi_m, \varphi_n)_{\mathcal{H}}|^2.$$

d) Counterexample:
Let $\mathcal{N} = \mathbf{N}$ be the set of natural numbers and $\check{\Phi} := (\check{\varphi}_n)_{n\in\mathbf{N}}$ be an orthonormal basis of $\mathcal{H}$. Define $\Phi := (\varphi_n)_{n\in\mathbf{N}}$ by $\varphi_n := n\check{\varphi}_n$. Then, $\Phi$ is linear independent and complete in $\mathcal{H}$, hence a Schauder basis. But for $f = \varphi_m$, where $m \in \mathbf{N}$ arbitrary, one has

$$\sum_{n\in\mathbf{N}} |(\varphi_m, \varphi_n)_{\mathcal{H}}|^2 = m^2 \sum_{n\in\mathbf{N}} n^2 |(\check{\varphi}_m, \check{\varphi}_n)_{\mathcal{H}}|^2 = m^4,$$

hence, there exists no upper bound in $(II.3)$, i.e. $\Phi$ is not a frame.

e) Counterexample:
Take $\mathcal{H} = \mathbf{C}^2$, $\varphi_1 = (1,0)$, $\varphi_2 = (-\frac{1}{2}, -\frac{\sqrt{3}}{2})$, $\varphi_3 = (-\frac{1}{2}, \frac{\sqrt{3}}{2})$. Then $\forall\ x = (x_1, x_2) \in \mathbf{C}^2$

$$\sum_{n=1}^{3} |(x, \varphi_n)_{\mathbf{C}^2}|^2 = |x_1|^2 + |-\frac{1}{2}x_1 - \frac{\sqrt{3}}{2}x_2|^2 + |-\frac{1}{2}x_1 + \frac{\sqrt{3}}{2}x_2|^2 =$$

$$= \frac{3}{2}\|x\|^2_{\mathbf{C}^2}.$$

By this, $\Phi$ is a frame. But, three vectors in $\mathbf{C}^2$ are linearly dependent. Hence, $\Phi$ is no basis. $\diamondsuit$

Definition. Let $\Phi$ be an $(A,B)$-frame for $\mathcal{H}$. The corresponding *coefficient operator* $T_\Phi$ is defined by

$$T_\Phi : \mathcal{H} \to l^2(\mathcal{N})$$

$$f \mapsto ((f, \varphi_n)_{\mathcal{H}})_{n \in \mathcal{N}}. \qquad (II.4)$$

THEOREM 2. *(Elementary properties of $T_\Phi$). $T_\Phi$ is a bounded linear operator and has a bounded inverse on its range, which is a closed subset of $l^2(\mathcal{N})$. More precisely,*

$$\sqrt{A} \leq \|T_\Phi\| \leq \sqrt{B}, \qquad (II.5)$$

$$\frac{1}{\sqrt{B}} \leq \|T_\Phi^{-1}\| \leq \frac{1}{\sqrt{A}}. \qquad (II.6)$$

*The adjoint of $T_\Phi$, $T_\Phi^* : \; l^2(\mathcal{N}) \to \mathcal{H}$, is given by*

$$c := (c_n)_{n \in \mathcal{N}} \mapsto T_\Phi^* c = \sum_{n \in \mathcal{N}} c_n \varphi_n, \qquad (II.7)$$

*where convergence holds in the norm of $\mathcal{H}$.*

*Proof.*

a) Linearity of $T_\Phi$ follows by the linearity of the scalar product in the first component.

b) Boundedness of $T_\Phi$:

$$\|T_\Phi\| = \sup_{\|f\|_{\mathcal{H}}=1} \|T_\Phi f\|_{l^2} = \sup_{\|f\|_{\mathcal{H}}=1} (\sum_{n \in \mathcal{N}} |(f, \varphi_n)_{\mathcal{H}}|^2)^{\frac{1}{2}} \leq \sqrt{B}$$

by $(II.3)$. Analogously $\|T_\Phi\| \geq \sqrt{A}$, i.e. $(II.5)$.

c) $Rg\ T_\Phi$ is closed:
Let $(c^m)_{m \in \mathbf{N}}$ be a Cauchy sequence in $Rg\ T_\Phi$, i.e.

$$\forall \epsilon > 0 \;\; \exists M \in \mathbf{N} : \; \forall \; m_1, m_2 > M \quad \|c^{m_1} - c^{m_2}\|_{l^2} < \epsilon \quad and$$
$$\forall \; c^m \quad \exists \; f^m \in \mathcal{H} : \quad c^m = T_\phi f^m.$$

Then $(f^m)_{m \in \mathbf{N}}$ is a Cauchy sequence in $\mathcal{H}$, because

$$\|f^{m_1} - f^{m_2}\|_{\mathcal{H}} \leq \frac{1}{A} \|c^{m_1} - c^{m_2}\|_{l^2} \;\; \forall \; m_1, m_2 \in \mathbf{N}$$

by $(II.3)$. Since $\mathcal{H}$ is complete, $(f^m)_{m \in \mathbf{N}}$ converges to an element $f \in \mathcal{H}$.
Since $T_\Phi$ is continuous by b), $T_\Phi f^m = c^m$ converges to $T_\Phi f$ in $l^2(\mathcal{N})$, i.e. each Cauchy sequence in $Rg\ T_\Phi$ possesses a limit therein.

d) $T_\Phi$ is injective, because the equation $T_\Phi f = 0$ implies that $\sum_{n\in\mathcal{N}} | < f, \varphi_n >_{\mathcal{H}} |^2 = 0$, hence $f = 0$, by definition of a frame.

e) c) and d) imply, by the bounded inverse theorem, that $T_\Phi$ has a bounded inverse on its range.

f) Inequality $(II.6)$:

$$T_\Phi^{-1} : \quad Rg\ T_\Phi \to \mathcal{H}, \qquad ((f, \phi_n)_{\mathcal{H}})_{n\in\mathcal{N}} \mapsto f$$

fulfills because of $(II.3)$

$$\|T_\Phi^{-1}\| = \sup_{\|((f,\varphi_n)_{\mathcal{H}})_{n\in\mathcal{N}}\|_{l^2}=1} \|f\|_{\mathcal{H}} \leq \frac{1}{\sqrt{A}} \quad (\geq \frac{1}{\sqrt{B}} \ \ resp.).$$

g) For $c \in l^2(\mathcal{N}),\ f \in \mathcal{H}$ :

$$(T_\Phi f, c)_{l^2} = \sum_{n\in\mathcal{N}} (f, \varphi_n)_{\mathcal{H}} \overline{c_n} =$$

$$= \sum_{n\in\mathcal{N}} (f, c_n\varphi_n)_{\mathcal{H}} = (f, \sum_{n\in\mathcal{N}} c_n\varphi_n) = (f, T_\Phi^* c),$$

so $T_\Phi^*$ is given by $(II.7)$.

h) Norm convergence of $(II.7)$:
Let $(\mathcal{N}_m)_{m\in\mathbf{N}}$ denote a sequence of finite subsets of $\mathcal{N}$ so that

$$\mathcal{N}_{m_1} \subseteq \mathcal{N}_{m_2} \quad if \quad m_1 \leq m_2, \qquad \bigcup_{m\in\mathbf{N}} \mathcal{N}_m = \mathcal{N}.$$

Assertion 1: $(\sum_{n\in\mathcal{N}_m} c_n\varphi_n)_{m\in\mathbf{N}}$ is convergent in $\mathcal{H}$.

*Proof of assertion 1:* If $m_1 \leq m_2 \leq m_3$, then

$$\| \sum_{n\in\mathcal{N}_{m_3}} c_n\varphi_n - \sum_{n\in\mathcal{N}_{m_2}} c_n\varphi_n \|_{\mathcal{H}} = \sup_{f\in\mathcal{H}, \|f\|_{\mathcal{H}}=1} |( \sum_{n\in\mathcal{N}_{m_3}\setminus\mathcal{N}_{m_2}} c_n\varphi_n, f)_{\mathcal{H}}| \leq$$

$$\leq \sup_{f\in\mathcal{H}, \|f\|_{\mathcal{H}}=1} ( \sum_{n\in\mathcal{N}_{m_3}\setminus\mathcal{N}_{m_2}} |c_n|^2)^{\frac{1}{2}} ( \sum_{n\in\mathcal{N}} |(\varphi_n, f)_{\mathcal{H}}|^2)^{\frac{1}{2}} \leq$$

$$\leq ( \sum_{n\in\mathcal{N}\setminus\mathcal{N}_{m_1}} |c_n|^2)^{\frac{1}{2}} B^{\frac{1}{2}} \cdot 1,$$

by definition of a frame, and this tends to zero, for $m_1 \to \infty$, because $c \in l^2(\mathcal{N})$. Hence $(\sum_{n\in\mathcal{N}_m} c_n\varphi_n)_{m\in\mathbf{N}}$ is a Cauchy sequence in $\mathcal{H}$, and because of the completeness of $\mathcal{H}$ it possesses a limit $g \in \mathcal{H}$.

Assertion 2: $g = T_\Phi^* c$.

*Proof of assertion 2:* $\forall\ f \in L^2(\mathbf{R})$

$$< f, g >_{\mathcal{H}} = \lim_{m\to\infty} (f, \sum_{n\in\mathcal{N}_m} c_n\varphi_n)_{\mathcal{H}} =$$

$$= \lim_{m\to\infty} \sum_{n\in\mathcal{N}_m} \overline{c}_n (f,\varphi_n)_{\mathcal{H}} = \sum_{n\in\mathcal{N}} \overline{c_n}(f,\varphi_n)_{\mathcal{H}} = (f, T_\Phi^* c),$$

by the continuity of the scalar product. $\diamondsuit$

DEFINITION. Let $\Phi$ be a frame for $\mathcal{H}$. The *frame operator* $S_\Phi$, corresponding to $\Phi$, is defined by

$$S_\Phi := T_\Phi^* T_\Phi : \quad \mathcal{H} \to \mathcal{H}$$

$$f \mapsto \sum_{n\in\mathcal{N}} (f,\varphi_n)_{\mathcal{H}}\varphi_n. \tag{II.8}$$

THEOREM 3. *(Elementary properties of* $S_\Phi$ *)*
$S_\Phi$ is a positive, bounded, linear operator from $\mathcal{H}$ *onto* $\mathcal{H}$, *with a bounded inverse. More precisely.*

$$A \cdot Id_{\mathcal{H}} \le S_\Phi \le B \cdot Id_{\mathcal{H}}, \tag{II.9}$$

$$\frac{1}{B} \cdot Id_{\mathcal{H}} \le S_\Phi^{-1} \le \frac{1}{A} \cdot Id_{\mathcal{H}}. \tag{II.10}$$

*Proof.*

a) The linearity of $S_\Phi$ follows from the linearity of $T_\Phi$ and $T_\Phi^*$.

b) $\forall\ f \in \mathcal{H}$

$$(S_\Phi f, f)_{\mathcal{H}} = \sum_{n\in\mathcal{N}} (f,\varphi_n)_{\mathcal{H}} (\varphi_n, f)_{\mathcal{H}} = \sum_{n\in\mathcal{N}} |(f,\varphi_n)_{\mathcal{H}}|^2,$$

hence, by definition of a frame,

$$A\|f\|_{\mathcal{H}}^2 \leq \ (S_\Phi f, f)_{\mathcal{H}} \leq B\|f\|_{\mathcal{H}}^2,$$

which implies $(II.9)$ and therefore boundedness and positivity of $S_\Phi$.

c) $Rg\ S_\Phi$ is a closed subspace of $\mathcal{H}$. This can be shown analogously to theorem 2, c).

d) $Rg\ S_\Phi = \mathcal{H}$, otherwise, there would exist a $f \in \mathcal{H}$ with $(f, S_\Phi g)_{\mathcal{H}} = 0 \quad \forall\ g \in \mathcal{H}$. Then, in particular, $(f, S_\Phi f)_{\mathcal{H}} = 0$, and so by $(II.9)$ $A\|f\|_{\mathcal{H}}^2 = 0$, hence $f = 0$. So $(Rg\ S_\Phi)^\perp = 0$. Together with c), it follows that $Rg\ S_\Phi = \mathcal{H}$.

e) $S_\Phi$ is injective: Assume, there exists $0 \neq f \in \mathcal{H}$ such that $S_\Phi f = 0$. Then $(S_\Phi f, f)_{\mathcal{H}} = 0$, which leads to the same contradiction as in d).

f) c),d) and e) together imply by the bounded inverse theorem that $S_\Phi^{-1} : \mathcal{H} \to \mathcal{H}$ is well-defined and bounded.

g) Proof of inequality $(II.10)$: $S_\Phi^{-1}$ is positive, since $\forall\ f \in \mathcal{H}$

$$(S_\Phi^{-1} f, f)_{\mathcal{H}} = (S_\Phi^{-1} f, S_\Phi(S_\Phi^{-1} f))_{\mathcal{H}} \geq A\|S_\Phi^{-1} f\|_{\mathcal{H}} \geq 0,$$

because of $(II.9)$ and the positivity (and therefore self-adjointness) of $S_\Phi$. So, multiplication of $(II.9)$ by $S_\Phi^{-1} \circ S_\Phi^{-1}$ gives

$$A \cdot S_\Phi^{-1} \circ S_\Phi^{-1} \leq S_\Phi^{-1} \leq B \cdot S_\Phi^{-1} \circ S_\Phi^{-1},$$

i.e $A \cdot S_\Phi^{-1} \leq Id_{\mathcal{H}}, \quad B \cdot S_\Phi^{-1} \geq Id_{\mathcal{H}}$, hence $(II.10)$. $\diamondsuit$

Corollary. *(Frame description by $S_\Phi$)*
*Equivalent.*

i) $\Phi = (\varphi_n)_{n\in\mathcal{N}} \in \mathcal{H}^{\mathcal{N}}$ *is an* $(A, B)$*-frame for* $\mathcal{H}$.

ii) $S_\Phi\ :\ \mathcal{H} \to \mathcal{H}, \qquad f \mapsto \sum_{n\in\mathcal{N}} (f, \varphi_n)_{\mathcal{H}} \varphi_n$, *is a bounded linear operator with*

$$A \cdot Id_{\mathcal{H}} \leq S_\Phi \leq B \cdot Id_{\mathcal{H}}.$$

*Proof.*
i) implies ii), by theorem 3.
ii) implies i), since

$$\forall f \in \mathcal{H} \quad (Af, f)_{\mathcal{H}} \leq \ (S_\Phi f, f)_{\mathcal{H}} \leq \ (Bf, f)_{\mathcal{H}},$$

and this is equivalent to

$$A\|f\|^2_{\mathcal{H}} \leq \sum_{n\in\mathcal{N}} |(f, \varphi_n)|^2 \leq B\|f\|^2_{\mathcal{H}}. \qquad \diamondsuit$$

THEOREM 4. *(Dual frame)*
*Let $\Phi$ be a frame for $\mathcal{H}$, $S_\Phi$ the corresponding frame operator. Define $\tilde{\Phi} := (\widetilde{\varphi_n})_{n\in\mathcal{N}}$ by.*

$$\widetilde{\varphi_n} := S_\Phi^{-1}\varphi_n. \qquad (II.11)$$

*Then, $\tilde{\Phi}$ is a $(\frac{1}{B}, \frac{1}{A})$-frame for $\mathcal{H}$, $\tilde{\tilde{\Phi}} = \Phi$, and*

$$T^*_{\tilde{\Phi}} = \begin{cases} T_\Phi^{-1}c, & c \in T_\Phi\mathcal{H}, \\ 0, & c \in (T_\Phi\mathcal{H})^\perp. \end{cases} \qquad (II.12)$$

$\tilde{\Phi}$ is called the *dual frame* of $\Phi$.

*Proof.*
1. $\tilde{\Phi}$ is $(\frac{1}{B}, \frac{1}{A})$-frame:
For any $f \in \mathcal{H}$, one has

$$(f, \widetilde{\varphi_n})_{\mathcal{H}} = (f, S_\Phi^{-1}\varphi_n)_{\mathcal{H}} = (S_\Phi^{-1}f, \varphi_n)_{\mathcal{H}},$$

because $S_\Phi$ and hence $S_\Phi^{-1}$ are self-adjoint, being positive operators. Therefore,

$$\sum_{n\in\mathcal{N}} |(f, \widetilde{\varphi_n})_{\mathcal{H}}|^2 = \|T_\Phi S_\Phi^{-1} f\|^2_{l^2} =$$

$$= (T_\Phi S_\Phi^{-1}f, T_\Phi S_\Phi^{-1}f)_{l^2} = (S_\Phi^{-1}f, T_\Phi^* T_\Phi S_\Phi^{-1}f)_{l^2} = (S_\Phi^{-1}f, f)_{l^2}, \ (II.13)$$

and $\frac{1}{B}\|f\| \leq (II.13) \leq \frac{1}{A}\|f\|$ by $(II.10)$.

2. Proof of $(II.12)$:

$$\forall f \in \mathcal{H} \quad T_{\tilde{\Phi}} f = ((f, \widetilde{\varphi_n})_{\mathcal{H}})_{n\in\mathcal{N}} = ((f, S_\Phi^{-1}\varphi_n)_{\mathcal{H}})_{n\in\mathcal{N}} =$$
$$= ((S_\Phi^{-1} f, \varphi_n)_{\mathcal{H}})_{n\in\mathcal{N}},$$

by the self-adjointness of $S_\Phi^{-1}$. Therefore,

$$T_{\tilde{\Phi}} = T_\Phi S_\Phi^{-1} : \quad \mathcal{H} \to T_\Phi(\mathcal{H})$$

is surjective, because of the bijectivity of $S_\Phi^{-1}$.
Case 1: $\exists f \in \mathcal{H} : \ c = T_\Phi f$.

$$T_{\tilde{\Phi}}^* c = (T_\Phi S_\Phi^{-1})^* c = S_\Phi^{-1} T_\Phi^* T_\Phi f = f.$$

Case 2: $c \in (T_\Phi \mathcal{H})^\perp$.

$$\forall f \in \mathcal{H} : \quad (T_{\tilde{\Phi}}^* c, f)_{\mathcal{H}} = (c, T_{\tilde{\Phi}} f)_{l^2} = (c, T_\Phi S_\Phi^{-1} f)_{l^2} = (c, T_\Phi g)_{l^2} = 0,$$

because $g = S_\Phi^{-1} f \in \mathcal{H}$. So $T_{\tilde{\Phi}}^*|_{(T_\Phi\mathcal{H})^\perp} = 0$.
3. $\tilde{\tilde{\Phi}} = \Phi$ :

$$\forall n \in \mathcal{N} : \quad S_{\tilde{\Phi}}^{-1} \widetilde{\varphi_n} = (T_{\tilde{\Phi}}^* T_{\tilde{\Phi}})^{-1} \widetilde{\varphi_n} = ((T_\Phi S_\Phi^{-1})^* (T_\Phi S_\Phi^{-1}))^{-1} \widetilde{\varphi_n} =$$
$$= (S_\Phi^{-1} T_\Phi^* T_\Phi S_\Phi^{-1})^{-1} \widetilde{\varphi_n} = S_\Phi \widetilde{\varphi_n} = \varphi_n. \qquad \diamondsuit$$

Corollary 1. *(Frame expansions)*
*Let $\Phi$, $\tilde{\Phi}$ as in above theorem. Then holds:* $\forall\ f \in \mathcal{H}$

$$f = \sum_{n\in\mathcal{N}} (f, \varphi_n)_{\mathcal{H}} \widetilde{\varphi_n} = \sum_{n\in\mathcal{N}} (f, \widetilde{\varphi_n})_{\mathcal{H}} \varphi_n. \qquad (II.14)$$

*Proof.* By theorem 4 and the definition of $T_\Phi$,

$$f = T_{\tilde{\Phi}} T_\Phi f = \sum_{n\in\mathcal{N}} (f, \varphi_n)_{\mathcal{H}} \widetilde{\varphi_n}.$$

The second assertion follows by $\tilde{\tilde{\Phi}} = \Phi$. $\diamondsuit$

Corollary 2. *(Optimality of $\tilde{\Phi}$ )*
*Let $\Phi$ be a frame for $\mathcal{H}$.*

a) *Define*

$$\mathcal{R}_\Phi := \{\check{\Phi} := (\check{\varphi}_n)_{n\in\mathcal{N}} \in \mathcal{H}^{\mathcal{N}} : \ \ \forall\, f \in \mathcal{H} : \\ f = \sum_{n\in\mathcal{N}} < f, \varphi_n >_{\mathcal{H}} \check{\varphi}_n\}.$$

*Then,* $\check{\Phi} = \tilde{\Phi}$ *is the unique element of* $\mathcal{R}_\Phi$ *such that*

$$\forall\ c = (c_n)_{n\in\mathcal{N}} \in (Rg\ T_\Phi)^\perp : \qquad \sum_{n\in\mathcal{N}} c_n \check{\varphi}_n = 0. \qquad (II.15)$$

b) *If* $f \in \mathcal{H}$ *possesses an expansion* $f = \sum_{n\in\mathcal{N}} c_n\varphi_n$, *where* $c \in l^2(\mathcal{N})$ *arbitrary, then*

$$\sum_{n\in\mathcal{N}} |c_n|^2 = \sum_{n\in\mathcal{N}} |(f, \widetilde{\varphi_n})_{\mathcal{H}}|^2 + \sum_{n\in\mathbf{N}} |(f, \widetilde{\varphi_n})_{\mathcal{H}} - c_n|^2,$$

*i.e. the dual basis coefficients have minimal* $l^2$*-norm.*

*Proof.* Ad a): $(II.15)$ holds for $\tilde{\Phi}$, by $(II.12)$. For $\check{\Phi} \in \mathcal{R}_\Phi$ arbitrary, write

$$\check{\varphi}_n =: \widetilde{\varphi_n} + u_n.$$

Then holds for all $f \in \mathcal{H}$

$$f = \sum_{n\in\mathcal{N}} (f, \varphi_n)_{\mathcal{H}} \widetilde{\varphi_n} + \sum_{n\in\mathcal{N}} (f, \varphi_n)_{\mathcal{H}} u_n,$$

and therefore $\sum_{n\in\mathcal{N}} (f, \varphi_n)_{\mathcal{H}} u_n = 0 \quad \forall\ f \in \mathcal{H}$, i.e. $\sum_{n\in\mathcal{N}} c_n u_n = 0 \ \ \forall\ c \in Rg\ T_\Phi$.

If $\check{\Phi}$ would additionally satisfy $(II.15)$

$$\sum_{n\in\mathcal{N}} c_n u_n = 0 \quad \forall c \in (Rg\ T_\Phi)^\perp,$$

then

$$\sum_{n\in\mathcal{N}} c_n u_n = 0 \quad \forall c \in l^2(\mathcal{N}).$$

Therefore $u_n = 0 \ \ \forall\ n \in \mathcal{N}$, i.e $\check{\Phi}$ unique with $(II.15)$.

Ad b):

$$\sum_{n\in\mathcal{N}} |(f, \widetilde{\varphi_n})_{\mathcal{H}}|^2 = (f, S_\Phi^{-1} f)_{\mathcal{H}} = \sum_{n\in\mathcal{N}} c_n (\varphi_n, S_\Phi^{-1} f)_{\mathcal{H}} =$$

$$= \sum_{n \in \mathcal{N}} c_n (\widetilde{\varphi_n}, f)_{\mathcal{H}}.$$

Hence

$$\sum_{n \in \mathcal{N}} |(f, \widetilde{\varphi_n})_{\mathcal{H}}|^2 + \sum_{n \in \mathcal{N}} |(f, \widetilde{\varphi_n})_{\mathcal{H}} - c_n|^2 =$$

$$= \sum_{n \in \mathcal{N}} |(f, \widetilde{\varphi_n})_{\mathcal{H}}|^2 + \sum_{n \in \mathcal{N}} (|(f, \widetilde{\varphi_n})_{\mathcal{H}}|^2 -$$

$$-(f, \widetilde{\varphi_n})_{\mathcal{H}} \overline{c_n} - \overline{(f, \widetilde{\varphi_n})} c_n + |c_n|^2) = \sum_{n \in \mathcal{N}} |c_n|^2. \qquad \diamondsuit$$

The previous results allow the reconstruction of $f \in \mathcal{H}$ from the discrete values $(f, \varphi_n)_{\mathcal{H}}$ $(\varphi_n \in \Phi)$, if the dual frame $\tilde{\Phi}$ of $\Phi$ (and so especially $S_\Phi^{-1}$) is known. The following theorem gives an explicit construction of $S_\Phi^{-1}$.

THEOREM 5. *(An explicit expression for $S_\Phi^{-1}$)*
*Let $\Phi$ be an $(A, B)$-frame for $\mathcal{H}$. Then*

$$S_\Phi^{-1} = \frac{2}{A+B} \sum_{k=0}^{\infty} (Id_{\mathcal{H}} - \frac{2S_\Phi}{A+B})^k, \qquad (II.16)$$

*where convergence holds in operator norm.*

*Proof.*

$$S_\Phi^{-1} = \frac{2}{A+B} [Id_{\mathcal{H}} - (Id_{\mathcal{H}} - \frac{2S_\Phi}{A+B})]^{-1}.$$

By theorem 3, $S_\Phi \geq A \cdot Id_{\mathcal{H}}$, hence

$$Id_{\mathcal{H}} - \frac{2S_\Phi}{A+B} \leq Id_{\mathcal{H}} - \frac{2A}{A+B} \cdot Id_{\mathcal{H}} = \frac{B-A}{B+A} \cdot Id_{\mathcal{H}}. \qquad (II.17)$$

Again from theorem 3 it follows that $S_\Phi \leq B \cdot Id_{\mathcal{H}}$, hence

$$Id_{\mathcal{H}} - \frac{2S_\Phi}{A+B} \geq Id_{\mathcal{H}} - \frac{2B}{A+B} \cdot Id_{\mathcal{H}} = \frac{A-B}{A+B} \cdot Id_{\mathcal{H}}. \qquad (II.18)$$

$(II.17)$ and $(II.18)$ together imply

$$\|Id_{\mathcal{H}} - \frac{2S_\Phi}{A+B}\| \leq \frac{\frac{B}{A} - 1}{\frac{B}{A} + 1} < 1 \qquad (II.19)$$

So $(II.16)$ converges uniformly in operator sense. $\diamondsuit$

COROLLARY 1. *(Approximative reconstruction)*

i) *If $\Phi$ is an (A,B)-frame for $\mathcal{H}$, then one has $\forall\ f \in \mathcal{H}$*

$$f = \frac{2}{A+B} \sum_{n\in\mathcal{N}} < f, \varphi_n >_{\mathcal{H}} \varphi_n + R_\Phi f, \tag{II.20}$$

*where*

$$\|R_\Phi\|^2 \le \frac{\frac{B}{A}-1}{\frac{B}{A}+1}. \tag{II.21}$$

ii) *If $\Phi$ is a tight A-frame for $\mathcal{H}$, then one has $\forall\ f \in \mathcal{H}$*

$$f = \frac{1}{A} \sum_{n\in\mathcal{N}} (f, \varphi_n)_{\mathcal{H}} \varphi_n, \tag{II.22}$$

*where convergence holds in the norm of $\mathcal{H}$.*

*Proof.*
i) $(II.20)$ and $(II.21)$ are immediate consequences of $(II.19)$ and the definition of $S_\Phi$.
ii) $(II.22)$ follows from i), because $\frac{B}{A} = 1$, in the tight case. $\diamondsuit$

COROLLARY 2. *(Reconstruction algorithm by Duffin-Schaeffer)* [DufS52]

*Let $\Phi$ be an $(A, B)$-frame for $\mathcal{H}$, $f \in \mathcal{H}$. Assume, $T_\Phi f = ((f, \varphi_n)_{\mathcal{H}})_{n\in\mathcal{N}}$ is known. Define recursively*

$$f^{(0)} := 0, \quad f^{(1)} := \frac{2}{A+B} \sum_{n\in\mathcal{N}} (f, \varphi_n)_{\mathcal{H}},$$

$$f^{(k)} := 2f^{(k-1)} + f^{(k-2)} + \frac{2}{A+B} S_\Phi (f^{(k-1)} - f^{(k-2)}). \tag{II.23}$$

*Then,*

$$\|f - f^{(k)}\|_{\mathcal{H}} \le \left(\frac{\frac{B}{A}-1}{\frac{B}{A}+1}\right)^k \|f\|_{\mathcal{H}}. \tag{II.24}$$

*Proof.* $(II.23)$ is equivalent to the following definition:

$$f^{(k)} = \frac{2}{A+B} S_\Phi (r^{(0)} + r^{(1)} + \ldots + r^{(k-1)}),$$

where $r^{(0)} := f$,

$$r^{(k)} := r^{(k-1)} - \frac{2}{A+B} S_\Phi r^{(k-1)} \quad (k \in \mathbf{N}).$$

Assertion:

$$r^{(k)} = f - f^{(k)}. \tag{$II.25$}$$

*Proof.* Induction over $k$.

$k = 1:$

$$r^{(1)} = r^{(0)} - \frac{2}{A+B} S_\Phi r^{(0)} = f - \frac{2}{A+B} S_\Phi f,$$

where

$$f^{(1)} = \sum_{n \in \mathcal{N}} \frac{2}{A+B} (f, \varphi_n)_{\mathcal{H}} \varphi_n = \frac{2}{A+B} S_\Phi f.$$

$k - 1 \to k:$

$$r^{(k)} = r^{(k-1)} - \frac{2}{A+B} S_\Phi r^{(k-1)} =$$

$$(by\ assumption)$$

$$= f - f^{(k-1)} - \frac{2}{A+B} S_\Phi (f - f^{(k-1)}). \tag{$II.26$}$$

$$f^{(k)} = \sum_{n \in \mathcal{N}} \frac{2}{A+B} (r^{(0)} + \ldots + r^{(k-1)}, \varphi_n)_{\mathcal{H}} \varphi_n =$$

$$= f^{(k-1)} + \frac{2}{A+B} \sum_{n \in \mathcal{N}} (r^{(k-1)}, \varphi_n)_{\mathcal{H}} \varphi_n =$$

$$= f^{(k-1)} + \frac{2}{A+B} S_\Phi r^{(k-1)} =$$

$$(by\ assumption)$$

$$= f^{(k-1)} + \frac{2}{A+B} S_\Phi (f - f^{(k-1)}). \tag{$II.27$}$$

Inserting $(II.27)$ into $(II.26)$, yields $(II.25)$.

$$\|f - f^{(k)}\|_{\mathcal{H}} = \|f - \sum_{n\in\mathcal{N}} \frac{2}{A+B}(r^{(0)} + \ldots + r^{(k-1)}, \varphi_n)_{\mathcal{H}}\varphi_n\|_{\mathcal{H}} =$$

$$= \|f - f^{(k-1)} - \frac{2}{A+B} S_\Phi r^{(k-1)}\|_{\mathcal{H}} =$$

$$= \|f - f^{(k-1)} - S_\Phi(f - f^{(k-1)})\|_{\mathcal{H}} \leq \left(\frac{\frac{B}{A} - 1}{\frac{B}{A} + 1}\right) \|f - f^{(k-1)}\|_{\mathcal{H}}$$

by $(II.19)$. Repeating these arguments $(k-2)$-times, results in $(II.24)$. $\diamondsuit$

The last corollary shows that the size of $|\frac{B}{A} - 1|$ is a good measure for the speed of convergence of the reconstruction algorithm. The less $|\frac{B}{A} - 1|$, the faster the convergence. Frames with $|\frac{B}{A} - 1| \approx 0$ are called *snug frames*. Tight frames are optimal in that respect, but not always available in praxis.

Theorem 6. *(Behaviour of frames under reduction)*
*If one removes one element from a frame, there remains either a frame or an uncomplete set in $\mathcal{H}$.*

*Proof.* Let $\Phi$ be an $(A, B)$-frame for $\mathcal{H}$. Remove the element $\varphi_{n_0}$ from $\Phi$ ($n_0 \in \mathcal{N}$ arbitrary). By $(II.14)$ one has

$$\varphi_{n_0} = \sum_{n\in\mathcal{N}} < \varphi_{n_0}, \widetilde{\varphi_n} >_{\mathcal{H}} \varphi_n.$$

On the other hand $\varphi_{n_0} = \sum_{n\in\mathcal{N}} \delta_{n_0 n}\varphi_n$. Therefore application of corollary 2 to theorem 4 results in

$$\sum_{n\in\mathcal{N}} |\delta_{n_0 n}|^2 = \sum_{n\in\mathcal{N}} |(\varphi_{n_0}, \widetilde{\varphi_n})_{\mathcal{H}}|^2 + \sum_{n\in\mathcal{N}} |(\varphi_{n_0}, \widetilde{\varphi_n})_{\mathcal{H}} - \delta_{n_0 n}|^2. \quad (II.28)$$

Case 1: $(\varphi_{n_0}, \widetilde{\varphi_{n_0}})_{\mathcal{H}} = 1$.
Then, $(II.28)$ implies

$$1 = 1 + 2\sum_{n\in\mathcal{N}} |(\varphi_{n_0}, \widetilde{\varphi_n})_{\mathcal{H}}|^2,$$

and so $(\varphi_{n_0}, \widetilde{\varphi_n})_{\mathcal{H}} = 0 \;\; \forall\; n \neq n_0$, i.e. $(\widetilde{\varphi_{n_0}}, \varphi_n)_{\mathcal{H}} = 0 \;\; \forall\; n \neq n_0$, by the selfadjointness of $S_\Phi^{-1}$, but $\widetilde{\varphi_{n_0}} \neq 0$, because $(\widetilde{\varphi_{n_0}}, \varphi_{n}0)_{\mathcal{H}} = 1$ by assumption. So

$$\overline{span}\{\varphi_n : \; n \neq n_0\}^{\perp} \neq \{0\},$$

and $\Phi \setminus \{\varphi_{n_0}\}$ is not complete.

Case 2: $(\varphi_{n_0}, \widetilde{\varphi_{n_0}})_{\mathcal{H}} \neq 1$.

Here $(II.14)$ implies

$$\varphi_{n_0} = \frac{1}{1 - (\varphi_{n_0}, \widetilde{\varphi_{n_0}})_{\mathcal{H}}} \sum_{n \in \mathcal{N}, n \neq n_0} (\varphi_{n_0}, \widetilde{\varphi_n})_{\mathcal{H}} \varphi_n.$$

Therefore, $\forall\; f \in \mathcal{H}$

$$|(f, \varphi_{n_0})_{\mathcal{H}}|^2 = |(f, \frac{1}{1 - (\varphi_{n_0}, \widetilde{\varphi_{n_0}})} \sum_{n \in \mathcal{N}, n \neq n_0} (f, \widetilde{\varphi_n})_{\mathcal{H}})|^2 \leq$$

$$\leq \frac{1}{|1 - (\varphi_{n_0}, \widetilde{\varphi_{n_0}})_{\mathcal{H}}|^2} \sum_{n \in \mathcal{N}, n \neq n_0} |(\varphi_{n_0}, \widetilde{\varphi_n})_{\mathcal{H}}|^2 \sum_{n \in \mathcal{N}, n \neq n_0} |(f, \varphi_n)_{\mathcal{H}}|^2.$$

Summation over $\mathcal{N}$ yields

$$\sum_{n \in \mathcal{N}, n \neq n_0} |(f, \varphi_n)_{\mathcal{H}}|^2 \leq \sum_{n \in \mathcal{N}, n \neq n_0} |(f, \varphi_n)_{\mathcal{H}}|^2 +$$

$$+ \frac{1}{|1 - (\varphi_{n_0}, \widetilde{\varphi_{n_0}})_{\mathcal{H}}|^2} \sum_{n \in \mathcal{N}, n \neq n_0} |(\varphi_{n_0}, \widetilde{\varphi_n})_{\mathcal{H}}|^2 \sum_{n \in \mathcal{N}, n \neq n_0} |(f, \varphi_n)_{\mathcal{H}}|^2$$

$$=: C \sum_{n \in \mathcal{N}, n \neq n_0} |(f, \varphi_n)_{\mathcal{H}}|^2.$$

Since $\Phi$ is a frame, it follows:

$$\frac{A}{C} \|f\|^2 \leq \sum_{n \in \mathcal{N}, n \neq n_0} |(f, \varphi_n)_{\mathcal{H}}|^2 \leq \sum_{n \in \mathcal{N}} |(f, \varphi_n)_{\mathcal{H}}|^2 \leq B \|f\|^2.$$

Hence, $\Phi \setminus \{\varphi_{n_0}\}$ is an $(\frac{A}{C}, B)$-frame. $\diamondsuit$

Definition. A frame $\Phi$ for $\mathcal{H}$ is called *exact* or *minimal*, if $\forall n_0 \in \mathcal{N} : \;\; \Phi \setminus \{\varphi_{n_0}\}$ is *no* frame for $\mathcal{H}$.

COROLLARY. *(Biorthogonality)*
*If $\Phi$ is an exact frame, $\tilde{\Phi}$ its dual frame, then*

$$(\varphi_m, \widetilde{\varphi_n}) = \delta_{mn} \quad \forall m, n \in \mathcal{N}.$$

*Proof.* The statement "$\Phi$ exact" corresponds to case 1, in the proof of the last theorem, with $n_0 \in \mathcal{N}$ arbitrary. $\diamondsuit$

DEFINITION. A linearly independent subset $\Phi = (\varphi_n)_{n\in\mathcal{N}}$ of $\mathcal{H}$ is called an $(A, B)$-*Riesz basis* of $\mathcal{H}$, if

i) $\overline{span}\{(\varphi_n)_{n\in\mathcal{N}}\} = \mathcal{H}$.

ii) $\exists\ A, B \in \mathbf{R},\ 0 < A \leq B < \infty$ such that $\forall\ c := (c_n)_{n\in\mathcal{N}} \in l^2(\mathcal{N})$

$$A\|c\|_{l^2}^2 \leq \|\sum_{n\in\mathcal{N}} c_n\varphi_n\|_{\mathcal{H}}^2 \leq B\|c\|_{l^2}^2.$$

THEOREM 7. *(Equivalent definition of a Riesz basis)*

*The following two statements are equivalent.*

i) $\Phi = (\varphi_n)_{n\in\mathcal{N}}$ *is a Riesz basis of* $\mathcal{H}$.

ii) $\Phi = (\varphi_n)_{n\in\mathcal{N}} \in \mathcal{H}^{\mathcal{N}}$ *is such that for all orthonormal bases* $E = (e_n)_{n\in\mathcal{N}}$ *of* $\mathcal{H}$ *the linear operator* $M_{\Phi E} : \ \mathcal{H} \to \mathcal{H}$ *defined by* $\varphi_n \mapsto e_n \ \ (n \in \mathcal{N})$ *is bounded and invertible on* $\mathcal{H}$.

*Proof. i)* $\Longrightarrow$ *ii)* :
Let $\Phi$ be an $(A, B)$-Riesz basis, $E$ an arbitrary orthonormal basis of $\mathcal{H}$. A linear operator is well-defined by its values on the basis elements. Hence $M_{\Phi E} :\ \varphi_n \mapsto e_n$ and $M_{E\Phi} :\ e_n \mapsto \varphi_n \quad (n \in \mathcal{N})$ are well-defined linear operators from $\mathcal{H}$ onto $\mathcal{H}$. The operators $M_{\Phi E}$ and $M_{E\Phi}$ are bounded, because

$$\sqrt{A} \leq \|\varphi_n\|_{\mathcal{H}} \leq \sqrt{B}, \quad \|e_n\|_{\mathcal{H}} = 1, \tag{II.29}$$

more exactly,

$$\|M_{\Phi E}\| = \frac{1}{\sqrt{A}}, \qquad \|M_{E\Phi}\| = \sqrt{B}.$$

By definition and the completeness of both systems $\Phi$ and $E$, one has

$$M_{\Phi E} \circ M_{E\Phi} = M_{E\Phi} \circ M_{\Phi E} = Id_{\mathcal{H}}.$$

therefore $M_{\Phi E}^{-1} = M_{E\Phi}$ is bounded.

$ii) \Longrightarrow i)$ :

Let $\Phi$ be as in ii). $\Phi$ is complete in $\mathcal{H}$, by the bijective correspondance of $\Phi$ to an orthonormal basis.

$$< f, g >_M := (M_{\Phi E} f, M_{\Phi E} g)_{\mathcal{H}} \qquad (f, g \in \mathcal{H})$$

defines a scalar product on $\mathcal{H}$. Let $\|\cdot\|_M$ denote the corresponding norm. It follows that $\forall\ f \in \mathcal{H}$

$$\|f\|_M = \|M_{\Phi E} f\|_{\mathcal{H}} \leq \|M_{\Phi E}\| \|f\|_{\mathcal{H}},$$

$$\|f\|_{\mathcal{H}} = \|M_{\Phi E} \circ M_{\Phi E}^{-1} f\|_{\mathcal{H}} = \|M_{\Phi E}^{-1} f\|_M \leq \|M_{\Phi E}^{-1}\| \|f\|_M.$$

Since $\|M_{\Phi E}\|$, $\|M_{\Phi E}^{-1}\|$ are bounded, one can define $A := \|M_{\Phi E}\|^{-2}$, $B := \|M_{\Phi E}^{-1}\|^2$, so

$$\frac{1}{\sqrt{B}} \|f\|_{\mathcal{H}} \leq \|f\|_M \leq \frac{1}{\sqrt{A}} \|f\|_{\mathcal{H}}. \tag{$II.30$}$$

Hence $(\cdot,\cdot)_{\mathcal{H}}$ and $(\cdot,\cdot)_M$ define equivalent norms on $\mathcal{H}$.

Let $f$ be an arbitrary element of $\mathcal{H}$, $f = \sum_{n\in\mathcal{N}} c_n \varphi_n$. (Such an expansion always exists, because of the completeness of $\Phi$ in $\mathcal{H}$.) $(II.30)$ implies

$$\frac{1}{\|M_{\Phi E}\|^2} \|\sum_{n\in\mathcal{N}} c_n \varphi_n\|_M^2 \leq \|\sum_{n\in\mathcal{N}} c_n \varphi_n\|_{\mathcal{H}}^2 \leq \|M_{\Phi E}^{-1}\|^2 \|\sum_{n\in\mathcal{N}} c_n \varphi_n\|_M^2,$$

where

$$\|\sum_{n\in\mathcal{N}} c_n \varphi_n\|_M^2 = \|\sum_{n\in\mathcal{N}} c_n M_{\Phi E} \varphi_n\|_{\mathcal{H}}^2 = \|\sum_{n\in\mathcal{N}} c_n e_n\|_{\mathcal{H}}^2 = \sum_{n\in\mathcal{N}} |c_n|^2$$

by Parseval's relation. This yields $\forall\ c \in l^2(\mathcal{N})$

$$\frac{1}{\|M_{\Phi E}\|^2}\|c\|_{l^2} \leq \|\sum_{n\in\mathcal{N}} c_n\varphi_n\|^2_{\mathcal{H}} \leq \|M^{-1}_{\Phi E}\|^2\|c\|^2_{l^2},$$

respectively,

$$A\|c\|_{l^2} \leq \|\sum_{n\in\mathcal{N}} c_n\varphi_n\|^2_{\mathcal{H}} \leq B\|c\|^2_{l^2},$$

hence $\Phi$ is a $(A, B)$-Riesz-basis. ♢

Corollary.
*The following two statements are equivalent.*

i) $\Phi = (\varphi_n)_{n\in\mathcal{N}}$ *is a Riesz basis for* $\mathcal{H}$.

ii) $\sum_{n\in\mathcal{N}} c_n\varphi_n$ *converges in* $\mathcal{H}$, *for arbitrary sequences* $c \in l^2(\mathcal{N})$.

*Proof.* Theorem 7 ii) is equivalent to: $\sum_{n\in\mathcal{N}} c_n\varphi_n$ converges iff $\sum_{n\in\mathcal{N}} c_n e_n$ converges, for any orthonormal basis $E$ of $\mathcal{H}$.
But, the last statement exactly holds if $c \in l^2(\mathcal{N})$. ♢

Theorem 8. *(Connection between frames and Riesz bases)*
$\Phi$ *is a Riesz basis of* $\mathcal{H}$ *iff* $\Phi$ *is an exact frame for* $\mathcal{H}$.

*Proof.* [You80] "$\Longrightarrow$"
Let $\Phi$ be an $(A, B)$-Riesz basis for $\mathcal{H}$. Choose an arbitrary orthonormal basis $E$. Define $M_{\Phi E}$ as in the last theorem. $M_{\Phi E}$ and $M^{-1}_{\Phi E}$ are bounded linear operators, and so are $M^*_{\Phi E}$ and $(M^{-1}_{\Phi E})^*$. Define

$$T_E : \mathcal{H} \to l^2(\mathcal{N}).$$

$$f \mapsto ((f, e_n)_{\mathcal{H}})_{n\in\mathcal{N}}.$$

Then holds

$$T_E f = ((f, M_{\Phi E}\varphi_n)_{\mathcal{H}})_{n\in\mathcal{N}} = ((M^*_{\Phi E} f, \varphi_n)_{\mathcal{H}})_{n\in\mathcal{N}},$$

and $T_E$ is linear and bounded. Therefore

$$T_\Phi = T_E \circ {M^*_{\Phi E}}^{-1} : \mathcal{H} \to l^2(\mathcal{N})$$

$$f \mapsto ((f,\varphi_n)_{\mathcal{H}})_{n\in\mathcal{N}}$$

is linear and bounded, i.e. there exists a constant $C > 0$ such that

$$\sum_{n\in\mathcal{N}} |(f,\varphi_n)_{\mathcal{H}}|^2 \leq C\|f\|^2_{\mathcal{H}},$$

more exactly, $C = B$ because of $(II.29)$ and $E$ orthonormal basis. $T_\Phi$ is injective, by the completeness of $\Phi$, even surjective, by the last lemma. From the bounded inverse theorem, one, therefore, has that $T_\Phi^{-1}$ is bounded. Therefore, $\exists\ D > 0$ :

$$\|f\|^2 \leq D \sum_{n\in\mathcal{N}} | < f,\varphi_n >_{\mathcal{H}} |^2,$$

more exactly, $D = A$, because $T_\Phi^{-1}|_{T_\Phi(\mathcal{H})} = M^*_{\Phi E} \circ T_E^{-1}|_{T_\Phi(\mathcal{H})}$ and $(II.29)$ holds. It remains to show: $\Phi$ exact. This is clear by the fact that the removal of a basis vector from a basis results in an uncomplete system.

"$\Longleftarrow$"

Let $\Phi$ be an exact frame for $\mathcal{H}$ with frame bounds $A, B$.
Assertion 1: $\Phi$ is a basis of $\mathcal{H}$.

*Proof of assertion 1*: One has to show: Every $f \in \mathcal{H}$ possesses a unique representation

$$f = \sum_{n\in\mathcal{N}} c_n\varphi_n. \tag{$II.31$}$$

By $(II.14)$ one has:

$$f = \sum_{n\in\mathcal{N}} (f,\tilde{\varphi}_n)_{\mathcal{H}}\varphi_n.$$

For any other representation of the type $(II.31)$ results:
$\forall\ m \in \mathcal{N} \quad (f,\widetilde{\varphi_m})_{\mathcal{H}} = \sum_{n\in\mathcal{N}} c_n(\varphi_n,\widetilde{\varphi_m})_{\mathcal{H}} = c_m$, because of the biorthogonality. This yields the uniqueness of the coefficients $c_m$.
Assertion 2: $\Phi$ Riesz basis. This means -by the corollary to theorem 7- $\sum_{n\in\mathcal{N}} c_n\varphi_n$ converges iff $c \in l^2(\mathcal{N})$.

*Proof of assertion 2*: Assume $\sum_{n\in\mathcal{N}} c_n\varphi_n$ converges to $f \in \mathcal{H}$. By assertion 1, the $c_n$ are unique,

$$c_n = (f,\widetilde{\varphi_n})_{\mathcal{H}} = (S_\Phi^{-1}f,\varphi_n)_{\mathcal{H}},$$

and so

$$\sum_{n\in\mathcal{N}} |c_n|^2 = \sum_{n\in\mathcal{N}} |(S_\Phi^{-1} f, \varphi_n)_{\mathcal{H}}|^2 \leq B\|S_\Phi^{-1} f\|_{\mathcal{H}}^2 \leq \frac{B}{A^2}\|f\|^2,$$

where we used $(II.5)$ and $(II.10)$. So $c \in l^2(\mathcal{N})$.

Assume $c \in l^2(\mathcal{N})$. Assume $\mathcal{N} = \mathbf{N}$ (which causes no problem, since $\mathcal{N}$ is countable.) Let $n \geq m \in \mathbf{N}$. Then

$$\|\sum_{i=m}^{n} c_i\varphi_i\|_{\mathcal{H}}^2 \leq B \sum_{i=m}^{n} |c_i|^2.$$

For $m, n \to \infty$, this finite sum tends to 0, so the partial sums $(\sum_{i=1}^{n} c_i\varphi_i)_{n\in\mathcal{N}}$ constitute a Cauchy sequence, which converges by the completeness of $\mathcal{H}$. $\diamondsuit$

Corollary 1. *(Estimation of the frame bounds)*

i) *If* $\Phi$ *is an* $(A, B)$*-Riesz basis*, then $\Phi$ *is an exact* $(A, B)$*-frame.*

ii) *If* $\Phi$ is an exact $(A, B)$*-frame, then* $\Phi$ *is an* $(\frac{A^2}{B}, B)$*-Riesz basis.*

Proof follows immediately, by the proof of the last theorem, using that $T_\Phi$ is surjective, in this case. $\diamondsuit$

Definition. A family $\Phi = (\varphi_n)_{n\in\mathcal{N}}$ in $\mathcal{H}$ is called *linearly* $l^2$*-independent,* if $\sum_{n\in\mathcal{N}} \varphi_n = 0 \ \ \forall\, c \in l^2(\mathcal{N})$ implies $c = 0$.

Corollary 2.
$\Phi$ *is a Riesz basis of* $\mathcal{H}$ *iff* $\Phi$ *is linearly* $l^2$*-independent frame.*

*Proof.* Because of the last theorem, it remains to show:
$\Phi$ exact frame iff $\Phi$ linearly $l^2$-independent frame.
"$\Longrightarrow$"
Assume, $\Phi$ is an exact frame, and there exists a sequence $0 \not\equiv c \in l^2(\mathcal{N})$ : $\sum_{n\in\mathcal{N}} c_n\varphi_n = 0$. Then, there exists $m \in \mathcal{N}$ such that $c_m \neq 0$. By the exactness of $\Phi$ follows $\varphi_n \neq 0 \ \ \forall n \in \mathcal{N}$. So,

$$\varphi_m = -\frac{1}{c_m} \sum_{n\in\mathcal{N}, n\neq m} c_n\varphi_n,$$

i.e. $\varphi$ can be removed from $\Phi$, and $\Phi$ remains a frame. This contradicts theorem 6.
"$\Longleftarrow$"
Let $\Phi$ be a linearly $l^2$-independent frame. Assume, there exists a $m \in \mathcal{N}$ such that $\Phi \setminus \varphi_m$ is still a frame. Hence, $\varphi_m = \sum_{n\in\mathcal{N},n\neq m} \widetilde{c_n}\varphi_n$ and $\tilde{c} \in l^2(\mathcal{N})$. Define

$$c_n = \begin{cases} \widetilde{c_n} & n \neq m \\ -1 & n = m. \end{cases}$$

$\sum_{n\in\mathcal{N}} c_n\varphi_n = 0$, but $\sum_{n\in\mathcal{N}} |c_n|^2 \geq 1 > 0$, contradicting $\Phi$ linearly $l^2$-independent. $\diamondsuit$

C.K. Chui [Chui92b,p.71] proves this corollary for the special case of a wavelet frame, directly, without using theorem 7.

COROLLARY. *(Equivalent characterizations of ONB)*
*Let $\Phi = (\varphi_n)_{n\in\mathcal{N}}$ be an orthonormal family in $\mathcal{H}$. Then are equivalent.*

a) $\Phi$ *orthonormal basis of* $\mathcal{H}$.

b) $\Phi$ *complete in* $\mathcal{H}$.

c) $\Phi$ *exact* $(1,1)$*-frame for* $\mathcal{H}$.

d) $\Phi$ $(1,1)$*-Riesz basis of* $\mathcal{H}$.

*Proof.*
a) equivalent b), by definition.
c) equivalent d), by previous theorems.
d) equivalent a), by definition of a Riesz basis and Parseval's relation.
$\diamondsuit$

## II.2. Wavelet Frames.

Now, as announced in II.0, the general theory of frames will be applied to the discretization of the CWT. There exists a standard choice for the set $S$ in $(II.1)$, namely, for fixed discretization parameters $a_0 > 1,\ b_0 > 0$,

$$S := \{(a,b) \in \mathbf{R}^+ \times \mathbf{R}: \quad a = a_0^{-j},\ b = kb_0a_0^{-j}, \quad j,k \in \mathbf{Z}\}. \quad (II.32)$$

Given an analyzing wavelet $\psi$, define

$$\psi_{jk}^{a_0b_0}(x) := \psi_{a_0^{-j},kb_0a_0^{-j}}(x) = a_0^{\frac{j}{2}}\psi(a_0^jx - kb_0) \quad (j,k \in \mathbf{Z}). \quad (II.33)$$

An analyzing wavelet $\psi$ is called a *frame wavelet*, a *tight frame wavelet*, a *Riesz wavelet*, or an *orthonormal wavelet for the discretization parameters* $a_0, b_0$, if $(\psi_{jk}^{a_0b_0})_{j,k\in\mathbf{Z}}$ is a frame, a tight frame, a Riesz basis, or an orthonormal basis for $L^2(\mathbf{R})$, respectively. We talk about an $(A,B)$*-frame wavelet*, if $(\psi_{jk}^{a_0b_0})_{j,k\in\mathbf{Z}}$ is an $(A,B)$-frame, etc. Conversely, a frame or basis in $L^2(\mathbf{R})$ is said to be a *wavelet frame* or a *wavelet basis*, in case its elements are of the form $(II.33)$.

The most favourite choice, for the discretization parameters, is the *dyadic* one:

$$a_0 = 2, \quad b_0 = 1.$$

In this case, we just speak of *frame wavelets*, and so on, dropping the upper index $a_0b_0$, in $(II.33)$.

In case $\psi$ is a frame wavelet for $a_0, b_0$, the results of the previous section II.1 can be applied to $(\psi_{jk}^{a_0b_0})_{j,k\in\mathbf{Z}}$, leading to the following theorem:

THEOREM 1. *(Wavelet frame expansions)*

a) *If $\psi$ is a tight $A$-frame wavelet for $a_0, b_0$, then*

$$f = \frac{1}{A}\sum_{j,k\in\mathbf{Z}} (f, \psi_{jk}^{a_0b_0})\psi_{jk}^{a_0b_0} \quad \forall f \in L^2(\mathbf{R}).$$

b) *If $\psi$ is an orthonormal wavelet for $a_0, b_0$, then*

$$f = \sum_{j,k\in\mathbf{Z}} (f, \psi_{jk}^{a_0b_0})\psi_{jk}^{a_0b_0} \quad \forall f \in L^2(\mathbf{R}).$$

c) *If $\psi$ is an arbitrary (A,B)-frame wavelet for $a_0, b_0$, then*

$$f = \sum_{j,k \in \mathbf{Z}} (f, \widetilde{\psi_{jk}^{a_0 b_0}}) \psi_{jk}^{a_0 b_0} = \sum_{j,k \in \mathbf{Z}} (f, \psi_{jk}^{a_0 b_0}) \widetilde{\psi_{jk}^{a_0 b_0}} \quad \forall f \in L^2(\mathbf{R}), \tag{II.34}$$

*where $(\widetilde{\psi_{jk}^{a_0 b_0}})_{j,k \in \mathbf{Z}}$ denotes the dual frame of $(\psi_{jk}^{a_0 b_0})_{j,k \in \mathbf{Z}}$, introduced in $(II.11)$.*

Let $\psi$ be a frame wavelet for $a_0, b_0$. A function $\tilde{\psi} \in L^2(\mathbf{R})$ is called a *dual wavelet of $\psi$, for $a_0, b_0$*, if $\tilde{\psi}_{jk}^{a_0 b_0} = \widetilde{\psi_{jk}^{a_0 b_0}} \quad \forall j, k \in \mathbf{Z}$, i.e., if the dual frame of $(\psi_{jk}^{a_0 b_0})_{j,k \in \mathbf{Z}}$ is a *wavelet* frame for $a_0, b_0$, again. If $\psi$ possesses such a dual wavelet, for $a_0, b_0$, then equation $(II.34)$ specifies to

$$f = \sum_{j,k \in \mathbf{Z}} (f, \tilde{\psi}_{jk}^{a_0 b_0}) \psi_{jk}^{a_0 b_0} = \sum_{j,k \in \mathbf{Z}} (f, \psi_{jk}^{a_0 b_0}) \tilde{\psi}_{jk}^{a_0 b_0} \quad \forall f \in L^2(\mathbf{R}).$$

REMARKS.

i) $\tilde{\psi}$ is the dual wavelet of $\psi$, for $a_0, b_0$, iff $\psi$ is the dual wavelet of $\tilde{\psi}$, for $a_0, b_0$.

ii) If $\psi$ is an orthonormal wavelet, for $a_0, b_0$, then $\psi$ is the dual wavelet of itself, for $a_0, b_0$.

iii) Every frame wavelet $\psi$, for $a_0, b_0$ has at most one dual wavelet $\tilde{\psi}$, for $a_0, b_0$.

iv) If $\tilde{\psi}$ exists, the frame bounds of $(\tilde{\psi}_{jk}^{a_0 b_0})_{j,k \in \mathbf{Z}}$ are $\frac{1}{B}$ and $\frac{1}{A}$, by theorem 4 in II.1.

v) There exist frame (even Riesz) wavelets $\psi$, for $a_0, b_0$, without dual wavelets, for $a_0, b_0$.

The following example for v) was given by C.K. Chui [Chui92b,p.13]:

Consider the dyadic case. Let $\psi$ be an orthonormal wavelet, $z \in \mathbf{C}$, $|z| < 1$. Define

$$\psi^z(x) := \psi(x) - \overline{z}\sqrt{2}\psi(2x).$$

By theorem 7 in II.1, $(\psi^z_{jk})_{j,k\in\mathbf{Z}}$ is still a Riesz basis, and so, by theorem 8 in II.1, an exact frame. Hence, the corollary to theorem 6 in II.1 implies that the unique dual frame of $(\psi^z_{jk})_{j,k\in\mathbf{Z}}$ is biorthogonal to it. Since $(\psi_{jk})_{j,k\in\mathbf{Z}}$ was assumed to be orthonormal, one gets, in particular, that

$$\widetilde{\psi^z_{00}}(x) = \sum_{l=0}^{\infty} \psi_{-l0}(x) z^l,$$

$$\widetilde{\psi^z_{01}}(x) = \psi_{01}(x).$$

Assume, for all $z \in \mathbf{C}$ with $|z| < 1$ exists a dual wavelet $\widetilde{\psi^z}$. This dual has to satisfy

$$\widetilde{\psi^z}_{00}(x) = \sum_{l=0}^{\infty} \psi_{-l0}(x) z^l, \quad \textit{and} \quad \widetilde{\psi^z}_{01}(x) = \psi_{01}(x),$$

which leads to

$$\psi(x) = \psi_{01}(x+1) = \widetilde{\psi^z}_{01}(x+1) = \widetilde{\psi^z}_{00}(x) = \sum_{l=0}^{\infty} \psi_{-l0}(x) z^l,$$

and, therefore, to

$$\sum_{l=1}^{\infty} \psi_{-l0}(x) z^l = 0 \quad \forall\ z \in \mathbf{C},\ |z| = 1,\ x \in \mathbf{R},$$

contradicting $\|\psi\| > 0$. $\diamondsuit$

This example shows that the dual frame of a wavelet frame is, in general, no *wavelet* frame. Nevertheless, it is not necessary to compute $S^{-1}_{\Psi^{a_0 b_0}} \psi^{a_0 b_0}_{jk}$ for all $j, k \in \mathbf{Z}$ explicitly, to get the dual frame. This is the content of the following proposition.

Proposition. *(Construction of the dual wavelet frame)*

*Let $\psi$ be a frame wavelet for $a_0, b_0$, $S_{\Psi^{a_0 b_0}}$ the corresponding frame operator. Then.*

$$\widetilde{\psi_{jk}^{a_0 b_0}}(x) = a_0^{\frac{j}{2}} \widetilde{\psi_{0k}^{a_0 b_0}}(a_0^j x) \qquad \forall j, k \in \mathbf{Z}.$$

(I.e. it suffices to compute $S_{\Psi^{a_0 b_0}}^{-1} \psi_{0k}^{a_0 b_0}$, $k \in \mathbf{Z}$, explicitly.)

*Proof.* For $j \in \mathbf{Z}$ define

$$D_j^{a_0} : \ L^2(\mathbf{R}) \to L^2(\mathbf{R})$$

$$f(\cdot) \mapsto a_0^{\frac{j}{2}} f(a_0^j x).$$

We have to show that $D_j^{a_0} \widetilde{\psi_{0k}^{a_0 b_0}}$ defines an element of the dual frame of $(\psi_{jk}^{a_0 b_0})_{j,k \in \mathbf{Z}}$ for all $j, k \in \mathbf{Z}$. By definition of the dual frame, $S_{\Psi^{a_0 b_0}}^{-1} \psi_{jk}^{a_0 b_0}$ is the dual element of $\psi_{jk}^{a_0 b_0}$ $\forall j, k \in \mathbf{Z}$. So, $D_j^{a_0} \psi_{0k}^{a_0 b_0}$ is certainly an element of the dual frame, in case $j = 0$. For $j \neq 0$,

$$S_{\Psi^{a_0 b_0}}^{-1} \psi_{jk}^{a_0 b_0} = S_{\Psi^{a_0 b_0}}^{-1} D_j^{a_0} \psi_{0k}^{a_0 b_0}.$$

So, it remains to show that $S_{\Psi^{a_0 b_0}}^{-1}$ and $D_j^{a_0}$ commute, or equivalently, $S_{\Psi^{a_0 b_0}}$ and $D_j^{a_0}$ commute. (Since $S_{\Psi^{a_0 b_0}} D_j^{a_0} = D_j^{a_0} S_{\Psi^{a_0 b_0}}$ is equivalent to $D_j^{a_0} = S_{\Psi^{a_0 b_0}}^{-1} D_j^{a_0} S_{\Psi^{a_0 b_0}}$, and this holds iff $D_j^{a_0} S_{\Psi^{a_0 b_0}}^{-1} = S_{\Psi^{a_0 b_0}}^{-1} D_j^{a_0}$.) By definition,

$$S_{\Psi^{a_0 b_0}} : L^2(\mathbf{R}) \to L^2(\mathbf{R})$$

$$f \mapsto \sum_{j,k \in \mathbf{Z}} (f, \psi_{jk}^{a_0 b_0}) \psi_{jk}^{a_0 b_0}.$$

For $f \in L^2(\mathbf{R})$ arbitrary, one has

$$S_{\Psi^{a_0 b_0}} D_j^{a_0} f(x) = S_{\Psi^{a_0 b_0}} a_0^{\frac{j}{2}} f(a_0^j x) =$$

$$= \sum_{i,k \in \mathbf{Z}} [\int_{-\infty}^{\infty} a_0^{\frac{j}{2}} f(a_0^j \tilde{x}) a_0^{\frac{i}{2}} \overline{\psi^{a_0 b_0}(a_0^i \tilde{x} - k)} d\tilde{x}] \psi_{ik}^{a_0 b_0}(x) =$$

$$= \sum_{i,k \in \mathbf{Z}} [\int_{-\infty}^{\infty} a_0^{-\frac{j}{2}} f(u) a_0^{\frac{i}{2}} \overline{\psi(a_0^i a_0^{-j} u - k)} du] \psi_{ik}^{a_0 b_0}(x) =$$

$$= \sum_{i,k\in\mathbf{Z}} (f, \psi_{i-j,k}^{a_0 b_0}) \psi_{ik}^{a_0 b_0}(x) = \sum_{i,k\in\mathbf{Z}} (f, \psi_{ik}^{a_0 b_0}) \psi_{i+j,k}^{a_0 b_0}(x) =$$

$$= \sum_{i,k\in\mathbf{Z}} (f, \psi_{ik}^{a_0 b_0}) D_j^{a_0} \psi_{i+j,k}^{a_0 b_0}(x) = D_j^{a_0} S_{\Psi^{a_0 b_0}} f.$$

$\diamondsuit$

Whether $\psi$ is an $(A, B)$-frame wavelet, for some given discretization parameters $a_0, b_0$, depends on $\psi$, as well as on the parameters $a_0, b_0$. A necessary condition, for $(\psi_{jk}^{a_0 b_0})_{j,k\in\mathbf{Z}}$ to constitute an $(A, B)$-frame, is given by the next theorem, due to I. Daubechies [Dau90].

THEOREM 2. *(Necessary frame condition)*
*If $\psi$ is a $(A, B)$-frame wavelet for $a_0, b_0$, then*

$$A \leq \frac{2\pi}{b_0 \ln a_0} \int_0^\infty |\hat{\psi}(\omega)|^2 \frac{d\omega}{\omega} \leq B$$

and

$$A \leq \frac{2\pi}{b_0 \ln a_0} \int_{-\infty}^0 |\hat{\psi}(\omega)|^2 \frac{d\omega}{|\omega|} \leq B. \tag{II.35}$$

*Sketch of proof*, following [Dau90,p.974] and [Dau92,p.63].

Step 1: For all positive trace class operators $T : L^2(\mathbf{R}) \to L^2(\mathbf{R})$, holds

$$A \cdot TrT \leq \sum_{j,k\in\mathbf{Z}} (T\psi_{jk}^{a_0 b_0}, \psi_{jk}^{a_0 b_0}) \leq B \cdot TrT, \tag{II.36}$$

where $TrT := \sum_{n\in\mathbf{N}} (Te_n, e_n)$, and $(e_n)_{n\in\mathbf{N}}$ an arbitrary orthonormal basis of $L^2(\mathbf{R})$.

($TrT$ is independent of the choice of a special orthonormal basis, by definition of a trace class operator.)

This is an immediate consequence of the definition of a frame and the fact that each trace class operator $T$ allows an expansion

$$T = \sum_{n\in\mathbf{N}} (Te_n, e_n)(\,\cdot\,, e_n).$$

Step 2: Let $h \in H^2(\mathbf{R})$ admissible, $h_{ab}(x) := \frac{1}{\sqrt{a}} h(\frac{x-b}{a}), \quad a \in \mathbf{R}^+, b \in \mathbf{R}$ and

$$c(a,b) := \begin{cases} w(\frac{|b|}{a}) & 1 \leq a \leq a_0, \\ 0 & otherwise, \end{cases}$$

where $w$ denotes a positive integrable function, and define

$$C: \ f \mapsto \int_0^\infty \int_{-\infty}^\infty c(a,b) h_{ab}(f, h_{ab}) \frac{dadb}{a^2}.$$

Then, one can show: $C$ is a positive trace-class operator. Hence, $(II.36)$ applies and results in

$$A \cdot 2 \ln a_0 [\int_0^\infty w(s) ds] \|h\|^2 \leq$$

$$\leq \int_0^\infty \int_{-\infty}^\infty \sum_{n \in \mathbf{N}} w(\frac{|b_0 + n b_0|}{a}) |(\psi, h_{ab})|^2 \frac{dadb}{a^2} \leq$$

$$\leq B \cdot 2 \ln a_0 [\int_0^\infty w(s) ds] \|h\|^2. \qquad (II.37)$$

Step 3: Choose a special function $w$, in $(II.37)$:

$$w(s) := w_\lambda(s) := \lambda e^{-\lambda^2 \pi^2 s^2},$$

where $\lambda \in \mathbf{R}^+$. This function satisfies

$$\sum_{n \in \mathbf{N}} w(\frac{|b + n b_0|}{a}) = \frac{a}{b_0} + \rho(a,b), \qquad (II.38)$$

where

$$|\rho(a,b)| \leq \lambda \ \ and \ \ \int_0^\infty w(s) ds = \frac{1}{2}. \qquad (II.39)$$

(The crucial point to prove $(II.38)$ is that $w$ possesses a unique maximum and is monotone, otherwise.)

Inserting $(II.39)$ and $(II.38)$ in $(II.37)$, yields

$$A \|h\|^2 \ln a_0 \leq \frac{2\pi}{b_0} \|h\|^2 \int_0^\infty |\hat{\psi}(\omega)|^2 \frac{d\omega}{\omega} + R \leq B \|h\|^2 \ln a_0, \quad (II.40)$$

where $|R| \leq \lambda c_h \|\psi\|^2$, where $c_h$ is defined as in $(I.19)$.

Dividing $(II.40)$ by $\|h\|^2 \ln a_0$ and letting $\lambda \to 0$, results in the first assertion, in $(II.35)$. The second assertion follows analogously. $\diamondsuit$

COROLLARY.

*If $\psi$ is an $(A, B)$-frame wavelet, for $a_0, b_0,$, then, $\psi$ is a strongly admissible analyzing wavelet such that*

$$2A \cdot b_0 \ln a_0 \leq c_\psi \leq 2B \cdot b_0 \ln a_0,$$

*where $c_\psi$ is defined as in $(I.19)$.*

*Especially.*

*If $\psi$ is a dyadic orthonormal wavelet, then holds*

$$c_\psi = 2 \ln 2.$$

THEOREM 3. *(Sufficient frame conditions)*

*Assumptions.*

*$\psi$ analyzing wavelet, $a_0 > 1$,*

$$\inf_{1 \leq |\omega| \leq a_0} \sum_{j \in \mathbf{Z}} |\hat{\psi}(a_0^{-j}\omega)|^2 > 0, \tag{II.41}$$

$$\sup_{1 \leq |\omega| \leq a_0} \sum_{j \in \mathbf{Z}} |\hat{\psi}(a_0^{-j}\omega)|^2 < \infty \tag{II.42}$$

*and*

$$\beta(s) := \sup_{\omega \in \mathbf{R}} |\hat{\psi}(a_0^{-j}\omega)||\hat{\psi}(a_0^{-j}\omega + s)| \tag{II.43}$$

*decays at least as fast as $(1 + |s|)^{-(1+\epsilon)}$, where $\epsilon > 0$.*

*Assertion.*

*There exists a $b_0^{max} > 0$ such that for all $b_0 < b_0^{max}$ $\psi$ is an $(A, B)$-frame wavelet, where*

$$A \geq \frac{2\pi}{b_0}\{\inf_{1 \leq |\omega| \leq a_0} \sum_{j \in \mathbf{Z}} |\hat{\psi}(a_0^{-j}\omega)|^2 - \sum_{m \in \mathbf{Z}\setminus\{0\}} [\beta(\frac{2\pi}{b_0}m)\beta(-\frac{2\pi}{b_0}m)]^{\frac{1}{2}}\},$$

$$B \leq \frac{2\pi}{b_0}\{\sup_{1 \leq |\omega| \leq a_0} \sum_{j \in \mathbf{Z}} |\hat{\psi}(a_0^{-j}\omega)|^2 + \sum_{m \in \mathbf{Z}\setminus\{0\}} [\beta(\frac{2\pi}{b_0}m)\beta(-\frac{2\pi}{b_0}m)]^{\frac{1}{2}}\}.$$

*Sketch of proof,* following [Dau90,p.983], [Dau92,p.69].

Step 1: The frame definition can be reformulated, as follows:

$$A\|f\|^2 \leq \frac{2\pi}{b_0}\int_{-\infty}^{\infty}|\hat{f}(\omega)|^2\sum_{j\in\mathbf{Z}}|\hat{\psi}(a_0^{-j}\omega)|^2 d\omega + \mathcal{R}(f) \leq B\|f\|^2, \tag{II.44}$$

where

$$\mathcal{R}(f) :=$$

$$= \frac{2\pi}{b_0}\sum_{j,m\in\mathbf{Z},m\neq 0}\int_{-\infty}^{\infty}\hat{f}(\omega)\overline{\hat{f}(\omega+m\frac{2\pi}{b_0a_0^{-j}})}\,\overline{\hat{\psi}(a_0^{-j}\omega)}\hat{\psi}(a_0^{-j}\omega+m\frac{2\pi}{b_0})d\omega.$$

Step 2: The first summand, in the middle term of $(II.44)$, is always bounded, by assumption. Applying Cauchy-Schwarz's inequality several times, yields the following estimation for the rest $\mathcal{R}$:

$$|\mathcal{R}(f)| \leq \frac{2\pi}{b_0}\|f\|^2\sum_{m\in\mathbf{Z},m\neq 0}[\beta(\frac{2\pi}{b_0}m)\beta(-\frac{2\pi}{b_0}m)]^{\frac{1}{2}}, \tag{II.45}$$

where $\beta$ is defined as in $(II.43)$.

Step 3: The decay condition on $\beta$ ensures that

$$\sum_{m\in\mathbf{Z},m\neq 0}[\beta(\frac{2\pi}{b_0}m)\beta(-\frac{2\pi}{b_0}m)]^{\frac{1}{2}} \leq \sum_{m\in\mathbf{Z}}(1+|\frac{2\pi}{b_0}|)^{-(1+\epsilon)} < \infty,$$

at last for $\frac{2\pi}{b_0} \geq 2$, i.e. $b_0 \leq \pi$, since in this case, $\sum_{m\in\mathbf{Z}} 2^{-m}$ is a majorant. $\diamondsuit$

The conditions of the last theorem are already satisfied, if

$$|\hat{\psi}(\omega)| \leq C|\omega|^{\alpha}(1+|\omega|)^{-\gamma},$$

for some $\alpha > 0,\ \gamma > \alpha + 1$.

For examples of wavelet frames, cf. [Dau92,p.73].

## II.3. Wavelet Orthonormal Bases (WONB).

**General Remark.**

We confine ourselves to the *dyadic* discretization $a_0 = 2,\ b_0 = 1$, for the rest of the chapter.

Theorem 1 b), in the last section, implies that the reconstruction of a function $f \in L^2(\mathbf{R})$ from its discrete wavelet coefficients $(f, \psi_{jk})_{j,k \in \mathbf{Z}}$ is the most efficient one, in case that $\psi$ is a orthonormal wavelet, since then, the numerically expensive computation of the dual frame ceases.

But, discretizing CWT is not the only way, leading to WONBs. Orthonormal bases of such a simple structure have been of interest, for a long time, in various branches of pure and applied mathematics, such as there are theory of function spaces, approximation theory and mathematical physics.

The first construction of a WONB goes back to A. Haar, in 1910. But, not before the late eighties, a unifying theory was created. This was finally done by Y. Meyer and S. Mallat by the concept of *multiresolution analysis*, which will be represented in section II.4.

EXAMPLES FOR ORTHONORMAL WAVELETS.

a) *Haar wavelet* [Ha10]:

$$\psi_H(x) := \begin{cases} 1, & 0 \leq x < \frac{1}{2} \\ -1, & \frac{1}{2} \leq x > 1 \\ 0, & otherwise. \end{cases}$$

$\psi_H$ is compactly supported, but discontinuous.

$\hat{\psi}_H$ is continuous, but decays not faster as $\frac{1}{|\omega|}$.

[Dau92]

b) *Littlewood-Paley wavelet* [Mey90], defined via its Fourier transform by

$$\hat{\psi}_{LP}(\omega) := \begin{cases} \frac{1}{\sqrt{2\pi}}, & \pi \le |\omega| \le 2\pi \\ 0, & otherwise. \end{cases}$$

$\psi_{LP}$ is continuous, but decays not faster as $\frac{1}{|x|}$. $\hat{\psi}_{LP}$ is compactly supported, but discontinuous.

c) *Modified Littlewood-Paley wavelet* [Mal89b], defined via its Fourier transform by

$$\hat{\psi}_{LPm}(\omega) := \begin{cases} \frac{1}{\sqrt{2\pi}}, & 4\frac{\pi}{7} \le |\omega| \le \pi \ or \\ & 4\pi \le |\omega| \le 4\pi + 4\frac{\pi}{7} \\ 0, & otherwise. \end{cases}$$

Properties similiar to b).

d) *Meyer wavelets* [Mey89a], [Mey86b], defined via its Fourier transform as

$$\hat{\psi}_{M\nu}(\omega) := \begin{cases} \frac{e^{\frac{i\omega}{2}}}{\sqrt{2\pi}} \sin[\frac{\pi}{2}\nu(\frac{3}{2\pi}|\omega| - 1)], & \frac{2\pi}{3} \le |\omega| \le \frac{4\pi}{3} \\ \frac{e^{\frac{i\omega}{2}}}{\sqrt{2\pi}} \cos[\frac{\pi}{2}\nu(\frac{3}{4\pi}|\omega| - 1)], & \frac{4\pi}{3} \le |\omega| \le \frac{8\pi}{3} \\ 0, & otherwise, \end{cases}$$

where $\nu \in C^k(\mathbf{R})$, $k \in \mathbf{N}$ arbitrary, satisfies $\nu(x)+\nu(1-x) = 1$ and

$$\nu(x) = \begin{cases} 0, & x \le 0 \\ 1, & x \ge 1. \end{cases}$$

$\psi_{M\nu} \in \mathcal{S}(\mathbf{R})$ (i.e. $C^\infty$ with polynomial decay), $\hat{\psi}_{M\nu}$ compactly supported, $\hat{\psi}_{M\nu} \in C^k(\mathbf{R})$, where $k \in \mathbf{N}$ is determined by $\nu$.

[Dau92]

e) *Battle-Lemarié wavelets/ Spline wavelets*
[Bat87], [Bat88], [Chui92b], [Chui92d], [Dau92], [Lem88]: For a precise definition, see for example [Dau92,p.146]. $\psi_{BL,k} \in C^{k-2}(\mathbf{R})$ and $\exists \gamma_k, C_k > 0$ such that $|\psi_{BL,k}(x)| \le C_k e^{-\gamma_k |x|}$, where $\gamma_k$ is the bigger, the smaller the parameter $k$.

$$\int_{-\infty}^{\infty} x^m \psi_{BL,k}(x) dx = 0, \quad m = 0, 1, \ldots, k-1.$$

[Dau92]

f) *Daubechies wavelets* [Dau88a]: For a precise definition, see section II.5. $\psi_D$ is compactly supported and Hölder continuous. The degree of Hölder-continuity increases, with increasing support-length.

[Dau92]

Let $\psi$ be an orthonormal wavelet. For $j \in \mathbf{Z}$, let

$$W_j := \overline{span}\{\psi_{jk} : \ k \in \mathbf{Z}\}. \qquad (II.46)$$

Then, obviously

$$L^2(\mathbf{R}) = \bigoplus_{j \in \mathbf{Z}} W_j.$$

Following Lemarié [Lem90b], we say that *for $\psi$ happens the miracle of low frequencies*, if there exists a function $\varphi \in L^2(\mathbf{R})$ such that $(\varphi(\,\cdot\, - k))_{k \in \mathbf{Z}}$ is an orthonormal set, and for

$$V_0 := \overline{span}\{\phi(\,\cdot\, - k) : \ k \in \mathbf{Z}\}$$

holds

$$L^2(\mathbf{R}) = \bigoplus_{j \in \mathbf{N}_0} W_j \oplus V_0.$$

The function $\varphi$ is called a *father function of $\psi$*. By this, one has an alternative orthonormal basis for $L^2(\mathbf{R})$, possessing a simple structure, namely

$$\{\psi_{jk},\ \varphi(\,\cdot\, - k),\ j \in \mathbf{N}, k \in \mathbf{Z}\}.$$

One can show that the miracle of low frequencies happens for all the orthonormal wavelets, presented in the previous collection, except for the modified Littlewood-Paley wavelet c).

The corresponding father functions $\varphi$ are given by a)

$$\varphi_H(x) := \begin{cases} 1, & 0 \le x < 1 \\ 0, & otherwise. \end{cases}$$

[Dau92]

b) $$\varphi_{LP}(x) = \tfrac{\sin \pi x}{\pi x}.$$

c)

$$\hat{\varphi}_{M\nu}(\omega) := \begin{cases} \frac{1}{\sqrt{2\pi}}, & |\omega| \le \frac{2\pi}{3} \\ \frac{1}{\sqrt{2\pi}} \cos[\frac{\pi}{2}\nu(\frac{3}{4\pi}|\omega| - 1)], & \frac{2\pi}{3} \le |\omega| \le \frac{4\pi}{3} \\ 0, & otherwise. \end{cases}$$

[Dau92]

d) $\varphi_{BLk}$ is the $k^{th}$ order cardinal B-spline.

[Dau92]

e) $\varphi_D$ is compactly supported and possesses the same regularity as $\psi_D$. As for $\psi_D$, the regularity of $\varphi_D$ is the higher, the bigger its support.

[Dau92]

The non-existence of a father function, for $\psi_{LPm}$ in example c), will be proved in the following section.

## II.4. Multiresolution Analysis (MRA) and Quadrature Mirror Filters (QMF).

Question. Is there a simple concept to describe those wavelet orthonormal bases, for which the miracle of low frequencies, described in II.3 (p.72), happens? The answer rests on the following

Definition. A *multiresolution analysis (MRA)* is a sequence $(V_j)_{j\in\mathbf{Z}}$ of closed subspaces in $L^2(\mathbf{R})$ satisfying the following conditions:

i) $V_{j-1} \subseteq V_j \quad \forall j \in \mathbf{Z}$,[9]

ii) $\overline{\bigcup_{j\in\mathbf{Z}} V_j} = L^2(\mathbf{R})$,

iii) $\bigcap_{j\in\mathbf{Z}} V_j = \{0\}$,

iv) $f \in V_{j-1} \iff f(2\,\cdot\,) \in V_j$,

v) $f \in V_j \iff f(\,\cdot\, - 2^{-j}k) \in V_j \ \ \forall k \in \mathbf{Z}$,

vi) $\exists g \in V_0 : \quad (g(\cdot - k))_{k\in\mathbf{Z}}$ Riesz basis of $V_0$. The function $g$ is called *generating function of the MRA*.

Lemma. *The Riesz basis, in vi), can always be chosen as an orthonormal basis.*

*Proof.* [Mey90c,p.27]. We will prove: For an arbitrary generating function $g$, the following function defines the Fourier transform of a function $\varphi$ such that $(\varphi(\cdot - k))_{k\in\mathbf{Z}}$ is an ONB of $L^2(\mathbf{R})$ :

$$\hat{\varphi}(\omega) := \frac{1}{\sqrt{2\pi}} \frac{\hat{g}(\omega)}{(\sum_{k\in\mathbf{Z}} |\hat{g}(\omega + 2k\pi)|^2)^{\frac{1}{2}}}. \qquad (II.47)$$

First, it will be shown that $(II.47)$ is well-defined, i.e. the denominator does not vanish, almost everywhere. Since $(g(\cdot - k))_{k\in\mathbf{Z}}$ is a Riesz basis of $V_0$, by assumption, there exist some constants $0 < A \le B < \infty$ with

$$A\|c\|_{l^2} \le \|\sum_{k\in\mathbf{Z}} c_k g(\cdot - k)\| \le B\|c\|_{l^2} \quad \forall c = (c_k)_{k\in\mathbf{Z}} \in l^2(\mathbf{Z}). \quad (II.48)$$

Denote by $C(\omega) := \sum_{k\in\mathbf{Z}} c_k e^{-ik\omega} \quad (\omega \in \mathbf{R})$ the Fourier series, corresponding to the Fourier coefficients $(c_k)_{k\in\mathbf{Z}}$. The function $C$

9 We choose, here, the convention of A. Cohen, C. Chui, K. Gröchenig/W.R. Madych, P.G. Lemarié and Y. Meyer. Be aware that there exists an alternative definition of MRA in the literature, where $V_j \subseteq V_{j-1}$, c.f. [AkH92], [Alp92a,b], [Ber91a], [Bey92], [BeyCR91], [Dau88a,92], [HeilW89], [Wil92].

is $2\pi$-periodic. So, in case that $c$ is a finite sequence, Plancharel's theorem implies

$$\frac{1}{\sqrt{2\pi}}A\|C\|_{L^2([0,2\pi[)} \leq \left(\int_{-\infty}^{\infty}|C(\omega)|^2|\hat{g}(\omega)|^2 d\omega\right)^{\frac{1}{2}} \leq$$

$$\leq \frac{1}{\sqrt{2\pi}}B\|C\|_{L^2([0,2\pi[)}. \qquad (II.49)$$

For infinite sequences, $(II.49)$ follows by continuity. The middle term of $(II.49)$ can be written as

$$\left(\int_0^{2\pi}|C(\omega)|^2 w(\omega)d\omega\right)^{\frac{1}{2}},$$

where

$$w(\omega) := \sum_{k\in\mathbf{Z}}|\hat{g}(\omega+2k\pi)|^2 \in L^1([0,2\pi[). \qquad (II.50)$$

Since $A$ and $B$ are independent of $C$, one can consider special functions $(C_N(\omega))_{N\in\mathbf{N}}$, namely $|C_N(\omega)|^2 := K_N(\omega)$, where $K_N(\omega)$ denotes the Fejer kernel. For $K_N$ holds:

$$\int_0^{2\pi}K_N(\omega)d\omega = 2\pi, \quad \lim_{N\to\infty}(K_N * w(\omega)) = w(\omega) \;\; a.e. \qquad (II.51)$$

So, substituting $C_n$ in $(II.49)$ using $(II.50)$ and $(II.51)$ results in

$$\sqrt{2\pi}A \leq \lim_{N\to\infty}(K_N * w)^{\frac{1}{2}}(\omega) = (w(\omega))^{\frac{1}{2}} \leq \sqrt{2\pi}B \;\; a.e.,$$

hence

$$\sqrt{2\pi}A \leq (\sum_{k\in\mathbf{Z}}|\hat{g}(\omega+2k\pi)|^2)^{\frac{1}{2}} \leq \sqrt{2\pi}B \;\; a.e., \qquad (II.52)$$

which proves that $(II.47)$ is well-defined.

Next we will show:

$$U : V_0 \to L^2([0,2\pi[)$$

$$h(x) = \sum_{k\in\mathbf{Z}}c_k g(x-k) \mapsto C(\omega)(w(\omega))^{\frac{1}{2}}$$

is an isometrical isomorphism.

The isometry follows by Plancharel's theorem. $U$ is injective, because of the linear independence of the sequence $(e^{-ikx})_{k\in\mathbf{Z}}$. It is surjective, since any $q \in L^2([0,2\pi[)$ can be written as $q(\omega) = C(\omega)(w(\omega))^{\frac{1}{2}}$, where $C(\omega) = \sum_{k\in\mathbf{Z}} c_k e^{-ik\omega}$, and $(c_k)_{k\in\mathbf{Z}}$ denotes the Fourier coefficients of $C(\omega)(w(\omega))^{-\frac{1}{2}}$. So, there exists a function $h \in V_0$ with $Uh = q$, namely $h = \sum_{k\in\mathbf{Z}} c_k g_{0k}$.

The last assertion, together with $U\tau_k = \chi_k U \quad \forall\ k \in \mathbf{Z}$ (where $\tau_k f(x) := f(x-k)$, $\chi_k f(x) := e^{-ikx} f(x)$), allows to transfer the problem of orthonormalizing translations of a function in $f \in L^2(\mathbf{R})$ to the well known problem of orthonormalizing modulations of a function, namely standard discrete Fourier theory. From there, it follows that the functions $\chi_k q(\omega)$ with $q(\omega) = \frac{1}{\sqrt{2\pi}}$ *a.e.* constitute an ONB for $L^2([0,2\pi[)$, and so $\hat{\varphi}$ defined by $(II.47)$ leads to an ONB. $\diamondsuit$

There is no uniqueness, for the function $\varphi$, to have the property that $\{\varphi(\cdot - k) : k \in \mathbf{Z}\}$ is an ONB of $V_0$. Any such $\varphi$, is called a *father function of the MRA* $(V_j)_{j\in\mathbf{Z}}$.

Corollary. *(Characterization of orthonormality)*
If $(\varphi(x-k))_{k\in\mathbf{Z}}$ is an orthonormal, then

$$\sum_{k\in\mathbf{Z}} |\hat{\varphi}(\omega + 2k\pi)|^2 = \frac{1}{2\pi} \qquad a.e. \tag{II.53}$$

*Proof.* Follows from substituting $A = B = 1$, in $(II.52)$.

Proposition. *(Refinement equation[10])*
*For* $(V_j)_{j\in\mathbf{Z}}$ *multiresolution analysis,* $\varphi$ *a corresponding father function, holds.*

$$\varphi(x) = \sum_{k\in\mathbf{Z}} h(k)\sqrt{2}\varphi(2x-k), \tag{II.54}$$

10 Such equations are fundamental for *subdivision algorithms* in *computer aided geometric design*, cf. [DahmM90a,b].

where

$$h(k) := \sqrt{2}\int_{-\infty}^{\infty} \varphi(x)\overline{\varphi(2x-k)}dx. \qquad (II.55)$$

*Proof.* Follows from properties i) and iv) of a MRA and the fact that the functions $(\varphi_{1k})_{k\in\mathbf{Z}}$ constitute an orthonormal basis of $V_1$.

$\diamondsuit$

For $j \in \mathbf{Z}$ arbitrary, define the subspace $W_j$ of $L^2(\mathbf{R})$ by

$$V_j \oplus W_j := V_{j+1}. \qquad (II.56)$$

The spaces $W_j$ are called *detail spaces* of the MRA. Their definition implies that

$$\bigoplus_{j\in\mathbf{Z}} W_j = L^2(\mathbf{R}). \qquad (II.57)$$

So, knowing an ONB for each of the spaces $W_j$ and uniting these ONBs, results in an ONB for $L^2(\mathbf{R})$. This idea leads to the construction of WONBs, as the next theorem shows.

THEOREM 1. *(Construction of a WONB from a MRA)*
*For every MRA $(V_j)_{j\in\mathbf{Z}}$, there exists an orthonormal wavelet $\psi$ such that $(\psi_{0k})_{k\in\mathbf{Z}}$ is an ONB of $W_0$, and for which the miracle of low frequencies happens. Any father function $\varphi$ of the MRA is a father function of $\psi$, in the sense of ch.II.3.*

*Proof.* [Mal89b], [Daub92,p.135]. Let $\varphi$ be a father function of the MRA.

Assertion 1: Define $H(\omega) := \sum_{k\in\mathbf{Z}} h(k)e^{-ik\omega}$ with $(h(k))_{k\in\mathbf{Z}}$ as in $(II.55)$. Then holds a.e.

$$|H(\omega)|^2 + |H(\omega+\pi)|^2 = 2. \qquad (II.58)$$

*Proof of assertion 1*: Fourier transform of $(II.54)$ results in

$$\hat{\varphi}(\omega) = \frac{1}{\sqrt{2}}H(\frac{\omega}{2})\hat{\varphi}(\frac{\omega}{2}) \quad a.e. \qquad (II.59)$$

Inserting $(II.59)$ into $(II.53)$ yields, after substitution $\xi := \frac{\omega}{2}$,

$$\frac{1}{\pi} = \sum_{k\in\mathbf{Z}} |H(\xi + k\pi)|^2 |\hat{\varphi}\xi + k\pi)|^2 =$$

$$= \sum_{l\in\mathbf{Z}} |H(\xi + 2l\pi)|^2 |\hat{\varphi}(\xi + 2l\pi)|^2 +$$

$$+ \sum_{m\in\mathbf{Z}} |H(\xi + 2m\pi + \pi)|^2 |\hat{\varphi}(\xi + 2m\pi + \pi)|^2 =$$

$$= \frac{1}{2\pi}[|H(\omega)|^2 + |H(\omega + \pi)|^2] \quad a.e.,$$

where we applied $(II.53)$ in the last step.

Assertion 2:

$$f \in W_0 \iff \hat{f}(\omega) = \frac{1}{\sqrt{2}} e^{\frac{i\omega}{2}} \nu_f(\omega) \overline{H(\frac{\omega}{2} + \pi)} \hat{\varphi}(\frac{\omega}{2}), \qquad (II.60)$$

where $\nu_f$ denotes a $2\pi$-periodic function.

If $f \in W_0$, then $f \in V_1$ and $f \perp V_0$. From the first fact it follows that

$$f(x) = \sum_{k\in\mathbf{Z}} (f, \varphi_{1k})\varphi_{1k}(x) = \sum_{k\in\mathbf{Z}} g_f(k)\varphi_{1k}(x), \qquad (II.61)$$

and so, after Fourier transforming,

$$\hat{f}(\omega) = \frac{1}{\sqrt{2}} G_f(\frac{\omega}{2}) \hat{\varphi}(\frac{\omega}{2}), \qquad (II.62)$$

where

$$G_f(\omega) := \sum_{k\in\mathbf{Z}} g_f(k) e^{-ik\omega}.$$

From the fact that $f \perp V_0$ and Plancharel's formula, one has $\forall\, k \in \mathbf{Z}$

$$0 = \int_{-\infty}^{\infty} f(x)\overline{\varphi_{0k}(x)}dx = \int_{-\infty}^{\infty} \hat{f}(\omega)\overline{\hat{\varphi}(\omega)} e^{ik\omega} d\omega =$$

$$= \int_0^{2\pi} e^{ik\omega} \sum_{l\in\mathbf{Z}} \hat{f}(\omega + 2l\pi)\overline{\hat{\varphi}(\omega + 2l\pi)} d\omega, \qquad (II.63)$$

where convergence of this series holds in $L^1([0,2\pi[)$. Substituting $(II.59)$ and $(II.61)$ into $(II.63)$, and splitting in a sum over odd and even integers (so that $(II.53)$ can be applied), results in

$$G_f(\omega)\overline{H(\omega)} + G_f(\omega+\pi)\overline{H(\omega+\pi)} = 0 \quad a.e. \tag{$II.64$}$$

Now, $(II.57)$ excludes that $H(\omega) = H(\omega+\pi) = 0 \quad a.e.$, so there must exist a $2\pi$-periodic function $\lambda_f(\omega)$ with

$$G_f(\omega) = \lambda_f(\omega)\overline{H(\omega+\pi)} \quad a.e. \tag{$II.65$}$$

and

$$\lambda_f(\omega) + \lambda_f(\omega+\pi) = 0 \quad a.e.$$

Hence

$$\lambda_f(\omega) = e^{i\omega}\nu_f(2\omega), \tag{$II.66$}$$

where $\nu_f(\omega)$ is $2\pi$-periodic, again. Substituting $(II.66)$ in $(II.65)$, $(II.65)$ in $(II.62)$, results in the assertion $(II.60)$. The inverse statement follows, by inverse Fourier transform.

Assertion 3:

$$\hat{\psi}(\omega) = \frac{1}{\sqrt{2}} e^{\frac{i\omega}{2}} \overline{H(\frac{\omega}{2}+\pi)}\hat{\varphi}(\frac{\omega}{2}) \tag{$II.67$}$$

is the Fourier transform of a function $\psi$ with $(\psi_{0k})_{k\in\mathbf{Z}}$ ONB of $W_0$.

*Proof of assertion 3*: $\psi$ is of the type $(II.60)$, hence $\psi \in W_0$. The functions $(\psi_{0k})_{k\in\mathbf{Z}}$ are orthogonal, since Plancharel's formula implies that

$$\int_{-\infty}^{\infty} \psi(x)\overline{\psi(x-k)}dx = \int_{-\infty}^{\infty} e^{ik\omega}|\hat{\psi}(\omega)|^2 = \int_0^{2\pi} e^{ik\omega} \sum_{l\in\mathbf{Z}} |\hat{\psi}(\omega+2l\pi)|^2,$$

where, by assumption,

$$\sum_{l\in\mathbf{Z}} |\hat{\psi}(\omega+2l\pi)|^2 = \sum_{l\in\mathbf{Z}} \frac{1}{2}|H(\frac{\omega}{2}+l\pi+\pi)|^2|\hat{\varphi}(\frac{\omega}{2}+l\pi)|^2 =$$

$$= \frac{1}{2}|H(\frac{\omega}{2}+\pi)|^2 \sum_{l\in\mathbf{Z}} |\hat{\varphi}(\frac{\omega}{2}+2l\pi)|^2 + \frac{1}{2}|H(\frac{\omega}{2})|^2 \sum_{l\in\mathbf{Z}} |\hat{\varphi}(\frac{\omega}{2}+2l\pi+\pi)|^2 =$$

$$= \frac{1}{4\pi}[|H(\frac{\omega}{2}+\pi)|^2 + |H(\frac{\omega}{2})|^2] = \frac{1}{2\pi},$$

by $(II.53)$ and $(II.58)$. So, $\int_{-\infty}^{\infty} \psi(x)\overline{\psi(x-k)}dx = 1$, for $k = 0$, 0 otherwise.

Completeness in $W_0$.

Let $f \in W_0$ be arbitrary. $(II.60)$ and $(II.67)$ imply that

$$\hat{f}(\omega) = \nu_f(\omega)\hat{\psi}(\omega),$$

where $\nu_f(\omega)$ is a $2\pi$-periodic function. If $\nu_f(\omega) \in L^2([0, 2\pi[)$, inverse Fourier transform results in the desired basis representation. By definition of $\nu_\psi$, one has

$$\int_0^{2\pi} |\nu_f(\omega)|^2 d\omega = 2\int_0^{\pi} |\lambda_f(\omega)|^2 d\omega.$$

On the other hand, by the orthonormality of the functions $(\varphi_{0k})_{k\in\mathbf{Z}}$,

$$\infty > 2\pi\|f\|^2 = \int_0^{2\pi} |G_f(\omega)|^2 d\omega = \int_0^{2\pi} |\lambda_f(\omega)|^2 |H(\omega+\pi)|^2 d\omega =$$

$$= \int_0^{\pi} |\lambda_f(\omega)|^2 [|H(\omega+\pi)|^2 + |H(\omega)|^2] d\omega = \int_0^{2\pi} 2|\lambda_f(\omega)| d\omega,$$

and therefore

$$\int_0^{2\pi} |\nu_f(\omega)|^2 d\omega < \infty,$$

so inverse Fourier transform yields the assertion.

Assertion 3, together with property iv) of a MRA, implies that $(\psi_{jk})_{k\in\mathbf{Z}}$ is an ONB of $W_j$ $\forall j \in \mathbf{Z}$, and therefore $(\psi_{jk})_{j,k\in\mathbf{Z}}$ WONB of $L^2(\mathbf{R})$.

The miracle of low frequencies happens for $\psi$ by construction; $\varphi$, $V_j$ and $W_j$ can be identified with $\varphi$, $V_j$ and $W_j$ in II.3. $\diamondsuit$

A function $\psi$, as in the theorem, is called a *mother wavelet, associated with the MRA* $(V_j)_{j\in\mathbf{Z}}$.

The last proof is constructive. Inverse Fourier transform of $(II.67)$ results in the following explicit expression for the mother wavelet $\psi$, constructed in the proof:

$$\psi(x) = \sum_{k\in\mathbf{Z}} g(k)\varphi_{1k}(x),$$

where

$$g(k) := (-1)^{k+1} h(-1-k). \qquad (II.68)$$

Note that a mother wavelet, associated with a MRA, is not unique. $(II.67)$ corresponds to the choice $\nu(\omega) \equiv 1$ in $(II.60)$. But any other choice of a $2\pi$-periodic function $\nu$ with $|\nu(\omega)| = 1$ *a.e.*, would do it, as well.

A variant of $(II.68)$, leading to a mother wavelet, too, which is frequently used in the literature, is

$$g(k) := (-1)^k h(2n+1-k),$$

for some fixed $n \in \mathbf{Z}$.

Corollary.

*For an orthonormal wavelet $\psi$, the miracle of low frequencies happens, iff $\psi$ is associated with a MRA.*

*Proof.* "$\Longleftarrow$" follows from the last theorem, "$\Longrightarrow$", because the father function of an orthonormal wavelet, for which the miracle of low frequencies happens, defines a MRA by $V_j := \overline{span}\{\varphi_{jk} : k \in \mathbf{Z}\}$. $\diamondsuit$

Now, one can explain, why for the example c), in the previous section, the miracle of low frequencies does *not* happen, or, equivalently, why this orthonormal wavelet is not associated with a MRA [Dau92,p.136].

Assume, for $\psi$ happens the miracle of low frequencies. By the last corollary, $\psi$ is associated with a MRA. Let $\varphi$ be a corresponding father function. Formulas $(II.59)$ and $(II.67)$, together with $(II.58)$, result in

$$|\hat{\varphi}(\omega)|^2 + |\hat{\psi}(\omega)|^2 = \frac{1}{2}|H(\frac{\omega}{2})|^2|\hat{\varphi}(\frac{\omega}{2})|^2 + \frac{1}{2}|H(\frac{\omega}{2}+\pi)|^2|\hat{\varphi}(\frac{\omega}{2})|^2 =$$

$$= |\hat{\varphi}(\frac{\omega}{2})|^2,$$

hence

$$|\hat{\varphi}(2\omega)|^2 + |\hat{\psi}(2\omega)|^2 = |\hat{\varphi}(\omega)|^2 \quad a.e.$$

Repeating this arguments, one gets, for $\omega \neq 0$:

$$|\hat{\varphi}(4\omega)|^2 + |\hat{\psi}(4\omega)|^2 + |\hat{\psi}(2\omega)|^2 = |\hat{\varphi}(\omega)|^2,$$

$$|\hat{\varphi}(8\omega)|^2 + |\hat{\psi}(8\omega)|^2 + |\hat{\psi}(4\omega)|^2 + |\hat{\psi}(2\omega)|^2 = |\hat{\varphi}(\omega)|^2,$$

$$\cdots$$

$$|\hat{\varphi}(2^n\omega)|^2 + |\hat{\psi}(2^n\omega)|^2 + |\hat{\psi}(2^{n-1}\omega)|^2 + \cdots + |\hat{\psi}(2\omega)|^2 = |\hat{\varphi}(\omega)|^2,$$

therefore,

$$|\hat{\varphi}(\omega)|^2 = \lim_{n\to\infty} |\hat{\varphi}(2^n\omega)|^2 + \sum_{j=1}^{n} |\hat{\psi}(2^j\omega)|^2.$$

$(II.53)$ results in

$$\lim_{n\to\infty} |\hat{\varphi}(2^n\omega)|^2 \to 0 \quad in \quad L^2(\mathbf{R}),$$

whence, $|\varphi(\omega)|^2 = \sum_{j=1}^{\infty} |\hat{\varphi}(2^j\omega)|^2$ a.e. Therefore, holds a.e.

$$|\hat{\varphi}(\omega)| = \begin{cases} \frac{1}{\sqrt{2\pi}}, & 0 \leq |\omega| \leq \frac{4\pi}{7} \quad or \\ & \pi \leq |\omega| \leq \frac{8\pi}{7} \quad or \\ & 2\pi \leq |\omega| \leq \frac{16\pi}{7}, \\ 0, & otherwise, \end{cases} \tag{II.69}$$

by definition of $\hat{\psi}$. Define $H$ as in the proof of the last theorem. For $\varphi$, as defined above, $|H(\omega)| = \sqrt{2} \ \ \forall\ \omega \in [-\frac{4\pi}{7}, \frac{4\pi}{7}]$, so by $2\pi$-periodicity,

$$|H(\omega)| = \sqrt{2} \quad \forall \omega \in [2\pi, \frac{18\pi}{7}].$$

But, $(II.59)$ and $(II.69)$ imply

$$\frac{1}{\sqrt{2\pi}} = \frac{1}{\sqrt{2}} |H(\omega)||\hat{\varphi}(\omega)| = |\hat{\varphi}(2\omega)| \ \ a.e. \ \ on \ \ [2\pi, \frac{16\pi}{7}],$$

in contradiction to $|\hat{\varphi}(2\omega)| = 0 \ \ a.e$ on this interval, by $(II.69)$. Hence, $\psi$ possesses no father function. $\diamondsuit$

The wavelet $\psi$, in the last example, is of slow decay. P.G. Lemarié proved in [Lem92b] that the miracle of low frequencies happens for all orthonormal wavelets, satisfying some smoothness and decay conditions.

THEOREM 2. *If $\psi$ is an orthonormal wavelet with*

i) $\psi$ *$\epsilon$-Hölder continuous for some* $\epsilon > 0$,

ii) $\omega(x) := \sup_{|h|\leq 1} \frac{|\psi(x)-\psi(x+h)|}{|h|^\epsilon}$ *satisfies* $x^k\omega(x) \in L^2(\mathbf{R}) \forall k \in \mathbf{N}$,

iii) *all zeros of* $\hat{\psi}$ *are of finite order.*

*Then, the miracle of low frequencies happens for* $\psi$.

*Proof* [Lem92b]. $\diamondsuit$

In particular, the assumptions of the last theorem are satisfied, for all compactly supported real-valued orthonormal wavelets which are $\epsilon$-Hölder continuous, for some $\epsilon > 0$. In this case, the father function $\varphi$ can be chosen compactly supported, as well.

**Extensions of the MRA-concept.**

The concept of MRA has been transferred to *different discretizations than the dyadic* one. Again, it leads to the construction of a corresponding wavelet orthonormal bases. (Cf. [Au92b] , [GröM92], [Stri93].)

Apart from this, one can change to *more than one dimension,* as sketched in the following.

By means of a MRA in $L^2(\mathbf{R})$, one can construct ONBs of wavelet type, for $L^2(\mathbf{R}^n)$ ($n \in \mathbf{N}$ arbitrary ) :

Let $\varphi$ be a father function, $\psi$ a corresponding mother wavelet of the MRA. Define

$$\psi^0 := \varphi, \ \psi^1 := \psi.$$

For $x = (x_1, x_2, \ldots, x_n) \in \mathbf{R}^n$, $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n) \in \{0,1\}^n \setminus \{(0,0,\ldots,0)\} =: E$, let

$$\Psi^\epsilon(x) := \psi^{\epsilon_1}(x_1) \cdot \psi^{\epsilon_2}(x_2) \cdot \ldots \cdot \psi^{\epsilon_n}(x_n).$$

($2^n - 1$ different functions.) Then,

$$\{2^{\frac{nj}{2}}\Psi^\epsilon(2^j x - k), \ j \in \mathbf{Z}, \ k \in \mathbf{Z}^n, \ \epsilon \in E\}$$

is an orthonormal basis for $L^2(\mathbf{R}^n)$. More explicitely, in case $n = 2$, one gets with

$$\Psi^{01}(x_1, x_2) := \varphi(x_1) \cdot \psi(x_2),$$

$$\Psi^{10}(x_1, x_2) := \psi(x_1) \cdot \varphi(x_2),$$

$$\Psi^{11}(x_1, x_2) := \psi(x_1) \cdot \psi(x_2)$$

that $\{2^j \Psi^\epsilon(2^j x_1 - k_1, 2^j x_2 - k_2),\ j \in \mathbf{Z},\ (k_1, k_2) \in \mathbf{Z}^2,\ \epsilon \in \{(0,1), (1,0), (1,1)\}\}$ is an orthonormal basis of $L^2(\mathbf{R}^2)$.

Defining a MRA $(V_j^n)_{j \in \mathbf{Z}}$ in $L^2(\mathbf{R}^n)$, analogous to a MRA in $L^2(\mathbf{R})$, as a nested sequence of subspaces in $L^2(\mathbf{R}^n)$, and imitating the construction of theorem 1, one always obtains $2^n - 1$ functions $\Psi^1, \ldots \Psi^{2n-1}$ such that

$$\{2^{\frac{nj}{2}} \Psi^q(2^j x - k),\ j \in \mathbf{Z},\ k \in \mathbf{Z}^n,\ q \in \{1, 2, \ldots, 2^n - 1\}\}$$

constitutes an ONB of $L^2(\mathbf{R})^n$. (See [Mey90,p.90].) The special bases, given above, appear, if one chooses the spaces $V_j^n$, in the MRA of $L^2(\mathbf{R}^n)$, as the n-fold tensor product of the spaces $V_j$ of a MRA of $L^2(\mathbf{R})$,

$$V_j^n := V_j \otimes V_j \otimes \ldots \otimes V_j \quad (n\ factors).$$

They are called *separable* MRA of $L^2(\mathbf{R}^n)$. Examples for *non*-separable MRAs, can be found in [Mey90c,p.86], or in [KoV92].

Be aware, that the wavelet orthonormal bases, constructed from a MRA of $L^2(\mathbf{R}^n)$, *cannot* be used to discretize the CWT in since they are built by more than one mother wavelet.

**The Signal Theoretical Background of MRA.**

Let $(V_j)_{j \in \mathbf{Z}}$ be a MRA of $L^2(\mathbf{R})$, let $(W_j)_{j \in \mathbf{Z}}$ be the corresponding sequence of detail spaces. For $f \in V_0$ arbitrary, define

$$f_{V_j} := orthogonal\ \ projection\ \ of\ \ f\ \ on\ \ V_j\ \ (j = -1, -2, -3, \ldots),$$

$$f_{W_j} := orthogonal\ \ projection\ \ of\ \ f\ \ on\ \ W_j\ \ (j = -1, -2, -3, \cdots).$$

Property i) of a MRA implies that

$$f_{V_{-1}}, f_{V_{-2}}, f_{V_{-3}}, \dots$$

constitutes a sequence of worse and worse approximations of $f$.

By $(II.56)$ holds: $V_j \oplus W_j = V_{j+1}$. So, in the decomposition $f_{V_j} = f_{V_{j-1}} + f_{W_{j-1}}$, the function $f_{W_{j-1}}$ contains all the detail information on $f$, lost by changing from $f_{V_j}$ to $f_{V_{j-1}}$ (hence, the name *detail space* for $W_j$).

Given some $N \in \mathbf{N}$, the function $f \in V_0$ can be reconstructed from the projections $f_{V_{-N}}, f_{W_{-N}}, f_{W_{-N+1}}, \dots, f_{W_{-1}}$ via:

$$f = f_{V_{-N}} + \sum_{i=0}^{N-1} f_{W_{-N+i}}.$$

If $\varphi$ is a father function, $\psi$ a corresponding mother wavelet of the MRA, the projections $f_{V_j}, f_{W_j}$ are completely characterized by the discrete coefficients $c^j := (c^j(k))_{k\in\mathbf{Z}}$ resp. $d^j := (d^j(k))_{k\in\mathbf{Z}}$, where

$$c^j(k) := (f, \varphi_{jk}) \quad and \quad d^j(k) := (f, \psi_{jk}) : \tag{II.70}$$

$$f_{V_j} = \sum_{k\in\mathbf{Z}} c^j(k)\varphi_{jk}, \quad f_{W_j} = \sum_{k\in\mathbf{Z}} d^j(k)\psi_{jk}.$$

Once $c^0$ is given, these sequences $c^j$, $d^j$ $(j = -1, -2, \dots)$ can be computed recursively, according to the following

LEMMA. *(Fast Wavelet Transform). Given* $(h(k))_{k\in\mathbf{Z}}$, *as in* $(II.55)$, $(g(k))_{k\in\mathbf{Z}}$, *as in* $(II.68)$, *and* $c^0$, *as in* $(II.70)$, *the sequences* $c^j$, $d^j$ $(j = -1, -2, \dots)$, *defined in* $(II.70)$, *can be computed, as follows*:

$$c^j(k) = \sum_{n\in\mathbf{Z}} h(n-2k)c^{j+1}(n),$$

$$d^j(k) = \sum_{n\in\mathbf{Z}} g(n-2k)c^{j+1}(n). \tag{II.71}$$

*Proof.* Induction over $j$:

$c: \quad j = -1:$

$$c^{-1}(k) = (f, \varphi_{-1k}) = (\sum_{n\in\mathbf{Z}} c^0(n)\varphi_{0n}, \varphi_{-1k}) =$$

$$= \sum_{n\in\mathbf{Z}} c^0(n)(\varphi_{0n},\varphi_{-1k}) = \sum_{n\in\mathbf{Z}} c^0(n)\int_{-\infty}^{\infty}\varphi(x-n)\frac{1}{\sqrt{2}}\overline{\varphi(2^{-j}x-k)}dx =$$

$$= \sum_{n\in\mathbf{Z}} c^0(n)\int_{-\infty}^{\infty}\sqrt{2}\varphi(2u-n)\overline{\varphi(u-k)}dx = \sum_{n\in\mathbf{Z}} c^0(n)h(n-2k).$$

($j \Longrightarrow j-1$ : completely analogous. )
$d: \quad j=-1:$

$$d^{-1}(k) = (f,\psi_{-1k}) = (\sum_{n\in\mathbf{Z}} c^0(n)\varphi_{0n},\psi_{-1k}) = \ (by\ (II.68))$$

$$= (\sum_{n\in\mathbf{Z}} c^0(n)\varphi_{0n}, \frac{1}{\sqrt{2}}\sum_{m\in\mathbf{Z}} g(m)\varphi_{1m}(2^{-1}x-k)) =$$

$$= (\sum_{n\in\mathbf{Z}} c^0(n)\varphi_{0n}, \sum_{m\in\mathbf{Z}} g(m)\varphi_{0,2k+m}(x)) = \sum_{n\in\mathbf{Z}} c^0(n)g(n-2k).$$

($j \Longrightarrow j-1$ : completely analogous. ) $\diamondsuit$

This leads to the so called **Mallat algorithm** [Mal89a,c] for the decomposition and reconstruction of functions $f \in V_0$ (represented by the corresponding discrete sequences $c^0 \in l^2(\mathbf{Z})$, defined in $(II.70)$):

Let

$$\begin{aligned} H:\ & l^2(\mathbf{Z}) \to l^2(\mathbf{Z}) \\ (c(k))_{k\in\mathbf{Z}} \mapsto & (\sum_{n\in\mathbf{Z}} h(n-2k)c(n))_{k\in\mathbf{Z}}, \\ G:\ & l^2(\mathbf{Z}) \to l^2(\mathbf{Z}) \\ (c(k))_{k\in\mathbf{Z}} \mapsto & (\sum_{n\in\mathbf{Z}} g(n-2k)c(n))_{k\in\mathbf{Z}}. \end{aligned} \tag{II.72}$$

Starting from $c^0 \in l^2(\mathbf{Z})$, define recursively

$$c^j := Hc^{j+1}, \quad d^j := Gc^{j+1} \qquad (j=-1,-2,\ldots).$$

Given some $N \in \mathbf{N}$, $c^0$ can be reconstructed from $c^{-N}, d^{-N}, d^{-N+1}$, $\ldots, d^{-1}$ via

$$c^{j+1} = H^*c^j + G^*d^j \qquad (j=-N,-N+1,-N+2,\ldots,-1).$$

Graphically:

Decomposition of $c^0$ :

Reconstruction of $c^0$:

**Data compression by Mallat algorithm.**

Assume, $c^0$ is a sequence of length $L \in \mathbf{N}$, representing a discrete signal which shall be transmitted. For $N \in \mathbf{N}, c^{-N}$ is a blurred version of $c^0$, just allowing to resolve the main features of $c^0$. $c^{-N}$ is a sequence of length $\frac{L}{2^N}$ and can therefore be transmitted more cheaply then $c^0$ itself. Each additional consideration of a detail sequence $d^{-N},\ d^{-N+1},\ d^{-N+2}, \ldots, d^{-1}$ improves the quality of the transmitted signal, and finally results in the original one. Since the length of $d^j \ \ (j = -N, -N+1, -N+2, \ldots, -1)$ is $\frac{L}{2^{-j}}$, maximally

$$\frac{L}{2^N} + \frac{L}{2^N} + \frac{L}{2^{N-1}} + \ldots + \frac{L}{2^{N-2}} + \ldots + \frac{L}{2} = L$$

data have to be transmitted. However, for applications, the quality of the signal reconstructed from $c^{-N}, d^{-N}, d^{-N+1}, \ldots, d^{-N+k}$, usually suffices, for some $k < N-1$. So, one can renounce the transmission of $d^{-N+k+1}, d^{-N+k+2}, \ldots, d^{-1}$, which means a saving of $\frac{L}{2^{N-k-1}} + \frac{L}{2^{N-k-2}} + \ldots + \frac{L}{2}$ data, i.e. *data compression.*

**Pyramid schemes for image processing.**

The (1-dim.) Mallat algorithm, presented above, can be extended to two dimensions, starting out from a two-dimensional MRA. This two-dimensional analog is an example of a class of so called *pyramid schemes*, developed for image processing by P. Burt and E. Adelson in 1983 [BurA83a,b]. These pyramid schemes were the basis for Mallat, to develop his algorithm, using wavelets. As a matter of fact, Mallat treated the one- as well as the two-dimensional case, creating the concept of MRA, on this occasion. This, in particular, explains the name of *multiresolution analysis.*

**Quadrature Mirror Filters (QMFs).**

Define $(h(k))_{k\in\mathbf{Z}}$, as in $(II.55)$. By $(II.58)$, $H(\omega) := \sum_{k\in\mathbf{Z}} h(k)e^{-ik\omega}$ satisfies

$$i)\quad |H(\omega)|^2 + |H(\omega)|^2 = 2. \tag{II.73}$$

If $\varphi \in L^1(\mathbf{R})$, integration of $(II.54)$ yields

$$ii)\quad H(0) = \sqrt{2}. \tag{II.74}$$

This corresponds to the definition of the so called *Quadrature Mirror Filters (QMFs)*, introduced by D. Esteban and C. Galand in 1985, to decompose an arbitrary signal in disjoint frequency bands, allowing perfect reconstruction. (See [Dau92,p.156] or [Ve92] for an introduction to QMFs.)

So, every MRA leads to a QMF. Under which conditions, conversely, a QMF leads to a MRA (and therefore to a WONB), is the content of the following theorems.

Theorem 3. (S.Mallat)
*Let* $H(\omega) = \sum_{k\in\mathbf{Z}} h(k)e^{-ik\omega}$ *satisfy* $(II.73), (II.74)$,

$$|h(k)| = O(1+k^2)^{-1} \quad \forall k \in \mathbf{Z}, \tag{II.75}$$

$$and\quad H(\omega) \neq 0 \quad in \quad [-\frac{\pi}{2}, \frac{\pi}{2}]. \tag{II.76}$$

*Define*

$$\hat{\varphi}(\omega) := \frac{1}{\sqrt{2\pi}} \prod_{k\in\mathbf{N}} \frac{1}{\sqrt{2}} H(2^{-k}\omega). \tag{II.77}$$

*Then, $\hat{\varphi}$ is the Fourier transform of a function $\varphi \in L^2(\mathbf{R})$ such that $(\varphi(x-k))_{k\in\mathbf{Z}}$ is the ONB of a closed subspace $V_0$ in $L^2(\mathbf{R})$. If $\varphi \in C^1(\mathbf{R})$ and $\varphi$ satisfies*

$$|\varphi(x)| \leq C_1(1+x^2)^{-1}, \quad |\varphi'(x)| \leq C_2(1+x^2)^{-1},$$

*for some constants $C_1, C_2 > 0$, then, the subspaces $(V_j)_{j\in\mathbf{Z}}$ defined by $f \in V_j \iff f(2^{-j}\cdot) \in V_0$, constitutes a MRA in $L^2(\mathbf{R})$.*

*Proof.* [Mal89b] $\diamondsuit$

THEOREM 4. (I. Daubechies)
*Let $H(\omega) = \sum_{k\in\mathbf{Z}} h(k)e^{-ik\omega}$ satisfy $(II.73), (II.74)$ and*

$$\sum_{k\in\mathbf{Z}} |h(k)||k|^{\epsilon} < \infty, \tag{II.78}$$

*for some $\epsilon > 0$. Assume, there exists a $N \in \mathbf{N}$ such that*

$$H(\omega) = \sqrt{2}[\frac{1}{2}(1+e^{i\omega})]^N[\sum_{k\in\mathbf{Z}} f(k)e^{-ik\omega}], \tag{II.79}$$

*where, for some $\eta > 0$,*

$$\sum_{k\in\mathbf{Z}} |f(k)||k|^{\eta} < \infty, \quad B := sup_{\omega\in\mathbf{R}}|\sum_{k\in\mathbf{Z}} f(k)e^{-ik\omega}| < 2^{N-1}.$$

*Then $\varphi$, defined as in $(II.77)$, is the father function of a MRA, satisfying*

$$|\hat{\varphi}(\omega)| \leq C(1+|\omega|)^{-N+\frac{logB}{log2}}.$$

*The same regularity holds, for the mother wavelet $\psi$, constructed from the MRA.*

*Proof.* [Dau88a].This proof has a graphical motivation. Remember that the coefficients $h(k)$ are the same as in the refinement equation $(II.54)$. Starting from the characteristic function of the unit interval, the iteration of the refinement equation results in the father function. $\diamondsuit$

For the next theorem, we need the following

DEFINITION. A compact set $K \subset \mathbf{R}$ is called *congruent to* $[-\pi, \pi]$ *modulo* $2\pi$, if for almost every $x \in [-\pi, \pi]$ there exists a unique $y \in K$ such that $x - y \in 2\pi\mathbf{Z}$.

THEOREM 5. (A. Cohen)
*Let* $H(\omega) = \sum_{k\in\mathbf{Z}} h(k)e^{-ik\omega}$ *satisfy* $(II.73)$ *and* $(II.74)$. *Define* $\varphi$ *as in* $(II.77)$. *Then the following statements are equivalent.*

i) *$H$ is an element of the Sobolev space $H^m$, for some $m \in \mathbf{N}$, such that* there exists a compact set $K$, congruent to $[-\pi, \pi]$ modulo $2\pi$, with $0 \in K$ and

$$H(\frac{\omega}{2^k}) \neq 0 \quad \forall k \in \mathbf{N},\ \omega \in K.$$

ii) *$\varphi$ is the father function of a MRA, satisfying $\hat{\varphi} \in H^m$. (Then, $\hat{\psi} \in H^m$, as well.)*

*Proof.* [Co90a,b], [Dau92,p.182]. $\diamondsuit$

## II.5. Compactly Supported Orthonormal Wavelets.

For applications, compactly supported orthonormal wavelets are of special interest. By theorem 2 in section II.4 and the following remarks, all such wavelets, which are in addition real-valued and $\epsilon$-Hölder continuous, for some $\epsilon > 0$, stem from a MRA, with a compactly supported father function. Therefore, the definition of the coefficients $h(k), k \in \mathbf{Z}$, characterizing the corresponding QMF (cf. $(II.55)$), implies that only finitely many of them can be different from zero. In other words: $H(\omega) = \sum_{k\in\mathbf{Z}} h(k)e^{-ik\omega}$ is a trigonometric polynomial. So, it is natural, to construct regular, compactly supported orthonormal wavelets, by starting from *finite* sequences $(h(k))_{k\in\mathbf{Z}}$ (resp. trigonometric *polynomials* $H$), which satisfy the conditions of one of the theorems, at the end of the previous section. This was done by I. Daubechies, in 1988, using theorem 4.

Obviously, any finite sequence obeys the appearing decay condition $(II.78)$. The class of trigonometric polynomials, fulfilling $(II.73)$,

$(II.74)$ and $(II.79)$, at the same time, (and therefore leading to a MRA by theorem 4), is described in the following

PROPOSITION. *$H$ is a trigonometric polynomial, satisfying $(II.73)$, $(II.74)$ and $(II.79)$ for some $N \in \mathbf{N}$, iff for the trigonometric polynomial*

$$F(\omega) := \sum_{k \in \mathbf{Z}} f(k) e^{-ik\omega},$$

*appearing in $(II.79)$, holds.*

$$|F(\omega)|^2 = P(sin^2 \frac{\omega}{2}),$$

where

$$P(x) := P_N(x) + x^N R(\frac{1}{2} - x), \tag{II.80}$$

$$P_N(x) := \sum_{k=0}^{N-1} \binom{N-1+k}{k} x^k$$

*and $R$ is an odd polynomial, chosen such that $P(x) \geq 0$ for $x \in [0,1]$.* [11]

*Sketch of Proof.* [Dau92,p.171]. Substituting $(II.79)$ into $(II.73)$, yields the following problem:

find a polynomial $P$ such that

$$(1-x)^N P(x) + x^N P(1-x) = 1.$$

The existence and uniqueness of a solution $P$ is guaranteed by a theorem of Bezout that can be proved constructively, using Euclid's algorithm. $\diamondsuit$

The explicit construction of $F$ (and therefore $H$) now follows, by a lemma of Riesz [Dau92,p.172], which gives a recipe, how to extract the "square root" of the trigonometric polynomial $|F(\omega)|^2$. (More explicitely, Riesz' recipe leads to the factorization of a polynomial, which can be performed numerically.)

11 In the first construction of compactly supported orthonormal wavelets [Dau88a], the choice was $R \equiv 0$.

The graphically motivated proof of theorem 4 in II.4 yields an algorithm (the so called *cascade algorithm*) to sketch the graphs of the wavelets, to be constructed, although no explicit formula for the wavelets is known, in general.

Note that the proposition retains the following freedoms in the construction of compactly supported orthonormal wavelets.

The parameter $N$ can be choosen arbitrary.
$R$ and the "square root" of $F$ are not uniquely determined.

So, there are different classes of compactly supported orthonormal wavelets, possessing different regularity properties (cf. [Daub92, Ch.7]), but, one can always observe an increase of regularity, with increasing support length.

$\psi$ is called *symmetric*, if there exists a $x_0 \in \mathbf{R}$ such that

$$\psi(x_0 + x) = \psi(x_0 - x) \quad \forall x \in \mathbf{R},$$

*antisymmetric*, if there exists a $x_0 \in \mathbf{R}$ such that

$$\psi(x_0 + x) = -\psi(x_0 - x) \quad \forall x \in \mathbf{R}.$$

A common property of all real-valued, compactly supported orthonormal wavelets is the missing of any symmetry or antisymmetry, stated in the following

THEOREM.
*If a real-valued, compactly supported orthonormal wavelet $\psi$ is symmetric or antisymmetric, then $\psi$ is the Haar wavelet.*

*Proof.* [Dau92,p.252]. The proof relies on the fact that, in case of compact support, the various father functions, corresponding to a given MRA, differ only by a translation. $\diamondsuit$

If one allows $\psi$ to obtain even *complex* values, compact support and (anti)symmetry are compatible, for orthonormal wavelets. As well, there exist *bi*orthogonal bases of (anti-)symmetric, real-valued, compactly supported wavelets [Dau92,p.259].

**II.6. Smoothness, Decay and Oscillation.**

DEFINITION. Let $m \in \mathbf{Z}$. A function $f : \mathbf{R} \to \mathbf{C}$ is said to

a) *decay of order* $m$, if there exist constants $C > 0,\ \alpha > m+1$ such that
$$|f(x)| \le C(1+|x|)^{-\alpha} \quad \forall x \in \mathbf{R};$$

b) *be smooth of order* $m$, if $\psi \in C^m(\mathbf{R})$ and $\psi^{(l)}$ is bounded $\forall\ 0 \le l \le m$;

c) *oscillate of order* $m$, if
$$\int_{-\infty}^{\infty} \psi(x)x^l dx = 0 \quad for \quad l = 0, 1, \cdots, m.$$

THEOREM. *(Oscillation theorem). Let $\psi \in L^2(\mathbf{R})$ be an arbitrary orthonormal wavelet. Assume, $\psi$ decays of order $m$ and is smooth of order $m$ $(m \in \mathbf{N})$. Then $\psi$ oscillates of order $m$.*

*Proof.* Cf. [Dau92,p.153] or [Mey90c,p.93], for a different proof.

Induction over $l$ :=number of vanishing moments.

$l = 0$ :

$\psi$ generates an ONB and so especially a dyadic frame with frame bounds $A = B = 1$. So, by $(II.35)$,
$$c_\psi = 2\pi \int_{-\infty}^{\infty} |\hat{\psi}(\omega)|^2 \frac{d\omega}{|\omega|} < \infty.$$

$\psi$ decays at least as $\frac{1}{(1+|x|)^{1+\epsilon}}$ $(\epsilon > 0)$, therefore $\psi \in L^1(\mathbf{R})$, i.e. $\hat{\psi}$ continuous.

Hence $\hat{\psi}(0) = \int_{-\infty}^{\infty} \psi(x)dx = 0$.

$l - 1 \to l$:

$\psi^{(l)}$ is continuous, $\psi^{(l)} \ne 0$, therefore exists an open interval $I$ with $\psi^{(l)} \ne 0$ on $I$. Since the dyadic rationals are dense in $\mathbf{R}$, there exists $J, K \in \mathbf{Z}$ with $2^{-J}K \in I$. Taylor expansion of $\psi$ up to order $l$ around $2^{-J}K$ gives: $\forall \epsilon > 0\ \ \exists\ 1 > \delta > 0$ such that $|x - 2^{-J}k| \le \delta$ implies
$$|\psi(x) - \sum_{n=0}^{l} \frac{1}{n!}\psi^{(n)}(2^{-J}K)(x - 2^{-J}K)^n| \le \epsilon|x - 2^{-J}k|^l. \quad (II.81)$$

Choose $j > 0$, $j > J$. Because of the orthonormality of $\psi_{00}$ and $\psi_{j2^{j-J}K}$ $(j \neq 0)$ we have

$$0 = \int_{-\infty}^{\infty} \psi(x)\overline{\psi(2^j x - 2^{j-J}K)}dx = \quad (by\ (II.81))$$

$$= \sum_{n=0}^{l} \frac{1}{n!}\psi^{(n)}(2^{-J}K)\int_{-\infty}^{\infty}(x - 2^{-J}K)^n\overline{\psi(2^j x - 2^{j-J}K)}dx + R, \tag{II.82}$$

where

$$R := \int_{-\infty}^{\infty}[\psi(x) - \sum_{n=0}^{l}\frac{1}{n!}\psi^{(n)}(2^{-J}K)(x - 2^{-J}K)^n]\overline{\psi(2^j x - 2^{j-J}K)}dx. \tag{II.83}$$

By assumption,

$$\int_{-\infty}^{\infty} x^n\psi(x)dx = 0 \quad \forall\ n < l,$$

so $(II.82)$ reduces to

$$\frac{1}{l!}\psi^{(l)}(2^{-J}K)\int_{-\infty}^{\infty}(x - 2^{-J}K)^l\overline{\psi(2^j x - 2^{j-J}K)}dx =$$

$$= \frac{1}{l!}\psi^{(l)}(2^{-J}K)(2^{-j})^{l+1}\int_{-\infty}^{\infty} u^l\overline{\psi(u)}du. \tag{II.84}$$

The $R$-term can be estimated as follows:

By assumption, $\psi^{(n)}$ is bounded $\forall\ n$ by some constant $\tilde{C}$. Together with $(II.81)$, one gets

$$|R| \leq \int_{|x-2^{-J}K|\leq\delta} \epsilon|x - 2^{-J}K|dx +$$

$$+\tilde{C}\int_{|x-2^{-J}K|>\delta} |(1 - (x - 2^{-J}K))^l||\psi(2^{-j}x - 2^{j-J}K)|dx$$

$$= \int_{|y|\leq\delta} \epsilon|y|^l|\psi(2^j y)|dy + \tilde{C}\int_{|y|>\delta}(1 + |y|)^l|\psi(2^{-j}y)|dy \leq$$

$$\leq 2\epsilon\int_0^{2^j\delta} |2^{-j}u|^l|\psi(u)|2^{-j}du + 2\tilde{C}C\int_\delta^\infty (1 + y)^l(1 + 2^j y)^{-\alpha}dy \leq$$

$$\leq 2\epsilon C(2^{-j})^{l+1}\int_0^\infty \frac{1}{(1+u)^\alpha}du + 2\tilde{C}C\int_\delta^\infty (1+y)^l \frac{1+\delta}{1+2^j\delta}\frac{1}{(1+y)^\alpha}dy,$$

because

$$\frac{1}{1+2^jy} = \frac{1}{2^j}\frac{1}{2^{-j}+y} \leq \frac{1+\delta}{2^j+2^j\delta}\frac{1}{2^{-j}+y} \leq \frac{1+\delta}{1+2^j\delta}\frac{1}{1+y} \quad for \;\; y \geq \delta,$$

so, altogether,

$$(II.83) \leq \hat{C}\epsilon(2^{-j})^{l+1} + \check{C}2^{-j\alpha}\frac{1+\delta}{\delta^\alpha(1+\delta)^{l-\alpha+1}},$$

where the constants only depend on $C, \alpha$ and $l$, not on $\epsilon, \delta$ and $j$. By $(II.82)$ and $(II.84)$, it follows that

$$|\int_{-\infty}^\infty u^l\psi(u)du| = \frac{1}{|\frac{1}{l!}\psi^{(l)}(2^{-J}K)(2^{-j})^{l+1}|}|R| \leq$$

$$\leq l!(\psi^{(l)}(2^{-J}K))^{-1}\epsilon\hat{C} + 2^{-j(\alpha-l-1)}C\delta^{-\alpha}(1+\delta)^{l+1}.$$

Because $\epsilon$ and $j$ were arbitrary, the right hand side term converges to 0. $\diamondsuit$

Definition. $\psi$ is said to be of *exponential decay*, if there exists a constant $\gamma > 0$ such that $e^{\gamma|x|}\psi(x)$ is bounded.

Corollary. *(Smoothness versus decay)*
*If $\psi$ is a dyadic orthonormal wavelet, the following properties exclude each other.*

i) *$\psi$ is smooth of order $m \quad \forall\, m \in \mathbf{N}$.*

ii) *$\psi$ is of exponential decay.*

*Proof.* Assume, there exists a dyadic, orthonormal wavelet $\psi \in L^2(\mathbf{R})$, wich satisfies both i) and ii). Exponential decay implies (polynomial) decay of order $m \quad \forall\, m \in \mathbf{N}$, so

$$\int_{-\infty}^\infty x^l\psi(x)dx = 0 \quad \forall\, l \in \mathbf{N},$$

and therefore

$$\frac{d^l}{d\omega^l}\hat{\psi}|_0 = 0 \quad \forall\, l \in \mathbf{N}.$$

On the other hand, the exponential decay of $\psi$ implies (by the Paley-Wiener theorem) that $\hat{\psi}$ is analytic in some strip in $\mathbf{C}$. So it is a well known consequence from complex analysis that $\hat{\psi} \equiv 0$, since all its derivatives (and therefore its power series) vanish at one point. So $\hat{\psi} \equiv 0$, and this cannot be a dyadic orthonormal wavelet. $\diamondsuit$

Compare with the examples in II.3.

**Uncertainty principle of Battle** [Bat89].

*Let $\psi \in L^2(\mathbf{R})$ be a dyadic orthonormal wavelet. Then, the following properties exclude each other.*

i) *$\psi$ is of exponential decay.*

ii) *$\hat{\psi}$ is of exponential decay.*

*Proof.* $\hat{\psi}$ is of exponential decay implies, by the Paley-Wiener theorem, that $\psi \in C^\infty(\mathbf{R})$, *supp* $\psi$ is compact. Hence, all derivatives are bounded, i.e. $\psi$ is smooth of order $m$. This is in contradiction to the oscillation theorem. $\diamondsuit$

Giving up orthonormality and just requiring "$\psi$ *frame* wavelet", allows exponential decay of $\psi$ *and* $\hat{\psi}$. The Mexican hat wavelet (cf.I.2.1) is an example, for this [Dau92,p.75].

## III. Group Theoretical Abstractions.

### III.0. Guide to the following chapter.

The *affine group* $(G_{aff}, \circ)$, defined by

$$G_{aff} := \mathbf{R}^* \times \mathbf{R}, \quad and \quad (a,b)\circ(a',b') = (a\cdot a', b + a\cdot b'), \quad (III.1)$$

can be identified with the group of affine transformations on the real line via the isomorphism

$$I: \ (a,b) \mapsto w_{ab}, \quad where \quad w_{ab}: \ \mathbf{R} \to \mathbf{R}, \quad x \mapsto ax+b. \quad (III.2)$$

If $G_{aff}$ is endowed with the product topology of $\mathbf{R}^* \times \mathbf{R}$, $(G_{aff}, \circ)$ becomes a locally compact topological group. For $(a,b) \in G_{aff}$ arbitrary, define the following unitary linear operator

$$U(a,b): \ L^2(\mathbf{R}) \to L^2(\mathbf{R})$$

$$\psi(x) \mapsto U(a,b)\psi(x) := \frac{\|\psi\|}{\|\psi(w_{ab}^{-1}(\cdot))\|}\psi(w_{ab}^{-1}(\cdot)) =$$

$$= \frac{1}{\sqrt{|a|}}\psi(\frac{x-b}{a}) = \psi_{ab}(x). \qquad (III.3)$$

The mapping

$$U: \ (a,b) \mapsto U(a,b),$$

from $G_{aff}$ to the set of unitary operators on $L^2(\mathbf{R})$, is called the *canonical representation of $G_{aff}$ on $L^2(\mathbf{R})$*. It turns out that for $\psi, \ f \in L^2(\mathbf{R})$ holds:

$$T_\psi f(a,b) = (f, U(a,b)\psi),$$

and it will be shown in III.2 that

$$c_\psi = \int_{G_{aff}} |(\psi, U(a,b)\psi)|^2 d\mu_L(a,b),$$

where $d\mu_L(a,b) := \frac{da\,db}{a^2}$ denotes a *left Haar measure* on $G_{aff}$ and $c_\psi$ is defined as in $(I.19)$. We know, from the orthogonality relation $(I.21)$, that for $c_\psi < \infty$ holds:

$$\forall f,g \in L^2(\mathbf{R}): \quad ((f, U(a,b)\psi), (g, U(a,b)\psi))_{L^2(G_{aff}, d\mu_L)} = c_\psi (f,g). \qquad (III.4)$$

In the following chapter, we will consider an *abstraction* of this situation to an arbitrary locally compact group $G$, instead of $G_{aff}$, and an arbitrary Hilbert space $\mathcal{H}$, instead of $L^2(\mathbf{R})$. In particular, we will see that $(III.4)$ holds, in a more general setting (cf.III.3).

## III.1. Preliminaries on Locally Compact Groups (LCGs).

In the following, $(G, \circ)$ denotes a *locally compact group (LCG)*, i.e.

i) $G$ is a group, with multiplication $\circ$, and

ii) $G$ is a locally compact topological space such that the mapping

$$\begin{aligned} P:\ G \times G &\to G \\ (g,h) &\mapsto g \circ h^{-1} \end{aligned}$$

is continuous.

For $g \in G$, the *left* (resp. *right*) *translation* $L_g$ (resp. $R_g$) is defined by

$$L_g:\ G \to G \qquad h \mapsto g \circ h,$$

resp.

$$R_g:\ G \to G \qquad h \mapsto h \circ g.$$

$\mu_L$ (resp. $\mu_R$) denotes a *left* (resp. *right*) *Haar measure* on $G$, i.e. $\mu_L$ (resp. $\mu_R$) is a Borel measure on $G$, and for all $g \in G$ and all Borel sets $M$ of $G$ holds:

$$\mu_L(L_g M) = \mu_L(M) \qquad (resp.\ \ \mu_R(R_g M) = \mu_R(M)),$$

where

$$L_g M := \{h \in G :\ \exists m \in M \quad h = L_g m\}$$

(resp. $R_g M := \{h \in G :\ \exists m \in M \quad h = R_g M\}$).

The existence of left and right Haar measures, for arbitrary locally compact groups $G$, was proved by A. Haar, J. von Neumann and A. Weil[12]. They showed, furthermore, that $\mu_L$ (resp. $\mu_R$) is *uniquely* determined, up to a constant factor. If $\mu$ is a left or right Haar measure with $\mu(G) < \infty$, one usually chooses this constant such that $\mu_L(G) = 1$.

In case that $G \subseteq \mathbf{R}^n$ and $L_g$ is a $C^1$-diffeomorphism with Jacobian $DL_g(h)$, independent of $h$, there exists a simple *construction method* for the corresponding left Haar measure:

$$d\mu_l(g) = const. \frac{1}{|det DL_g|} d^n g, \qquad (III.5)$$

12 See e.g E. Hewitt, K. Ross, *Abstract Harmonic Analysis I*, Berlin (1963), Ch. IV.

where $d^n g$ denotes the n-dimensional Lebesgue measure. Analogous conditions on $R_g$, result in

$$d\mu_R(g) = const.\frac{1}{|detDR_g|}d^n g. \qquad (III.6)$$

$G$ is said to be *unimodular*, if every left Haar measure on $G$ is a right Haar measure on $G$ as well, and viceversa.

Evidently, every abelian group is unimodular.

Examples.

a) $(G, \circ) = (\mathbf{R}^n, +)$ is unimodular with

$$d\mu_L(g) = d\mu_R(g) = d^n g,$$

since the Lebesgue measure is known to be translation invariant. (The construction, suggested above, leads to Lebesgue measure on $\mathbf{R}^n$, too.)

b) *Affine groups* $G_{aff}$, $G_{aff}^{+}$.

$G_{aff}$, with the product topology on $\mathbf{R}^* \times \mathbf{R}$.

$G_{aff}^{+}$ : topological subgroup of $G_{aff}$, consisting of $(a, b) \in G_{aff}$ with $a > 0$.

Since

$$DL_{(a,b)}(h) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$$

and

$$DR_{(a,b)}(h) = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

are independent of $h$, one can apply $(III.5)$ (resp. $(III.6)$) to construct the Haar measures, for these two groups:

$$d\mu_L(a, b) = const.\frac{1}{a^2}dadb, \qquad (III.7)$$

$$d\mu_R(a, b) = const.\frac{1}{a}dadb, \qquad (III.8)$$

where, in both cases, the constant is conventionally chosen equally to one. This shows that the affine groups $G_{aff}$ and $G^+_{aff}$ are *not* unimodular.

### III.2. Unitary Representations of LCGs.

**General assumption.**

$(G, \circ)$ locally compact group, with left Haar measure $d\mu_L$ and right Haar measure $d\mu_R$.
$\mathcal{H}$ complex Hilbert space, with scalar product $(\cdot,\cdot)_{\mathcal{H}}$, norm $\|\cdot\|_{\mathcal{H}}$.
$\mathcal{U}(\mathcal{H})$ set of unitary operators on $\mathcal{H}$.

DEFINITION. A *unitary representation* of $(G, \circ)$ on $\mathcal{H}$ is a homomorphism

$$U: \ G \to \mathcal{U}(\mathcal{H})$$
$$g \mapsto U(g).$$

$U$ is called *irreducible*, if $\forall f \in \mathcal{H}$

$$\overline{span}\{U(g)f : \ g \in G\} = \mathcal{H}.$$

An element $\psi \in \mathcal{H} \setminus \{0\}$ is called $U$-*admissible*, if

$$c_\psi^U := \int_G |(\psi, U(g)\psi)_{\mathcal{H}}|^2 d\mu_L(g) < \infty. \qquad (III.9)$$

(Then, $\int_G |(\psi, U(g)\psi)_{\mathcal{H}}|^2 d\mu_R(g)$ is finite, too. Cf. [Hol93b].)

$U$ is a *square integrable* unitary representation, if there exists a $U$-admissible $\psi \in \mathcal{H} \setminus \{0\}$.

Note that there exist *non*-square integrable irreducible unitary representations, e.g.

$$U: \ (G, \circ) := (\mathbf{R}, +) \to \mathcal{U}(\mathbf{T}), \quad g \mapsto U(g),$$

where

$$U(g): \ \mathbf{T} \to \mathbf{T}, \quad t \mapsto e^{ig}t.$$

In this case, $\int_G |(t, U(g)t)_{\mathcal{H}}|^2 d\mu_L(g) = \int_{-\infty}^{\infty} dg$.

If $G$ is a group of bijective mappings on a topological space $X$ with regular Borel measure $\lambda$ such that the neutral element $e$ of $G$ corresponds to the identity $id_x$ on $X$, one can construct an irreducible unitary representation of $G$ as follows.

Define $\mathcal{H} := L^2(X, \lambda)$,

$$U: \ G \to \mathcal{U}(\mathcal{H}), \quad g \mapsto U(g),$$

where

$$U(g): \ \mathcal{H} \to \mathcal{H},$$

$$f \mapsto U(g)f(\cdot) := \begin{cases} 0, & \|f\|_{\mathcal{H}} = 0 \\ \frac{\|f\|_{\mathcal{H}}}{\|f(L_{g^{-1}}\cdot)\|_{\mathcal{H}}} f(L_{g^{-1}}\cdot), & otherwise. \end{cases} \qquad (III.10)$$

For $\psi \in \mathcal{H} \setminus \{0\}$ arbitrary, define

$$\mathcal{H}_\psi := \overline{span}\{U(g)\psi: \ g \in G\} \subseteq \mathcal{H}.$$

Then, $U$ is an irreducible unitary representation of $G$ on $\mathcal{H}_\psi$, as a straightforward calculation shows.

If one chooses $\psi \in \mathcal{H} \setminus \{0\}$ such that

$$\int_G |(U(g)\psi, \psi)_{\mathcal{H}}|^2 d\mu_L(g) < \infty,$$

then $U$ is a square integrable representation of $G$ on $\mathcal{H}_\psi$.

Examples.

a) Every locally compact group $G$ can be regarded as a group of bijective mappings on itself, via the action of the left and right translation. In this case, the foregoing construction leads to the so called *left* and (resp. *right*) *regular representation* $\lambda$ (resp. $\rho$):

$$\lambda: \ G \to \mathcal{U}(L^2(G, d\mu_L)), \quad g \mapsto \lambda(g),$$

where

$$\lambda(g): \ L^2(G, d\mu_L) \to L^2(G, d\mu_R)$$

$$f \mapsto \begin{cases} 0, & \|f\|_{\mathcal{H}} = 0 \\ f(g^{-1} \circ \cdot), & otherwise, \end{cases} \qquad (III.11)$$

respectively,

$$\rho : \ G \to \mathcal{U}(L^2(G, d\mu_R)), \quad g \mapsto \rho(g),$$

where

$$\rho(g) : \ L^2(G, d\mu_R) \to L^2(G, d\mu_R)$$

$$f \mapsto \begin{cases} 0, & \|f\|_{\mathcal{H}} = 0 \\ f(\cdot \circ g), & otherwise. \end{cases} \qquad (III.12)$$

These representations are clearly irreducible as well as square integrable.

b) By $(III.2), G_{aff}$ can be identified with the group of affine transformations on the real line:

$$G_{aff} \simeq \{w_{ab} : \mathbf{R} \to \mathbf{R}, \ \ x \mapsto ax + b, \ where \ a \in \mathbf{R}^*, \ b \in \mathbf{R}\}.$$

Since, for arbitrary $f \in L^2(\mathbf{R})$, by $(III.10)$ holds

$$\|f(w_{ab}^{-1} \cdot)\| = \int_{-\infty}^{\infty} |f(\frac{x-b}{a})|^2 dx = |a| \|f\|^2,$$

one gets, for every $\psi \in L^2(\mathbf{R})$ with $\|\psi\| \neq 0$, that:

$$U : \ G_{aff} \to \mathcal{U}(L^2(\mathbf{R})), \ \ (a, b) \mapsto U(a, b),$$

where

$$U(a, b) : \ L^2(\mathbf{R}) \to L^2(\mathbf{R})$$

$$f \mapsto \begin{cases} 0, & \|f\| = 0 \\ \frac{1}{\sqrt{|a|}} f(\frac{x-b}{a}), & otherwise, \end{cases} \qquad (III.13)$$

is an irreducible, unitary representation of $G_{aff}$ on

$$\mathcal{H}_\psi := \overline{span}\{\frac{1}{\sqrt{|a|}} \psi(\frac{x-b}{a}) : \ a \in \mathbf{R}^*, \ b \in \mathbf{R}\}. \qquad (III.14)$$

ASSERTION. $\forall \psi \in L^2(\mathbf{R}),\ \|\psi\| > 0$ :

$$\mathcal{H}_\psi \equiv L^2(\mathbf{R}). \tag{$III.15$}$$

This can be seen as follows.

Assume, there exists a $\psi \in L^2(\mathbf{R})$ such that $\mathcal{H}_\psi$ is a closed subspace of $L^2(\mathbf{R})$, but $\mathcal{H}_\psi \neq L^2(\mathbf{R})$. Then, there exists a $f \neq 0$ in $\mathcal{H}_\psi^\perp$, i.e. $(f, \psi_{ab}) = 0 \ \ \forall (a,b) \in \mathbf{R}^* \times \mathbf{R}$. Recall that $(f, \psi_{ab}) = T_\psi f(a,b)$ as in $(I.2)$, and $T_\psi$ is injective, by the theorem in I.0. $\diamondsuit$

The function $\psi \in L^2(\mathbf{R})$ is U-admissible, if

$$\infty > \int_G |(\psi, U(a,b)\psi)|^2 d\mu_L(a,b) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |(\psi, \psi_{ab})|^2 \frac{dadb}{a^2} =$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Big(\int_{-\infty}^{\infty} \frac{a}{\sqrt{|a|}} \hat{\psi}(a\omega) e^{-ib\omega} \overline{\hat{\psi}(\omega)} d\omega\Big) \cdot$$

$$\cdot \Big(\int_{-\infty}^{\infty} \frac{a}{\sqrt{|a|}} \overline{\hat{\psi}(a\tilde{\omega})} e^{ib\tilde{\omega}} \hat{\psi}(\tilde{\omega}) d\tilde{\omega}\Big) \frac{dbda}{a^2} =$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\hat{\psi}(au)|^2 |\hat{\psi}(u)|^2 du \frac{da}{|a|} = \|\psi\|^2 \int_{-\infty}^{\infty} |\hat{\psi}(v)|^2 \frac{dv}{|v|},$$

i.e.

$$c_\psi^U := \int_{-\infty}^{\infty} |\hat{\psi}(v)|^2 \frac{dv}{|v|} < \infty.$$

Note that $c_\psi^U = c_\psi$, for $c_\psi$ defined in $(I.18)$. In other words: the U-admissible elements of $L^2(\mathbf{R})$ are just the admissible analyzing wavelets for CWT. Since we know that admissible analyzing wavelets exist (cf. the examples in I.2.1), $U$ is a square integrable representation of $G_{aff}$.

c) $G_{aff}^+$ can be identified with affine transformations on the real line as well, where, this time, $a$ is restricted to $\mathbf{R}^+$. By the same arguments as in b), we get that $U$, as defined in $(III.13)$, is a unitary representation of $G_{aff}^+$ on $\mathcal{H}_\psi$, given by $(III.14)$. But, in contrast to $(III.15)$, in this case, $\mathcal{H}_\psi \not\equiv L^2(\mathbf{R})$ (cf. I.2.5).

However, the restriction of the operators $U(a,b)$ to $H^2(\mathbf{R})$, as defined in I.2.6, yields an irreducible unitary representation of $G^+_{aff}$ on $H^2(\mathbf{R})$. Since, for $\psi \in H^2(\mathbf{R})$, the Fourier transform $\hat{\psi}$ vanishes for $\omega < 0$, the $U$-admissibility of $\psi$, $c^U_\psi < \infty$, now takes the form

$$\int_0^\infty |\hat{\psi}(\omega)|^2 \frac{d\omega}{\omega} < \infty,$$

so $c^U_\psi = c_\psi$ holds. Since there exist admissible analyzing wavelets in $H^2(\mathbf{R})$ (cf. Paul wavelets, I.2.1.ii)), the restriction of $U$, to $a > 0$, is a square integrable representation of $G^+_{aff}$ on $H^2(\mathbf{R})$.

## III.3. The Orthogonality Relation for Square Integrable Representations of LCGs.

The main result of this section is the following

THEOREM. *(Orthogonality relation)*

*Let $G$ be a locally compact group with left Haar measure $d\mu_L$, $\mathcal{H}$ a complex Hilbert space and $U$ a square integrable, irreducible, unitary representation of $G$ on $\mathcal{H}$. Define*

$$\mathcal{A}_U := \{\psi \in \mathcal{H} : \ \psi \ \ is \ \ U-admissible\}.$$

*Then, $\mathcal{A}_U$ is dense in $\mathcal{H}$, and there exists a unique positive operator $C_U : \mathcal{A}_U \to \mathcal{H}$ such that $\forall \psi, \Psi \in \mathcal{A}_U, \quad \forall f_1, f_2 \in \mathcal{H}$*

$$\int_G (f_1, U(g)\psi)_{\mathcal{H}} \overline{(f_2, U(g)\Psi)_{\mathcal{H}}} d\mu_L(g) = (C_U\Psi, C_U\psi)_{\mathcal{H}} (f_1, f_2)_{\mathcal{H}}. \qquad (III.16)$$

*If $G$ is unimodular, $C_U$ is a multiple of the identity.*

To prove $(III.16)$, several variants of the *lemma of Schur* will be needed. We will prove them, in advance.

**The classical lemma of Schur**. *(Characterization of irreducibility)*

*Let $G$ be a locally compact group, $\mathcal{H}$ s complex Hilbert space, and $U$ a unitary representation of $G$ on $\mathcal{H}$. Then $U$ is irreducible iff the only bounded linear operators $A$ on $\mathcal{H}$ with*

$$U(g)A = AU(g) \quad \forall g \in G \tag{III.17}$$

*are of the form*

$$A = \lambda \cdot Id_{\mathcal{H}} \quad (\lambda \in \mathbf{R}). \tag{III.18}$$

*Proof.* [13]
"$\Longleftarrow$"
Assume $A = \lambda \cdot Id_{\mathcal{H}}$ is the only solution of $(III.17)$. Let $\mathcal{M}$ be a closed subspace of $\mathcal{H}$ which is invariant under $U$, i.e. $\forall g \in G,\ f \in \mathcal{M}:\ U(g)f \in \mathcal{M}$. Then, $\mathcal{M}^{\perp}$ is invariant under $U$, as well. Because $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}$, for all $f \in \mathcal{H}$, there exists a unique decomposition $f = f_1 + f_2$, where $f_1 \in \mathcal{M}$, $f_2 \in \mathcal{M}^{\perp}$. Define the projection operator

$$P:\ \mathcal{H} \to \mathcal{M} \quad f \mapsto f_1.$$

Then, $P$ is a bounded operator on $\mathcal{H}$, and

$$U(g)P = PU(g) \quad \forall g \in G.$$

Hence, $P$ is a solution of $(III.17)$, and by assumption, there exists a $\lambda \in \mathbf{R}$ with $P = \lambda \cdot Id_{\mathcal{H}}$. We must distinguish two cases. First, if $\lambda = 0$, it follows that $\mathcal{M} \equiv \{0\}$, i.e. $\mathcal{M}^{\perp} = \mathcal{H}$. Second, if $\lambda \neq 0$, it follows that $\lambda = 1$, hence $P \equiv Id_{\mathcal{H}}$, and so $\mathcal{M} = \mathcal{H}$. In any case, we see that there cannot be an invariant, proper subspace of $\mathcal{H}$. Hence, $U$ is an irreducible representation.
"$\Longrightarrow$"
Let $U$ be an irreducible representation of $G$ on $\mathcal{H}$. Let $A$ be a solution of $(III.17)$. Then,

$$U(g)A^* = (A(U(g))^*)^* = (AU(g^{-1}))^* = (U(g^{-1})A)^* = A^*U(g),$$

13 Cf. A. Wawrzyńczyk, *Group representations and special functions*, Warsaw (1984), p.143.

hence $A^*$ is a solution of $(III.17)$, too. Define $B := A + A^*$, $C := i(A - A^*)$. Then, $B$ and $C$ are solutions of $(III.17)$, selfadjoint, and the corresponding spectral families consist of orthogonal projectors which are solutions of $(III.17)$. Now, the image of a projection operator which commutes with $U(g)$ $\forall g \in G$ is a $U(g)$-invariant subspace. Because $U$ is assumed to be an irreducible representation of $G$ on $\mathcal{H}$, this image must be all of $\mathcal{H}$. Hence, the only possible projection is the identity projection. Therefore

$$B = \mu \cdot Id_{\mathcal{H}}, \quad C = \lambda \cdot Id_{\mathcal{H}},$$

for suited real $\mu$, $\lambda$ and consequently $A = \nu \cdot Id_{\mathcal{H}}$ with $\nu \in \mathbf{R}$. $\diamondsuit$

Corollary.
*Let $G$ be a locally compact group, $\mathcal{H}$, $\mathcal{H}'$ complex Hilbert spaces, $U$ an irreducible unitary representation of $G$ on $\mathcal{H}$, $U'$ a unitary representation of $G$ on $\mathcal{H}'$. If $A$ is a bounded linear operator from $\mathcal{H}$ to $\mathcal{H}'$ with*

$$AU(g) = U'(g)A \quad \forall g \in G, \tag{III.19}$$

*then, $A$ is a multiple of an isometry.*

*Proof.* For every bounded linear operator $A$ holds:

$$(III.19) \iff (U'(g)A)^*AU(g) = (U'(g)A)^*U'(g)A$$

$$\iff (AU(g))^*AU(g) = A^*U'^{-1}(g)U'(g)A$$

$$\iff U^{-1}(g)A^*AU(g) = A^*A \iff A^*AU(g) = U(g)A^*A$$

$$\implies A^*A \ \textit{is a solution of} \ (III.17) \iff A^*A = \lambda \cdot Id_{\mathcal{H}}.$$

The last equivalence holds by Schur's lemma. Hence:

$$(III.19) \implies (Af, Ag)_{\mathcal{H}} = (f, A^*Ag)_{\mathcal{H}} = \lambda(f,g)_{\mathcal{H}}. \qquad \diamondsuit$$

The generalized lemma of Schur. *Let $G$, $\mathcal{H}$, $\mathcal{H}'$, $U$, $U'$ be as in the corollary. Let $\mathcal{D}$ be a dense $U$-invariant subset of $\mathcal{H}$ (i.e. $U\mathcal{D} \subseteq \mathcal{D}$, $\overline{\mathcal{D}} = \mathcal{H}$). If $A$ is a closed linear operator from $\mathcal{D}$*

*to $\mathcal{H}'$ satisfying $(III.19)$, then, $A$ is a multiple of an isometry, and $\mathcal{D} \equiv \mathcal{H}$.*

*Proof* [GrosMP85].
Step 1:
For $f, g \in \mathcal{D}$ define:

$$(f,g)_{\mathcal{D}} := (f,g)_{\mathcal{H}} + (Af, Ag)_{\mathcal{H}'}.$$

Then, $\mathcal{D}$ is a Hilbert space with scalar product $(\cdot,\cdot)_{\mathcal{D}}$, because $(\cdot,\cdot)_{\mathcal{H}}$ and $(\cdot,\cdot)_{\mathcal{H}'}$ are scalar products, and $\mathcal{D}$ is complete with respect to $\|\cdot\|_{\mathcal{D}}$, because of the closedness of $\mathcal{A}$.
Step 2:
$A: \ (\mathcal{D}, (\cdot,\cdot)_{\mathcal{D}}) \to \mathcal{H}'$ is bounded, because

$$\frac{\|Af\|^2_{\mathcal{H}'}}{\|f\|^2_{\mathcal{D}}} = \frac{\|Af\|^2_{\mathcal{H}'}}{\|Af\|^2_{\mathcal{H}'} + \|f\|^2_{\mathcal{H}}} \leq 1 \quad \forall f \in \mathcal{D}.$$

Step 3:

$$U(g): \ (\mathcal{D}, (\cdot,\cdot)_{\mathcal{H}}) \to (\mathcal{D}, (\cdot,\cdot)_{\mathcal{H}})$$

is unitary $\forall g \in G$, since

$$\|U(g)f\|^2_{\mathcal{D}} = \|U(g)f\|^2_{\mathcal{H}} + \|AU(g)\|^2_{\mathcal{H}'} =$$

$$= \|f\|^2_{\mathcal{H}} + \|U'(g)Af\|^2_{\mathcal{H}'} = \|f\|^2_{\mathcal{H}} + \|Af\|^2_{\mathcal{H}'} = \|f\|^2_{\mathcal{D}}.$$

Step 4:
$U(g)|_{\mathcal{D}}$ is surjective $\forall g \in G$, since

$$f = U(g)U(g^{-1})f \quad \forall f \in \mathcal{D},$$

and $\mathcal{D}$ is invariant under $U(g^{-1})$. Therefore, the previous corollary can be applied, if one replaces $(\mathcal{H}, (\cdot,\cdot))_{\mathcal{H}}$ by $(\mathcal{D}, (\cdot,\cdot)_{\mathcal{D}})$. It follows that $A$ is a multiple of an isometry from $\mathcal{D}$ to $\mathcal{H}$, i.e.

$$\forall f \in \mathcal{D}: \quad \|Af\|^2_{\mathcal{H}'} = \lambda \|f\|^2_{\mathcal{D}} = \lambda \|f\|^2_{\mathcal{H}} + \lambda \|Af\|^2_{\mathcal{H}'} \quad (\lambda \in \mathbf{R}).$$

Because $f \in D$ was arbitrary, we can assume $f \neq 0$ and $\lambda \neq 1$, hence,

$$\|Af\|^2_{\mathcal{H}'} = \frac{\lambda}{1-\lambda} \|f\|^2_{\mathcal{H}}.$$

So, $A$ is a multiple of an isometry from $\mathcal{D} \subset \mathcal{H}$ (now with $\mathcal{H}$-norm) to $\mathcal{H}'$. Since $\mathcal{D}$ is dense in $\mathcal{H}$, and $A$ is continuous, $A$ extends to a multiple of an isometry from $\mathcal{H}$ to $\mathcal{H}'$. Because $A$ is closed, by assumption, it follows that $\mathcal{D} = \mathcal{H}$. $\diamondsuit$

SKETCH OF THE PROOF OF THE ORTHOGONALITY RELATION. [GrosMP85]

Assertion 1:
Let $\psi \in \mathcal{A}_U$. Then,

$$\begin{aligned} T_\psi^U :\ \mathcal{H} &\to L^2(G, d\mu_L) \\ f &\mapsto (U(g)\psi, f)_{\mathcal{H}} \end{aligned} \qquad (III.20)$$

is a multiple of an isometry.

The proof of assertion 1 follows by the generalized lemma of Schur with

$$\mathcal{H}' = L^2(G, d\mu_L),\ U' = \lambda,\ A = T_\psi^U,$$
$$\mathcal{D} = \mathcal{D}_\psi := \{f \in \mathcal{H} :\ \int_G |(T_\psi^U f)(g)|^2 d\mu_L(g) < \infty\}.$$

By this, $(III.16)$ holds for $\psi = \Psi,\ f_1 = f_2$.

Assertion 2:
Let $\psi, \Psi_2 \in \mathcal{A}_U,\ f_1, f_2 \in \mathcal{H},\ T_\psi^U, T_\Psi^U$ as in assertion 1. Then, there exists a constant $c_{\psi\Psi} > 0$ such that

$$(T_\psi^U f_1, T_\Psi^U f_2)_{L^2(G,d\mu_L)} = c_{\psi\Psi}(f_1, f_2)_{\mathcal{H}}. \qquad (III.21)$$

The proof of assertion 2 follows by the classical lemma of Schur with $A = T_\psi^{U*} T_\Psi^U$.

Assertion 3:
$\exists_1\ C_U :\ \mathcal{A}_U \to \mathcal{H}$ positive such that

$$(C_U\Psi, C_U\psi)_{\mathcal{H}} = c_{\psi\Psi}. \qquad (II.22)$$

To prove of assertion 3, consider the quadratic form

$$q : \mathcal{A}_u \times \mathcal{A}_U \to \mathbf{C} \quad (\psi, \Psi) \mapsto c_{\psi_1\psi_2}.$$

$q$ is closed, symmetric and positive, so it satisfies the assumptions of a variant of Riesz' representation theorem[14], which guarantees the existence of a unique positive operator such that $(II.22)$ holds.

If $G$ is unimodular, $q(U(g)\psi, U(g)\Psi)_{\mathcal{H}} = q(\psi, \Psi)$, hence $C_U U(g) = U(g) C_U$. Applying the generalized Schur's lemma, once more, yields the assertion. $\diamondsuit$

COROLLARY 1. $\forall \psi \in \mathcal{A}_U, \quad \forall f \in \mathcal{H}$

$$\int_G |(f, U(g)\psi)_{\mathcal{H}}|^2 d\mu_L(g) = \|C_U \psi\|^2_{\mathcal{H}} \cdot \|f\|^2_{\mathcal{H}},$$

*with $C_U$ as in the orthogonality relation.*

*Proof.* Choose $\Psi = \psi, \ f_1 = f_2$ in $(III.16)$.

COROLLARY 2. $\forall \psi \in \mathcal{A}_U, \quad \forall f_1, f_2 \in \mathcal{H}$ *and* $T^U_\psi$, *as defined in* $(III.20)$:

$$(f_1, f_2)_{\mathcal{H}} = 0 \iff (T^U_\psi f_1, T^U_\psi f_2)_{L^2(G,d\mu)} = 0.$$

*Proof.*
"$\Longrightarrow$" $(III.16)$ with $\Psi = \psi$.
"$\Longleftarrow$" $(III.16)$, noting that $C_U$ is positive, hence $(C_U\Psi, C_U\psi)_{\mathcal{H}} \neq 0$ for $\psi \neq 0$. $\diamondsuit$

COROLLARY 3.
*The range of $T^U_\psi$, as defined in $(III.20)$, is a reproducing kernel Hilbert space.*

*Proof.* As in I.2.4, one can show that the reproducing kernel is given by

$$K^U_\psi(g, g') := \frac{1}{(C_U\psi, C_U\psi)_{\mathcal{H}}} (T^U_\psi U(g')\psi)(g).$$

$\diamondsuit$

14 Cf. T. Kato, *Perturbation Theory for Linear Operators*, Berlin (1976).

COROLLARY 4. *For a square integrable, irreducible, unitary representation $U$ of $G$ on $\mathcal{H}$, the $U$-admissible elements are dense in $\mathcal{H}$. If, in addition, $G$ is unimodular, then, all elements of $\mathcal{H}$ are $U$-admissible.*

(Compare with theorem 1 in I.2.1, for the affine case.)

## Appendix A. The Windowed Fourier Transform

Another transform, sharing many properties with the wavelet transform, has been introduced by D. Gabor, in 1946: the so called *windowed Fourier* or *Gabor* or *short time Fourier transform.* The following pages shall serve for a comparison between both transform methods.

### A.1. The Continuous Windowed Fourier Transform (CWFT)

Fix $g \in L^2(\mathbf{R}) \setminus \{0\}$. $g$ is called a *window function.*
For $(\omega, t) \in \mathbf{R} \times \mathbf{R}$, define

$$g_{\omega t}(x) := e^{i\omega x} g(x-t). \qquad (A.1)$$

The *continuous windowed Fourier transform (CWFT)* of a function $f \in L^2(\mathbf{R})$, with respect to the window function $g$, is given by the following function

$$W_g f : \mathbf{R} \times \mathbf{R} \to \mathbf{C}$$

$$(\omega, t) \mapsto W_g f(\omega, t) := \int_{-\infty}^{\infty} f(x) \overline{g_{\omega t}(x)} dx.$$

For a fixed value $(\omega, t) \in \mathbf{R} \times \mathbf{R}$, the complex number $W_g f(\omega, t)$ is called the *windowed Fourier coefficient* of $f$, with respect to the window function $g$, at the point $(\omega, t)$. The *continuous windowed Fourier transform operator*, with respect to the analyzing wavelet $g$, is given by the following integral operator

$$W_g : \ L^2(\mathbf{R}) \to \mathbf{C}^{\mathbf{R} \times \mathbf{R}}, \qquad f \mapsto W_g f.$$

**Interpretation**:

$$W_g f(\omega,t) = \int_{-\infty}^{\infty} f(x)\overline{g(x-t)}e^{-i\omega x}dx = \sqrt{2\pi}\,(f(\cdot)\overline{g(\cdot - t)})\widehat{\ }.$$

So, up to a factor $\sqrt{2\pi}$, $W_g f$ corresponds to the Fourier transform of the function $f(\cdot)\overline{g(\cdot - t)}$. Choose for example

$$g(x) := \mathbf{1}_{[-\frac{1}{2},\frac{1}{2}]}.$$

Then, the windowed Fourier transform coefficient $W_g f(\omega,t)$ describes the frequency content of $f$, restricted to the time-window $[t-\frac{1}{2},\frac{1}{2}]$, $t \in \mathbf{R}$. Hence, the name "windowed Fourier transform". In praxis, one usually uses smooth window functions, for example Gaussian functions $(I.17)$, as it was done, first, by D. Gabor.

**Fourier representation**.

$$W_g f(\omega,t) = e^{-it\omega}\int_{-\infty}^{\infty} \hat{f}(\xi)e^{it\xi}\overline{\hat{g}(\xi-\omega)}d\xi = e^{-it\omega}W_g\hat{f}(\omega,-t).$$

**Properties**.

*$W_g$ is an injective, bounded, linear operator from $L^2(\mathbf{R})$ to $L^\infty$ $(\mathbf{R}\times\mathbf{R})$, possessing the following invariance properties.*

i) $[W_g f(\cdot - x_0)](\omega,t) = e^{-i\omega x_0}W_g f(\omega, t-x_0)$ *(translation invariance)*;

ii) $[W_g f(e^{i\omega_0 \cdot} f(\cdot))](\omega,t) = W_g f(\omega-\omega_0,t)$ *(modulation invariance)*.

$W_g f$ is continuous as a function in $(\omega,t)$.

(Proof analogous to the wavelet case.)

For an *arbitrary* window function $g$, which does *not* satisfy an additional admissibility condition, as the analyzing wavelet in the CWT case, we have the following

**Orthogonality relation**:

$$\forall f_1, f_2 \in L^2(\mathbf{R}) : \quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[W_g f_1(\omega,t)\overline{W_g f_2(\omega,t)}]d\omega dt = \\ = 2\pi\|g\|^2(f_1, f_2). \qquad (A.2)$$

As a consequence,

$$\frac{1}{\sqrt{2\pi}\|g\|}W_g : \; L^2(\mathbf{R}) \to L^2(\mathbf{R}\times\mathbf{R}, d\omega dt),$$

is an isometry, and inversion formulas, similiar to those for CWT in I.2.3, are valid. In particular, $RgW_g$ is a reproducing kernel Hilbert space.

There is *no analogue* of the *time-/frequency-zooming* of CWT, in the CWFT case. This can be explained as follows: Defining the center and the radius of a function, as in $(I.6)$ and $(I.7)$, we get:

$$m_{g_{\omega t}} = m_g + t, \quad \Delta_{g_{\omega t}} = \Delta_g, \quad m_{\widehat{g_{\omega t}}} = m_{\hat{g}} + \omega, \quad \Delta_{\widehat{g_{\omega t}}} = \Delta_{\hat{g}}.$$

Consequently, $\forall c \geq 1$

$$\lambda(I(g_{\omega t}, c)) = 2c\Delta_g \quad \forall \omega, t \in \mathbf{R},$$

$$\lambda(I(\widehat{g_{\omega,t}}, c)) = 2c\Delta_{\hat{g}} \quad \forall \omega, t \in \mathbf{R},$$

where the interval $I$ is defined as in I.1.

Therefore, for no $\omega_0 \in \mathbf{R}$, the interval $I(g_{\omega t}, c)$ converges to some point of $\mathbf{R}$, as $\omega \to \omega_0$. A corresponding statement holds for $I(\widehat{g_{\omega t}}, c)$, with respect to $t$.

Illustration of the $(\omega, t)$-dependence of the time-frequency windows

$$R(g_{\omega t}, c) := I(g_{\omega t}, c) \times I(\widehat{g_{\omega t}}, c) \qquad (cf. I.3) :$$

[Chui92b]

### A.2. The Discrete Windowed Fourier Transform (DWFT)

Fix $\omega_0, t_0 \in \mathbf{R}^*$. Let $g$ be a window function. Define

$$g_{mn}^{\omega_0 t_0}(x) := g_{m\omega_0, nt_0}(x) = e^{im\omega_0 x} g(x - nt_0).$$

The *discrete windowed Fourier transform (DWFT)* of a function $f \in L^2(\mathbf{R})$, with respect to $g$ and the discretization parameters $\omega_0, t_0$, is given by

$$W_{g,\omega_0,t_0} f := ((f, g_{mn}^{\omega_0 t_0}))_{m,n \in \mathbf{Z}}.$$

To reconstruct $f$ from the discrete windowed Fourier coefficients $W_{g,\omega_0,t_0} f$ in a stable manner, one requires again that the functions $(g_{mn}^{\omega_0 t_0})_{m,n \in \mathbf{Z}}$ constitute a *frame* of $L^2(\mathbf{R})$.

It turns out that, in contrary to the wavelet case, there is an *upper bound for the product of the discretization parameters* $\omega_0, t_0$, in order that the functions $(g_{mn}^{\omega_0 t_0})_{m,n \in \mathbf{Z}}$ constitute a of $L^2(\mathbf{R})$, more precisely:

*There exists no window function* $g$ *such that* $(g_{mn}^{\omega_0 t_0})_{m,n\in\mathbf{Z}}$ *is a frame of* $L^2(\mathbf{R})$, *in case*

$$\omega_0 \cdot t_0 > 2\pi. \qquad (A.3)$$

(For information, see [Dau90,p.978].)

So let us restrict to $\omega_0 t_0 \leq 2\pi$, in the following.

Using the same arguments, as in the proof of theorem 2 in II.2, one can show that a *necessary condition* for $(g_{mn}^{\omega_0 t_0})_{m,n\in\mathbf{Z}}$, to be a *frame* of $L^2(\mathbf{R})$, is the existence of two constants $0 < A \leq B < \infty$ such that

$$A \leq \frac{2\pi}{\omega_0 t_0}\|g\|^2 \leq B. \qquad (A.4)$$

As for the CWFT, no additional admissibility condition on $g$ is needed.

A special consequence of $(A.4)$ is that "$(g_{mn}^{\omega_0 t_0})_{m,n\in\mathbf{Z}}$ *ONB* of $L^2(\mathbf{R})$" is only possible in case

$$\omega_0 \cdot t_0 = 2\pi, \qquad (A.5)$$

which is a critical case, due to $(A.3)$.

The following theorem shows that windowed Fourier frames, at the critical value $(A.4)$, necessarily possess bad time-frequency localization.

**Uncertainty principle of Balian-Low.** (1985)

*If* $(g_{mn}^{\omega_0 t_0})_{m,n\in\mathbf{Z}}$ *is a frame for* $L^2(\mathbf{R})$, *where* $\omega_0 t_0 = 2\pi$, *then either*

$$xg(x) \notin L^2(\mathbf{R})$$

or

$$\omega\hat{g}(\omega) \notin L^2(\mathbf{R}).$$

Symbolically:

$$\Delta_g \cdot \Delta_{\hat{g}} = \frac{1}{4c^2}\lambda^2(R(\psi, c)) = \infty \quad \forall c \geq 1.$$

*Proof.* [Bal81] or [Dau90,p.976]. $\diamondsuit$

This result should be compared with Battle's uncertainty principle for WONBs, as proved in II.5.

The previous results can be summarized in the following diagram:

[Dau92, p.113]

Another important difference between DWT and DWFT is that the *dual frame* of a windowed Fourier frame is always a *windowed Fourier frame*, with respect to the same discretization parameters $\omega_0, t_0$:
$(g_{mn}^{\omega_0 t_0})_{m,n\in\mathbf{Z}}$ *frame for* $L^2(\mathbf{R}) \Longrightarrow \exists \tilde{g} \in L^2(\mathbf{R})$ *such that*

$$\tilde{g}_{mn}^{\omega_0 t_0} = \widetilde{g_{mn}^{\omega_0 t_0}} \quad \forall m, n \in \mathbf{Z}.$$

(Cf., the proposition in II.2, for the wavelet case.)

**A.3. Relationship between CWFT and the Heisenberg Group** [BlaMW91], [HeilW89]

The *Heisenberg group* $(G_H, \circ)$ is the locally compact group, defined by $G_H := \mathbf{R}^3$,

$$(\omega, t, s) \circ (\omega', t', s') := (\omega + \omega', t + t', s + s' + \omega t').$$

Define the left and right translation $L_{(\omega,t,s)}$, $R_{(\omega,t,s)}$, as in III.1. Since

$$DL_{(\omega,t,s)}(\omega',t',s') = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & p \\ 0 & 0 & 1 \end{pmatrix} = 1,$$

independent of $(\omega',t',s')$, and

$$DR_{(\omega,t,s)}(\omega',t',s') = \begin{pmatrix} 1 & 0 & q \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 1,$$

independent of $(\omega',t',s')$, $(III.5)$ and $(III.6)$ can be applied to construct the left and right Haar measure on $G$, which results (by choosing the appearing constant to be one) in

$$d\mu_L(\omega,t,s) = d\mu_R(\omega,t,s) = d\omega dt ds,$$

i.e. the Heisenberg group is *unimodular*.

A square-integrable, irreducible, unitary representation of $(G_H, \circ)$ on $L^2(\mathbf{R})$ is given by

$$U:\ G_H \to \mathcal{U}(L^2(\mathbf{R})), \quad (\omega,t,s) \mapsto U(\omega,t,s), \quad where \qquad (A.5)$$

$$U(\omega,t,s):\ L^2(\mathbf{R}) \to L^2(\mathbf{R})$$

$$f \to U(\omega,t,s)f := \frac{1}{\sqrt{2\pi}} e^{-is} e^{i\omega\cdot} f(\cdot - t).$$

So, the orthogonality relation $(III.11)$ holds for $G = G_H$, $\mathcal{H} = L^2(\mathbf{R})$ and $U$, as defined in $(A.6)$. In particular, the unimodularity of $G_H$ ensures that every $g \in L^2(\mathbf{R})$ is $U$-admissible. Fixing $s = 0$, we get for $f, g \in L^2(\mathbf{R})$, $\omega, t \in \mathbf{R}$ :

$$(f, U(\omega,t,0)g) = (f, \frac{1}{\sqrt{2\pi}} e^{i\omega\cdot} g(\cdot - t)) = \frac{1}{\sqrt{2\pi}} W_g f(\omega,t),$$

and $(III.11)$ reduces to $(A.2)$. So the CWFT stems from a group representation as does the CWT.

**Concluding remark.**

Point 3 explains the parallels in the theory of CWT and CWFT, e.g. the orthogonality relations and their consequences, as well as their main difference, namely the necessity of an admissibility condition for CWT, in contrary to CWFT. The reason for this is that $G_H$ is unimodular, while $G_{aff}$ is not.

## Appendix B: Coherent States

### B.1. Definition and Basic Properties.

Let $\mathcal{L}$ be a topological space. A set $\{\psi_l : l \in \mathcal{L}\}$ of elements in a Hilbert space $\mathcal{H}$ is called a set of *coherent states* for $\mathcal{H}$ (in the sense of J.R. Klauder and B.-S. Skagerstam [KlSk85]), if the following two conditions are satisfied:

i) *Continuity.*

$$\lim_{l \to l_0} \|\psi_l - \psi_{l_0}\|_{\mathcal{H}} = 0 \qquad \forall l_0 \in \mathcal{L}. \tag{$B.1$}$$

ii) *Resolution of Unity.*

There exists a Borel measure $l$ on $\mathcal{L}$ such that $\forall f, g \in \mathcal{H}$

$$(f, g)_{\mathcal{H}} = \int_{\mathcal{L}} (f, \psi_l)_{\mathcal{H}} (\psi_l, g)_{\mathcal{H}} dl. \tag{$B.2$}$$

Examples.

a) $\mathcal{L} = G_{aff}$, $\mathcal{H} = L^2(\mathbf{R})$, $\psi_l = \psi_{ab}$, as in $(I.1)$, where $\psi$ is an admissible, analyzing wavelet, $\frac{1}{c_\psi} l$ left Haar measure on $G_{aff}$, i.e. $dl = \frac{1}{c_\psi} \frac{da db}{a^2}$. (Cf. I.0, Lemma 1ii) and $(I.21)$.)

b) $\mathcal{L} = G^+_{aff}$, $\mathcal{H} = H^2(\mathbf{R})$, $\psi_l$, $dl$ as in a). These are the *affine coherent states*, introduced by Aslaksen and Klauder [AslK68] to describe the movement (without reflection) of a particle, on the real line.

c) $\mathcal{L} = \mathbf{R}^2$, $\mathcal{H} = L^2(\mathbf{R})$, $\psi_l = g_{\omega t}$ as in $(A.1)$, where $g$ is a window function, $dl = \frac{1}{2\pi||g||^2} d\omega dt$.

Coherent states for $\mathcal{H}$ induce a *functional representation* of $\mathcal{H}$, consisting of *continuous* functions, as follows: For $f \in \mathcal{H}$, define $F : \mathcal{L} \to \mathbf{C}$ by

$$F(l) := (f, \psi_l)_{\mathcal{H}}.$$

$F$ is a continuous function of $l$, because of $(B.1)$ and the continuity of the scalar product.

$$\mathcal{C} := \{F : \mathcal{L} \to \mathbf{C} \;\mid\; F = (f, \psi_l)_{\mathcal{H}} \; \textit{for some} \;\; f \in \mathcal{H}\}$$

constitutes a reproducing kernel Hilbert space, with scalar product, given by $(F, G)_{\mathcal{C}} := \int_{\mathcal{L}} F(l)\overline{G(l)} dl$, and reproducing kernel $K(l, l') = (\psi_{l'}, \psi_l)_{\mathcal{H}}$.

In case of the examples, given above, $F$ corresponds to the CWT, resp. CWFT, of $f$. Hence, the foregoing explanation confirms the well-known fact that the ranges of these transforms are reproducing kernel Hilbert spaces.

**B.2. Canonical coherent states.**

In example b), choose

$$g := \pi^{-\frac{1}{4}} e^{-\frac{x^2}{2}}.$$

The functions $(g_{\omega t})_{(\omega,t)\in\mathbf{R}^2}$ are called *canonical coherent states.*[15]

**Properties.**

i) $g$ is an eigenfunction of

$$H := -\frac{d^2}{dx^2} + x^2, \tag{B.3}$$

with eigenvalue 1. Here, $H$ can be interpreteted as the Hamiltonian of a harmonic oscillator.

15 R.J. Glauber, in: C. DeWitt, A. Blandin, C. Cohen-Tannoudji (Eds.), *Quantum Optics and Electronics*, Gordon and Breach, New York, 1964.

ii) Consider $s$ as time-variable. Then, the standard solution of the time dependent Schrödinger equation

$$i\frac{\partial}{\partial s}\psi_s = H\psi_s, \quad \psi_0 = \psi, \tag{B.4}$$

for given $\psi$, is

$$\psi_s = e^{-iHs}\psi = \sum_{n=0}^{\infty}\frac{(-iHs)^n}{n!}\psi,$$

where convergence of the series holds in $L^2(\mathbf{R})$. For the special choice $\psi = g_{\omega t}$, where
$(\omega, t) \in \mathbf{R}^2$ is fixed, one gets:

$$\psi_s = g_{\omega t,s} = e^{ia_s}g^{\omega_s t_s},$$

where $\omega_s := \omega\cos 2s - t\sin 2s$, $t_s := \omega\sin 2s + t\cos 2s$, $a_s := \frac{1}{2}(\omega t - \omega_s t_s)$. I.e.: Under time-evolution, a coherent state remains a coherent state, up to a phase factor.

iii) The reproducing kernel Hilbert space $\mathcal{C}$ for $\psi_l = g_{\omega t}$, is a space of entire functions, the so called *Bargmann Hilbert space* $\mathcal{C}_{Barg}$, defined by

$$\mathcal{C}_{Barg} := \{F : \mathbf{R}^2 \to \mathbf{C} \ \mid \ F(\omega, t) = e^{-\frac{1}{4}(\omega^2+t^2)-\frac{i}{2}t}\varphi(\omega + it), \quad \varphi \ entire\}.$$

## Bibliography

(without claim for completeness)

**Topics**

1. *General Literature on Wavelets*.

**(I)** **I**ntroductions

**(S)** **S**urveys and Seminar Notes

**(M)** **M**onographs

**(P)** **P**roceedings and Lecture Notes

**(SI)** **S**pecial **I**ssues

2. *Theoretical Background*.

**(U)** **U**ncertainty Principles, Sampling Theorems and Methods of Irregular Sampling

**(G)** **G**roup Theoretical Background of Wavelet Theory, Coherent States

3. *Wavelet Transforms*.

**(C)** **C**ontinuous Wavelet Transform

**($D_1$)** **D**iscrete Wavelet Transform 1: Dilation equations, multiresolution analysis and the construction of wavelet orthonormal bases

**($D_2$)** **D**iscrete Wavelet Transform 2: *Special* wavelet orthonormal bases

**($D_3$)** **D**iscrete Wavelet Transform 3: Frames, especially wavelet frames, Riesz bases

**($D_4$)** **D**iscrete Wavelet Transform 4: Wavelet Packets

**(ZM)** Reconstruction of Signals from **Z**ero Crossing and **M**odulus **M**axima of Wavelet Transforms

**(R)** **R**elated Transform Methods, especially Gabor Transform, and Application of Wavelet Transform in the Theory of **R**adon Transform

**(N)** **N**umerical Methods to compute Wavelet Transforms

4. *Applications.*

**(Q)** **Q**uantum Physics

**(SA)** **S**ignal **A**nalysis in Seismology, Radar, Sonar, Medicine and Acoustics

**(SP)** **S**ignal and Image **P**rocessing, Data Compression and Decorrelation,

Pyramide Schemes and QMFs, Curve and Surface Generation

**(IA)** **I**nterpolation and Non Linear **A**pproximation, Function Spaces, Regularity Estimations

**(E)** Differential and Integral **E**quations

**(F)** **F**ractal Objects, Detection of Singularities, Turbulence, Stochastic Processes, Probability Theory

# Bibliography