Lineární algebra, desáté cvičení

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1 Determinanty-inverzní matice

1.1 Najděte determinant matice a determinant inverzní matice.

$$A = \begin{pmatrix} a & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 1 & 1 & a & 1 \end{pmatrix}.$$

$$\begin{matrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{matrix}$$ ZNAMÉNKA PRO DOPLŇKY

[Rozvojem podle čtvrtého sloupce a Sarrusovym pravidlem: $det A = \begin{vmatrix} a & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 3 \end{vmatrix} = 3a + 1 + 1 - 1 - 3 - a = 2a - 2,\ det A^{-1} = \frac{1}{2a-2},\ a \neq 1, A^{-1}$ neex, $a = 1$.]

$$1 = \det(E) = \det(A \cdot A^{-1}) = \det(A) \cdot \det(A^{-1}) \longrightarrow \det(A^{-1}) = \frac{1}{\det(A)}$$

1.2 Pomocí determinantů najděte, pokud existuje, inverzní matici.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 0 \\ -1 & 1 & 3 \\ 3 & -1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & 3 & 0 \\ -1 & 2 & 1 \\ -1 & 9 & 3 \end{pmatrix}.$$

VZOREC

$$A^{-1} = \frac{1}{\det(A)} \left(D_{ij} \right)^T$$

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \det(A) = -2, \underline{\underline{A^{-1}}} = \frac{1}{-2} \begin{pmatrix} 4 & -3 \\ -2 & 1 \end{pmatrix}^T = \underline{\underline{-\frac{1}{2} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}}}$$

ZKOUŠKA:

$$A \cdot A^{-1} = -\frac{1}{2} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}}$$

$$A = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 1 & 3 \\ 3 & -1 & 2 \end{pmatrix}, \quad \det(A) = 2+18+3+4 = \underline{27}$$

$$\underline{\underline{A^{-1}}} = \frac{1}{27}\begin{pmatrix} 5 & 11 & -2 \\ -4 & 2 & 7 \\ 6 & -3 & 3 \end{pmatrix}^T = \underline{\underline{\frac{1}{27}\begin{pmatrix} 5 & -4 & 6 \\ 11 & 2 & -3 \\ -2 & 7 & 3 \end{pmatrix}}}$$

1.3 Pomocí determinantů najděte vzhledem k $a \in \mathbb{R}$ inverzní matici.

$$\begin{pmatrix} 2 & a & 3 \\ -1 & 2 & 1 \\ 2 & 0 & 1 \end{pmatrix}.$$

[Použijeme vzorec $A^{-1} = \frac{1}{detA}(D_{i,j})^T$: $A^{-1} = \frac{1}{3a-8}\begin{pmatrix} 2 & -a & a-6 \\ 3 & -4 & -5 \\ -4 & 2a & 4+a \end{pmatrix}$, $a \neq \frac{8}{3}$]

2 Determinanty-soustavy.

2.1 Pomocí Cramerovy věty najděte , je-li to možné, řešení soustavy.

$$\left(\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ -1 & 2 & 0 & -1 \\ 3 & 4 & 2 & 0 \end{array}\right), \quad \left(\begin{array}{ccc|c} 2 & 3 & 0 & 0 \\ -1 & 2 & 1 & 1 \\ -1 & 9 & 3 & 3 \end{array}\right).$$

VZOREC (CRAMER)

$$\left(\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ -1 & 2 & 0 & -1 \\ 3 & 4 & 2 & 0 \end{array}\right) \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{D}\begin{pmatrix} D_x \\ D_y \\ D_z \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{-26}\begin{pmatrix} \begin{vmatrix} 1 & 0 & 3 \\ -1 & 2 & 0 \\ 0 & 4 & 2 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 & 3 \\ -1 & -1 & 0 \\ 3 & 0 & 2 \end{vmatrix} \\ \begin{vmatrix} 1 & 0 & 1 \\ -1 & 2 & -1 \\ 3 & 4 & 0 \end{vmatrix} \end{pmatrix} = -\frac{1}{26}\begin{pmatrix} -8 \\ -2+11 \\ -4-6+4 \end{pmatrix} = \underline{\underline{-\frac{1}{26}\begin{pmatrix} -8 \\ 9 \\ -6 \end{pmatrix}}}$$

2.2 Vzhledem k $p \in \mathbb{R}$ najděte řešení soustavy. Kde je to možné, použijte determinanty.

$$\left(\begin{array}{ccc|c} p & 2 & 2 & 1 \\ 1 & p & 0 & -1 \\ 1 & 0 & p & 2 \end{array}\right), \quad \left(\begin{array}{ccc|c} p & 1 & 1 & 1 \\ 1 & 1 & 3 & 0 \\ 1 & 1 & p & 1 \end{array}\right).$$

$[detA = (p-3)(p-1),\ p = 3,\ \left(\begin{array}{ccc|c} 3 & 1 & 1 & 1 \\ 1 & 1 & 3 & 0 \\ 1 & 1 & 3 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 3 & 1 & 1 & 1 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right),\ rankA = 2 \neq 3 = rank\bar{A}$, řešení neexistuje, $p = 1,\ \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)\ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ 0 \\ -\frac{1}{2} \end{pmatrix} + span\{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}\},\ p \neq 3$ a $p \neq 1$,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 1 & 1 & p \end{vmatrix}}{detA} \\ \frac{\begin{vmatrix} p & 1 & 1 \\ 1 & 0 & 3 \\ 1 & 1 & p \end{vmatrix}}{detA} \\ \frac{\begin{vmatrix} p & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix}}{detA} \end{pmatrix} = \begin{pmatrix} \frac{1}{p-3} \\ -\frac{4}{p-3} \\ \frac{1}{p-3} \end{pmatrix}]$$

$$\left(\begin{array}{ccc|c} p & 2 & 2 & 1 \\ 1 & p & 0 & -1 \\ 1 & 0 & p & 2 \end{array}\right); \quad det(A) = p^3 - 2p - 2p = p(p^2 - 4) = p(p+2)(p-2) = 0$$

1) $p = 0$

$$\left(\begin{array}{ccc|c} 0 & 2 & 2 & 1 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 0 & 2 & 2 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 3 \end{array}\right) R_3 - R_2$$

$RANK(A) = 2 \neq 3 = RANK(\bar{A}) \xRightarrow{F.V.}$ NŘ

2) $p = -2$

$$\left(\begin{array}{ccc|c} -2 & 2 & 2 & 1 \\ 1 & -2 & 0 & -1 \\ 1 & 0 & -2 & 2 \end{array}\right) \sim \left(\begin{array}{ccc|c} -2 & 2 & 2 & 1 \\ 0 & -2 & 2 & -1 \\ 0 & 2 & -2 & 3 \end{array}\right) \begin{array}{l} 2R_2 + R_1 \\ R_3 - R_2 \end{array} \sim \left(\begin{array}{ccc|c} -2 & 2 & 2 & 1 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & 0 & 2 \end{array}\right) R_3 + R_2 \quad \ldots\ldots \quad \text{NŘ}$$

3) $p = 2$

$$\left(\begin{array}{ccc|c} 2 & 2 & 2 & 1 \\ 1 & 2 & 0 & -1 \\ 1 & 0 & 2 & 2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 2 & 2 & 2 & 1 \\ 0 & 2 & -2 & -3 \\ 0 & -2 & 2 & 3 \end{array}\right) \begin{array}{l} 2R_2 - R_1 \\ R_3 - R_2 \end{array} \sim \left(\begin{array}{ccc|c} 2 & 2 & 2 & 1 \\ 0 & 2 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right) R_3 + R_2$$

$RANK(A) = 2 = 2 = RANK(\bar{A})$, ŘEŠ. EX., dimenze $= 3 - 2 = 1$

$$\underline{\underline{\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -\frac{3}{2} \\ 0 \end{pmatrix} + \mathrm{SPAN}\left\{\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}\right\}}}$$

$$2x + 2\left(-\tfrac{3}{2}\right) + 2\cdot 0 = 1 \to 2x = 4$$

$$2x + 2\cdot 1 + 2\cdot 1 = 0 \to 2x = -4$$

c) $p \notin \{0, -2, 2\}$

$$\left(\begin{array}{ccc|c} p & 2 & 2 & 1 \\ 1 & p & 0 & -1 \\ 1 & 0 & p & 2 \end{array}\right)$$

$$\underline{\underline{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}} = \frac{1}{\begin{vmatrix} p & 2 & 2 \\ 1 & p & 0 \\ 1 & 0 & p \end{vmatrix}} \begin{pmatrix} \begin{vmatrix} 1 & 2 & 2 \\ -1 & p & 0 \\ 2 & 0 & p \end{vmatrix} \\ \begin{vmatrix} p & 1 & 2 \\ 1 & -1 & 0 \\ 1 & 2 & p \end{vmatrix} \\ \begin{vmatrix} p & 2 & 1 \\ 1 & p & -1 \\ 1 & 0 & 2 \end{vmatrix} \end{pmatrix} = \frac{1}{p(p+2)(p-2)} \begin{pmatrix} p^2 - 2p \\ -p^2 + 6 - p \\ 2p^2 - 6 - p \end{pmatrix} =$$

$$= \underline{\underline{\begin{pmatrix} \frac{1}{p+2} \\ \frac{p+3}{p(p+2)} \\ \frac{2p+3}{p(p+2)} \end{pmatrix}}}$$

$$p^2 - p + 6 = (p+3)(p-2)$$

$$2p^2 - p - 6 = (p-2)(2p+3)$$