Lineární algebra, dvanácté cvičení

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1 Skalární součin

1.1 Je dán standartní skalární součin. Najděte skalární součin, velikosti a úhel vektorů. Ověřte Cauchy-Schwarz-Bunyakovského nerovnost ($|\langle\vec{x}|\vec{y}\rangle| \leq \sqrt{\langle\vec{x}|\vec{x}\rangle} \cdot \sqrt{\langle\vec{y}|\vec{y}\rangle}$).

$\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ a $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$[\langle\begin{pmatrix} 2 \\ 1 \end{pmatrix} | \begin{pmatrix} 1 \\ 2 \end{pmatrix}\rangle = 2 \cdot 1 + 1 \cdot 2 = 4,\ ||\begin{pmatrix} 2 \\ 1 \end{pmatrix}|| = \sqrt{\langle\begin{pmatrix} 2 \\ 1 \end{pmatrix} | \begin{pmatrix} 2 \\ 1 \end{pmatrix}\rangle} = \sqrt{5},$

$||\begin{pmatrix} 1 \\ 2 \end{pmatrix}|| = \sqrt{5},\ cos\phi = \frac{\langle\begin{pmatrix} 2 \\ 1 \end{pmatrix}|\begin{pmatrix} 1 \\ 2 \end{pmatrix}\rangle}{||\begin{pmatrix} 2 \\ 1 \end{pmatrix}||\cdot||\begin{pmatrix} 1 \\ 2 \end{pmatrix}||} = \frac{4}{5},\ 4 \leq 5\ ]$

$$\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \text{ a } \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

$$\left\langle \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \middle| \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \right\rangle = 3\cdot 1 + 2\cdot 2 + 1\cdot(-1) = \underline{\underline{6}}$$

$$\left\| \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \right\| = \sqrt{\left\langle \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \middle| \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \right\rangle} = \underline{\underline{\sqrt{14}}} \qquad \left\| \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \right\| = \sqrt{\left\langle \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \middle| \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \right\rangle} = \underline{\underline{\sqrt{6}}}$$

$$\langle \bar{u} | \bar{v} \rangle = \|\bar{u}\| \|\bar{v}\| \cos\varphi \quad \rightarrow \cos\varphi = \frac{\langle \bar{u} | \bar{v} \rangle}{\|\bar{u}\| \|\bar{v}\|}$$

$$\cos\varphi = \frac{6}{\sqrt{14}\sqrt{6}} = \frac{6}{2\sqrt{21}} = \underline{\underline{\frac{3}{\sqrt{21}}}}$$

$$\langle \bar{u} | \bar{v} \rangle \leq \|\bar{u}\| \cdot \|\bar{v}\|$$

$$6 \leq 2\sqrt{21} \qquad \underline{\underline{\text{PLATÍ}}}$$

1.2 Rozhodněte o pozitivní definitnosti matic. Pro ty pozitivně definitní napište příslušné skalární součiny

$\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$,

[Symetrická, $|1| = 1 > 0$, $\begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = 5 - 4 = 1 > 0$, tedy je pozitivně definitní,

$\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} | \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \rangle = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = x_1y_1 + 2x_1y_2 + 2x_2y_1 + 5x_2y_2$]

$\begin{pmatrix} 3 & 4 & 8 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{pmatrix}$,

[Není symetrická, tedy ani pozitivně definitní.]

$$\begin{pmatrix} 3 & 0 & 8 \\ 0 & 1 & -2 \\ 8 & -2 & 2 \end{pmatrix},$$

[Symetrická, $|3| = 3 > 0$, $\begin{vmatrix} 3 & 0 \\ 0 & 1 \end{vmatrix} = 3 > 0$, ale $\begin{vmatrix} 3 & 0 & 8 \\ 0 & 1 & -2 \\ 8 & -2 & 2 \end{vmatrix} = 6 - 64 - 12 < 0$, tedy není pozitivně definitní]

$$\begin{pmatrix} 3 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & -2 & 6 \end{pmatrix}.$$

[Symetrická, $|3| = 3 > 0$, $\begin{vmatrix} 3 & 0 \\ 0 & 1 \end{vmatrix} = 3 > 0$, ale $\begin{vmatrix} 3 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & -2 & 6 \end{vmatrix} = 18 - 1 - 12 > 0$, tedy je pozitivně definitní, $\langle \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} | \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \rangle = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \cdot \begin{pmatrix} 3 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & -2 & 6 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$]

$$\left\langle \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \middle| \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \right\rangle = 3x_1y_1 + 0x_1y_2 + 1x_1y_3 +$$
$$+ 0x_2y_1 + 1x_2y_2 - 2x_2y_3 +$$
$$+ 1x_3y_1 - 2x_3y_2 + 6x_3y_3$$

1.3 Najděte skalární součin, pro který jsou dané báze ortonormální.

$(\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix})$,

[Matici skalárního součinu najdeme takto: $G = (B^{-1})^T(B^{-1}) = \left(\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1}\right)^T \cdot \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix}$, kde sloupce B jsou prvky z dané báze.
Zkouška: $\begin{pmatrix} 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0$, $\begin{pmatrix} 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 1$, $\begin{pmatrix} 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1$]

$$-2+6+2$$

$$\left(\begin{pmatrix}3\\2\\0\end{pmatrix},\begin{pmatrix}1\\0\\-1\end{pmatrix},\begin{pmatrix}1\\2\\-1\end{pmatrix}\right).$$

$$G = (B^{-1})^T \cdot B^{-1} = \frac{1}{6}\begin{pmatrix}2 & 2 & -2\\0 & -3 & 3\\2 & -4 & -2\end{pmatrix}\frac{1}{6}\begin{pmatrix}2 & 0 & 2\\2 & -3 & -4\\-2 & 3 & -2\end{pmatrix} = \underline{\underline{\frac{1}{36}\begin{pmatrix}12 & -12 & 0\\-12 & 18 & 6\\0 & 6 & 24\end{pmatrix}}}$$

ZKOUŠKA:

$$(3\ 2\ 0)\frac{1}{6}\begin{pmatrix}2 & -2 & 0\\-2 & 3 & 1\\0 & 1 & 4\end{pmatrix}\begin{pmatrix}1\\0\\-1\end{pmatrix} = \frac{1}{6}(2\ 0\ 2)\begin{pmatrix}1\\0\\-1\end{pmatrix} = 0$$

$$(3\ 2\ 0)\frac{1}{6}\begin{pmatrix}2 & -2 & 0\\-2 & 3 & 1\\0 & 1 & 4\end{pmatrix}\begin{pmatrix}1\\2\\-1\end{pmatrix} = \frac{1}{6}(2\ 0\ 2)\begin{pmatrix}1\\2\\-1\end{pmatrix} = 0$$

$$(1\ 0\ -1)\frac{1}{6}\begin{pmatrix}2 & -2 & 0\\-2 & 3 & 1\\0 & 1 & 4\end{pmatrix}\begin{pmatrix}1\\2\\-1\end{pmatrix} = \frac{1}{6}(2\ -3\ -4)\begin{pmatrix}1\\2\\-1\end{pmatrix} = 0$$

$$(3\ 2\ 0)\frac{1}{6}\begin{pmatrix}2 & -2 & 0\\-2 & 3 & 1\\0 & 1 & 4\end{pmatrix}\begin{pmatrix}3\\2\\0\end{pmatrix} = \frac{1}{6}(2\ 0\ 2)\begin{pmatrix}3\\2\\0\end{pmatrix} = \frac{1}{6}6 = 1$$

$$(1\ 0\ -1)\frac{1}{6}\begin{pmatrix}2 & -2 & 0\\-2 & 3 & 1\\0 & 1 & 4\end{pmatrix}\begin{pmatrix}1\\0\\-1\end{pmatrix} = \frac{1}{6}(2\ -3\ -4)\begin{pmatrix}1\\0\\-1\end{pmatrix} = \frac{1}{6}6 = 1$$

$$(1\ 2\ -1)\frac{1}{6}\begin{pmatrix}2 & -2 & 0\\-2 & 3 & 1\\0 & 1 & 4\end{pmatrix}\begin{pmatrix}1\\2\\-1\end{pmatrix} = \frac{1}{6}(-2\ 3\ -2)\begin{pmatrix}1\\2\\-1\end{pmatrix} = \frac{1}{6}6 = 1$$

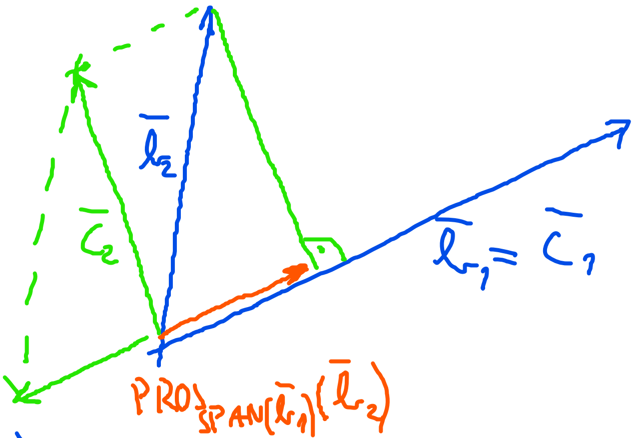
1.4 V $\mathbb{R}^3$ ortogonalizujte bázi $B = (\vec{b}_1, \vec{b}_2)$.

$$\vec{b}_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \vec{b}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

[$C = (\vec{c}_1, \vec{c}_2)$, kde $\vec{c}_1 = \vec{b}_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$, $\vec{c}_2 = \vec{b}_2 - proj_{span(\vec{c}_1)}\vec{b}_2 = \vec{b}_2 - \frac{\langle \vec{b}_2 | \vec{c}_1 \rangle}{\langle \vec{c}_1 | \vec{c}_1 \rangle}\vec{c}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} - \frac{1}{14}\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{11}{14} \\ -\frac{16}{14} \\ -\frac{1}{14} \end{pmatrix}$, bez narušení ortogonality lze psát $C = (\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 11 \\ -16 \\ -1 \end{pmatrix})$]

$$B = (\bar{b}_1, \bar{b}_2)$$

$$C = (\bar{c}_1, \bar{c}_2)$$



$$\bar{c}_1 = \bar{b}_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

$$\bar{c}_2 = \bar{b}_2 - PROJ_{SPAN(\bar{c}_1)}(\bar{b}_2) =$$

$$= \bar{b}_2 - \frac{\langle \bar{c}_1 | \bar{b}_2 \rangle}{\langle \bar{c}_1 | \bar{c}_1 \rangle}\bar{c}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} - \frac{\langle \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} | \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \rangle}{\langle \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} | \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \rangle}\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} - \frac{1}{14}\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \frac{1}{14}\begin{pmatrix} 11 \\ -16 \\ -1 \end{pmatrix}$$

TEDY ORTOGONÁLNÍ BÁZE NAPŘ.: $\underline{\underline{\left(\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 11 \\ -16 \\ -1 \end{pmatrix}\right)}}$

ORTONORMÁLNÍ: $\left(\frac{1}{\left\|\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}\right\|}\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \frac{1}{\left\|\begin{pmatrix} 11 \\ -16 \\ -1 \end{pmatrix}\right\|}\begin{pmatrix} 11 \\ -16 \\ -1 \end{pmatrix}\right) = \ldots$ ATD