Matematická analýza 1, 1. paralelka - obory EEM, EK, druhé cvičení

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1 Limita posloupnosti.

1.1 Najděte limitu posloupnosti.

NEDEFINOVÁNO

$\lim\limits_{n\to\infty} \frac{n+1}{n^2-8n+2}$ [0] $\approx \left\langle\!\left\langle \frac{\infty}{\infty-\infty} \right\rangle\!\right\rangle = \left\langle\!\left\langle \frac{\infty}{\infty} \right\rangle\!\right\rangle = \infty$

$\infty \cdots n^2 - 8n + 2 = n^2\left(1 - \frac{8}{n} + \frac{2}{n^2}\right) \to \infty$

$\doteq \lim\limits_{n\to\infty} \frac{n\left(1+\frac{1}{n}\right)}{n^2\left(1-\frac{8}{n}+\frac{2}{n^2}\right)} = \underline{\underline{0}}$

$\lim\limits_{n\to\infty} \frac{3n^2+(n+1)^2}{(n+2)^2+(n-2)^2}$ [2]

$\lim\limits_{n\to\infty} \frac{4n^2+2n+1}{2n^2+8} = \lim\limits_{n\to\infty} \frac{4+\frac{2}{n}+\frac{1}{n^2}}{2+\frac{8}{n^2}} = \underline{\underline{2}}$

TYP $\frac{\infty}{\infty}$, ZKRÁTÍM n^2

$\lim\limits_{n\to\infty} \frac{(n+1)!}{(n+2)!-(n-1)!}$ [0] $= \left\langle\!\left\langle \frac{\infty}{\infty-\infty} \right\rangle\!\right\rangle =$, UPRAVÍM A ZKRÁTÍM,

$= \lim\limits_{n\to\infty} \frac{(n+1)\,n\,\cancel{(n-1)!}}{(n+2)(n+1)\,n\,\cancel{(n-1)!} - \cancel{(n-1)!}} = \lim\limits_{n\to\infty} \frac{n^2+n}{(n^2+3n+2)n - 1} =$

$= \lim\limits_{n\to\infty} \frac{n^2+n}{n^3+3n^2+2n-1} = \lim\limits_{n\to\infty} \frac{1}{n} \cdot \frac{1+\frac{1}{n}}{1+\frac{3}{n}+\frac{2}{n^2}-\frac{1}{n^3}} = \underline{\underline{0}}$

$\lim\limits_{n\to\infty} \frac{5\ 3^n-3\ 2^{n+2}}{2^n-2\ 3^n}$ $[-\frac{5}{2}]$

$\lim\limits_{n\to\infty} \frac{5\cdot 3^n - 3\cdot 2^{n+2}}{2^n - 2\cdot 3^n} \cdot \frac{\frac{1}{3^n}}{\frac{1}{3^n}} = \lim\limits_{n\to\infty} \frac{5 - 3\left(\frac{2}{3}\right)^n\cdot 4}{\left(\frac{2}{3}\right)^n - 2} = \underline{\underline{-\frac{5}{2}}}$

$\lim\limits_{n\to\infty} \frac{2\ (-5)^n+3\ 4^n}{2^{n+3}+(-5)^n}$ [2]

$$\lim_{n\to\infty} \frac{2(-5)^n+3\cdot 4^n}{2^{n+3}+(-5)^n}\cdot\frac{\frac{1}{(-5)^n}}{\frac{1}{(-5)^n}} = \lim_{n\to\infty}\frac{2+3\underbrace{\left(-\frac{4}{5}\right)^n}_{\to 0}}{8\cdot\underbrace{\left(-\frac{2}{5}\right)^n}_{\to 0}+1} = \underline{\underline{2}}$$

$\lim\limits_{n\to\infty} \frac{n^3+1}{n+n^2}\sin n\pi$ [0]

$$\lim_{n\to\infty} \underbrace{n}_{\to\infty}\cdot\underbrace{\frac{1+\frac{1}{n^3}}{\frac{1}{n}+1}}_{\to\infty}\cdot\sin n\pi$$

SIN V ∞ LIMITU NEMÁ, CO TEĎ?

PLATÍ: $\forall n\in\mathbb{N};\ \sin n\pi = 0$

A TAKY $\frac{n^3+1}{n+n^2}\cdot 0 = 0$, TAKŽE $\lim\limits_{n\to\infty}\frac{n^3+1}{n+n^2}\sin n\pi = \lim\limits_{n\to\infty} 0 = \underline{\underline{0}}$

2 Limita funkce.

2.1 Najděte limitu funkce.

$\lim\limits_{x\to 1} \frac{2x^2-x-1}{x-1}$ [3]

NEDEFINOVÁNO

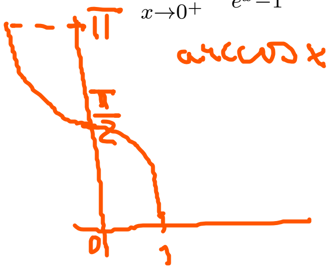
$$\lim_{x\to 1}\frac{2x^2-x-1}{x-1} = \left\langle\!\left\langle \frac{0}{0} \right\rangle\!\right\rangle = \lim_{x\to 1}\frac{\cancel{(x-1)}(2x+1)}{\cancel{x-1}} = \underline{\underline{3}}$$

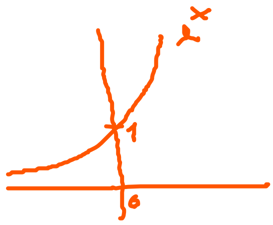
$2x^2-x-1$ MÁ KOŘEN 1 → ROZLOŽÍM: $(2x^2-x-1):(x-1) = 2x+1$

$\lim\limits_{x\to 0^-} (e^{1/x} + \frac{x+3}{x^2-1})$ [−3] $= \left\langle\!\left\langle e^{\frac{1}{0^-}} - 3 = e^{-\infty} - 3 = \frac{1}{e^{\infty}} - 3 = \frac{1}{\infty} - 3 \right\rangle\!\right\rangle = \underline{\underline{-3}}$

$\lim\limits_{x\to 1^+} e^{x+\ln(x-1)}$ [0] $= \left\langle\!\left\langle e^{1^+ + \ln(0^+)} = e^{-\infty} = \frac{1}{e^{\infty}} = \frac{1}{\infty} \right\rangle\!\right\rangle = \underline{\underline{0}}$

$\lim\limits_{x\to 0^+} \frac{\arccos(x)}{e^x-1}$ $[\infty]$ $= \left\langle\!\left\langle \frac{\frac{\pi}{2}^-}{1^+-1} = \frac{\frac{\pi}{2}^-}{0^+} \right\rangle\!\right\rangle = \underline{\underline{\infty}}$

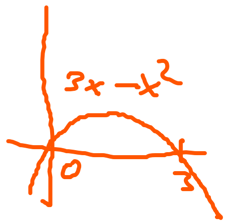




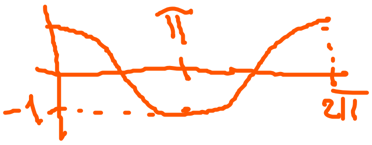
$\lim\limits_{x\to 3} \frac{7-x}{3x-x^2}$ $[neexistuje]$ $= \left\langle\!\left\langle \frac{4}{0\pm} \right\rangle\!\right\rangle = \underline{\underline{\text{NEEX}}}$?

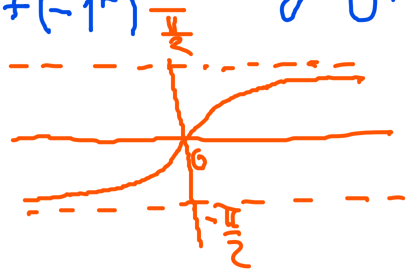
$\lim\limits_{x\to 3^+} \frac{7-x}{3x-x^2} = \left\langle\!\left\langle \frac{4}{0^-} \right\rangle\!\right\rangle = -\infty$

$\lim\limits_{x\to 3^-} \frac{7-x}{3x-x^2} = \left\langle\!\left\langle \frac{4}{0^+} \right\rangle\!\right\rangle = \infty$



$\lim\limits_{x\to \pi} \operatorname{arctg}\left(\frac{1}{1+\cos(x)}\right)$ $[\frac{\pi}{2}]$ $= \left\langle\!\left\langle \operatorname{arctg} \frac{1}{1+(-1^+)} = \operatorname{arctg} \frac{1}{0^+} = \operatorname{arctg} \infty \right\rangle\!\right\rangle = \underline{\underline{\frac{\pi}{2}}}$





$\lim\limits_{x\to -\infty} \left(\frac{7-x^2-x^3}{3x-x^2}\right)^3$ $[-\infty]$ $= \left\langle\!\left\langle \frac{-\infty}{\infty} \right\rangle\!\right\rangle =$

$= \lim\limits_{x\to -\infty} \left(x\, \frac{\frac{7}{x^3} - \frac{1}{x} - 1}{\frac{3}{x} - 1} \right)^3 = \left\langle\!\left\langle (-\infty \cdot 1)^3 \right\rangle\!\right\rangle = -\underline{\underline{\infty}}$

$\frac{7}{x^3} \to 0$, $\frac{1}{x} \to 0$, $\frac{3}{x} \to 0$

$\lim\limits_{x\to \infty} \left(\frac{7-x^2+3x^3}{3x-2x^3}\right)^3$ $[-\frac{27}{8}]$ $= \left(-\frac{3}{2}\right)^3 = -\underline{\underline{\frac{27}{8}}}$

$\sim \frac{3x^3}{-2x^3}$ (viz výše).

$\lim\limits_{x\to\infty} \frac{x^2-\sqrt{x^3-1}}{x+\sqrt{x^2+x}}$ [∞] $= \left\langle\!\left\langle \frac{\infty-\infty}{\infty+\infty} \right\rangle\!\right\rangle =$

$$= \lim_{x\to\infty} \frac{x^2-\sqrt{x^3-1}}{x+\sqrt{x^2+x}} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to\infty} \frac{1-\sqrt{\underbrace{\frac{x^3-1}{x^4}}_{\to 0}}}{\underbrace{\frac{1}{x}}_{\to 0+}+\sqrt{\underbrace{\frac{x^2+x}{x^4}}_{\to 0+}}} = \underline{\underline{\infty}}$$

$\lim\limits_{x\to-\infty} \frac{\sqrt{x^2+1}}{x}$ [-1] $= \left\langle\!\left\langle \frac{\infty}{-\infty} \right\rangle\!\right\rangle = \lim\limits_{x\to-\infty}\left(-\sqrt{\underbrace{\frac{x^2+1}{x^2}}_{\to 1}}\right) = \underline{\underline{-1}}$

POZOR! $\sqrt{x^2} = |x| \overset{x\to-\infty}{=} -x$

$\lim\limits_{x\to\infty} \left(\sqrt{x+1}-\sqrt{x}\right)$ [0] $= \left\langle\!\left\langle \infty-\infty \right\rangle\!\right\rangle =$

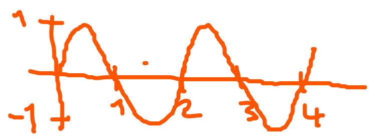
$$= \lim_{x\to\infty} \left(\sqrt{x+1}-\sqrt{x}\right) \cdot \frac{\sqrt{x+1}+\sqrt{x}}{\sqrt{x+1}+\sqrt{x}} = \lim_{x\to\infty} \frac{x+1-x}{\sqrt{x+1}+\sqrt{x}} = \left\langle\!\left\langle \frac{1}{\infty} \right\rangle\!\right\rangle = \underline{\underline{0}}$$

$$(a-b)\,\frac{a+b}{a+b} = \frac{a^2-b^2}{a+b}$$

$\lim\limits_{x\to\infty} \frac{x^3+1}{x+x^2} \sin \pi x$ [*neexistuje*]

$$\lim_{x\to\infty} \underbrace{\frac{x^3+1}{x+x^2}}_{\to\infty} \underbrace{\sin \pi x}_{\text{NEMÁ LIMITU}} = \underline{\underline{\text{NEEX.}}}$$

$\sin \pi x$



$\lim\limits_{x\to\infty} \frac{x+x^2}{x^3+1} \sin \pi x$ [0]

$$\lim_{x\to\infty} \underbrace{\frac{x+x^2}{x^3+1}}_{\to 0} \underbrace{\sin \pi x}_{\text{OMEZENÁ FUNKCE}} = \underline{\underline{0}}$$

$\lim\limits_{x\to\infty} \frac{x^3+1}{x+x^3}\sin \pi x \quad [neexistuje]$

$$\lim_{x\to\infty} \underbrace{\frac{x^3+1}{x+x^3}}_{\downarrow\ 1}\ \sin \pi x = \text{NEEX.}$$

NEMÁ LIMITU

ZA X DÁM $\frac{1}{x}$

$\lim\limits_{x\to 0} x\left[\frac{1}{x}\right] \quad [1]$

$$\frac{1}{x}-1 \le \left[\frac{1}{x}\right] \le \frac{1}{x} \quad / \cdot x$$

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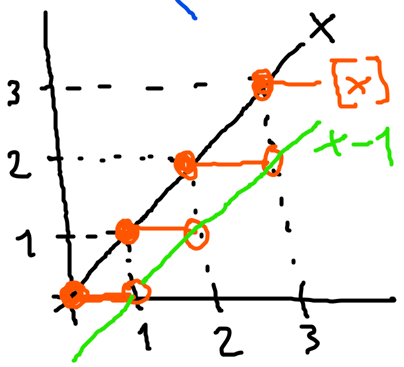
$$\underbrace{1-x}_{\downarrow\, x\to 0^+\ 1} \le x\left[\frac{1}{x}\right] \le \underbrace{1}_{\downarrow\, x\to 0^+\ 1}$$

$$\lim_{x\to 0^+} x\left[\frac{1}{x}\right] = 1$$



3 Zkoumání asymptot

3.1 Najděte definiční obor funkce a limity v krajních bodech jeho intervalů. Načrtněte asymptoty.

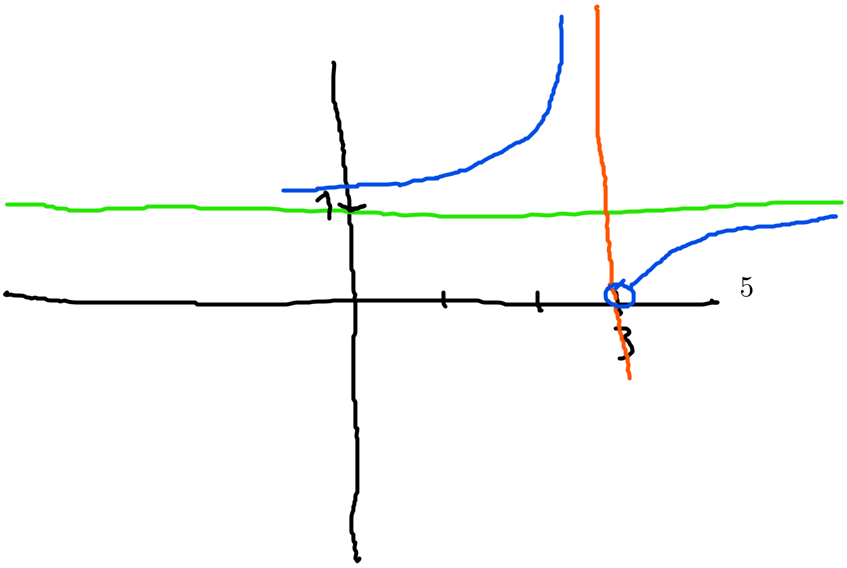
$f(x) = e^{-\frac{1}{x-3}}$

$$D(f) = \mathbb{R} \setminus \{3\}$$

$$\lim_{x\to-\infty} f(x) = \left\langle\!\left\langle e^{\frac{1}{\infty}} = e^{0} \right\rangle\!\right\rangle = 1$$

$$\lim_{x\to 3^{\mp}} f(x) = \left\langle\!\left\langle e^{-\frac{1}{0^{\mp}}} = e^{\pm\infty} \right\rangle\!\right\rangle = \left\langle \begin{matrix} \infty \\ 0 \end{matrix} \right\} \to \lim_{x\to 3} f(x) = \text{NEEX}$$

$$\lim_{x\to\infty} f(x) = \left\langle\!\left\langle e^{-\frac{1}{\infty}} = e^{0} \right\rangle\!\right\rangle = 1$$



SVISLÁ ASYMPTOTA: $x = 3$

VODOROVNÁ ASYMPTOTA: $y=1$

MOŽNÁ PODOBA GRAFU $f(x)$