Matematická analýza 1, 1.paralelka - obory EEM a EK, šesté cvičení

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1 Neurčitý integrál.

1.1 Spočtěte.

$\int \cos(x)e^{\sin(x)}\,dx$

SUBSTITUCE

$$\int f(g(x))g'(x)\,dx = \int f(g)\,dg$$ VZOREC

$$\int e^{\sin(x)}(\sin(x))'\,dx = \int e^{g}\,dg = e^{g} + C =$$

LZE UŽÍT TABULKU

$$\boxed{\begin{aligned} g &= \sin(x) \\ dg &= \cos(x)\,dx \end{aligned}}$$

A DOSADIT

$$= \underline{\underline{e^{\sin(x)} + C \quad ; \quad x \in \mathbb{R}}}$$

$\int \sinh(\sqrt{x})\frac{1}{\sqrt{x}}\,dx$ SUBSTITUCE

POMOCÍ VZORCE

$$\int \sinh(\sqrt{x})\frac{1}{\sqrt{x}}\,dx = 2\int \sinh(\sqrt{x})(\sqrt{x})'\,dx =$$

$$= 2\int \sinh(g)\,dg = 2\cosh(g) + C = \underline{\underline{2\cosh(\sqrt{x}) + C}}$$

POMOCÍ TABULKY

$$\int \sinh(\sqrt{x})\frac{1}{\sqrt{x}}\,dx = \left| \begin{aligned} g &= \sqrt{x} \\ dg &= \frac{1}{2\sqrt{x}}\,dx \\ 2dg &= \frac{1}{\sqrt{x}}\,dx \end{aligned} \right| = \int \sinh(g)\,2dg$$

... ATD

$\int \sqrt{1-x^2}\,dx$

SUBSTITUCE, VZOREC OBRÁCENĚ

$$\int f(x)\,dx = \int f(x(t))\,x'(t)\,dt$$

$$\int \sqrt{1-x^2}\,dx = \int \sqrt{1-\sin^2(t)}\,\cos(t)\,dt =$$

$$= \int \cos^2(t)\,dt = \int \frac{1+\cos(2t)}{2}\,dt = \frac{1}{2}\int 1 + \frac{1}{2}\int \cos(2t)\,dt =$$

$$= \frac{1}{2}t + \frac{1}{4}\sin(2t) + C = \frac{1}{2}\arcsin(x) + \frac{1}{4}\sin(2\arcsin(x)) + C$$

$$|x| \le 1$$

TABULKA: $\left| \begin{array}{l} x = \sin(t) \\ dx = \cos(t)\,dt \\ t = \arcsin(x) \end{array} \right|$

$\int (x+3)\cos(2x)\,dx$

PER PARTES: $\int u'v\,dx = uv - \int uv'\,dx$

$$\int (x+3)\cos(2x)\,dx = \left| \begin{array}{ll} u = x+3 & v' = \cos(2x) \\ u' = 1 & v = \frac{\sin(2x)}{2} \end{array} \right| =$$

$$= (x+3)\frac{\sin(2x)}{2} - \int \frac{\sin(2x)}{2}\,dx = (x+3)\frac{\sin(2x)}{2} + \frac{\cos(2x)}{4} + C$$

$\int x \ln x \, dx$

PER PARTES: $\int u v' \, dx = uv - \int u' v$

$$\int x \ln(x) \, dx = \left| \begin{matrix} u = \ln(x) & v' = x \\ u' = \frac{1}{x} & v = \frac{x^2}{2} \end{matrix} \right| = \frac{x^2}{2} \ln(x) - \int \frac{x}{2} \, dx =$$

$$= \underline{\underline{\frac{x^2}{2} \ln(x) - \frac{x^2}{4} + C \,, \quad x > 0}}$$

$\int e^{2x} \cos 3x \, dx$

PER PARTES

$$\underline{\int e^{2x} \cos(3x) \, dx} = \left| \begin{matrix} u = e^{2x} & v' = \cos(3x) \\ u' = 2e^{2x} & v = \frac{\sin(3x)}{3} \end{matrix} \right| =$$

$$= e^{2x} \frac{\sin(3x)}{3} - \frac{2}{3} \int e^{2x} \sin(3x) \, dx = \left| \begin{matrix} u = e^{2x} & v' = \sin(3x) \\ u' = 2e^{2x} & v = -\frac{\cos(3x)}{3} \end{matrix} \right| =$$

$$= e^{2x} \frac{\sin(3x)}{3} - \frac{2}{3} \left(-e^{2x} \frac{\cos(3x)}{3} + \frac{2}{3} \int e^{2x} \cos(3x) \, dx \right) =$$

$$= \underline{e^{2x} \frac{\sin(3x)}{3} + \frac{2}{9} e^{2x} \cos(3x) - \frac{4}{9} \int e^{2x} \cos(3x) \, dx} =$$

PLATÍ TEDY $I = e^{2x} \frac{\sin(3x)}{3} + \frac{2}{9} e^{2x} \cos(3x) - \frac{4}{9} I$

KDE I JE HLEDANÝ INTEGRÁL.

VYJÁDŘÍM $\underline{\underline{I = \frac{3}{13} e^{2x} \left(\sin(3x) + \frac{2}{3} \cos(3x) \right) + C \,, \quad x \in \mathbb{R}}}$

$\int 2\sin(x)\cos(x)\,dx = \int \sin(2x)\,dx = -\frac{\cos(2x)}{2} + C, \ x \in \mathbb{R}$

NEBO PER PARTES

$$\int 2\sin(x)\cos(x)\,dx = \begin{vmatrix} u = \sin(x) & v' = \cos(x) \\ u' = \cos(x) & v = \sin(x) \end{vmatrix} =$$

$$= 2\sin^2(x) - \int 2\sin(x)\cos(x)$$

PROČ TO NEVYŠLO STEJNĚ ?

VYJÁDŘÍM

$$\int 2\sin(x)\cos(x)\,dx = \sin^2(x) + C, \ x \in \mathbb{R}$$

NEBO SUBSTITUCE

$$\int 2\sin(x)\cos(x)\,dx = \begin{vmatrix} u = \sin(x) \\ du = \cos(x)\,dx \end{vmatrix} = \int 2u\,du = u^2 + C = \sin^2(x) + C, \ x \in \mathbb{R}$$

$\int \sin(3x)\cos(5x)\,dx$

$$\left.\begin{aligned} \sin(\alpha+\beta) &= \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) \\ \sin(\alpha-\beta) &= \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta) \end{aligned}\right\} \text{SEČTU}$$

$$\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin(\alpha)\cos(\beta)$$

$$\int \sin(3x)\cos(5x)\,dx = \frac{1}{2}\int \sin(8x)\,dx + \frac{1}{2}\int \sin(-2x)\,dx =$$

$$= -\frac{1}{16}\cos(8x) + \frac{1}{4}\cos(2x) + C, \ x \in \mathbb{R}$$

$\int (x+1)^5 x\,dx = \int t^5(t-1)\,dt = \int (t^6 - t^5)\,dt = \frac{t^7}{7} - \frac{t^6}{6} + C =$

SUBSTITUCE

$$\left| \begin{array}{l} t = x+1 \\ dt = dx \\ x = t-1 \end{array} \right|$$

$= \underline{\underline{\frac{(x+1)^7}{7} - \frac{(x+1)^6}{6} + C, \ x \in \mathbb{R}}}$

$\int \frac{4x}{\sqrt{2x-1}}\,dx = \int \frac{2(t^2+1)}{t}\, t\,dt = \int 2t^2 + 2\,dt =$

SUBSTITUCE

$$\left| \begin{array}{l} t = \sqrt{2x-1} \\ t^2 = 2x-1 \\ x = \frac{t^2+1}{2} \\ dx = t\,dt \end{array} \right|$$

$= 2\frac{t^3}{3} + 2t + C = \underline{\underline{2\frac{\sqrt{(2x-1)^3}}{3} + 2\sqrt{(2x-1)} + C}}$

$\underline{\underline{x > \frac{1}{2}}}$

NEBO SUBSTITUCE

$$\left| \begin{array}{l} t = 2x-1 \\ x = \frac{t+1}{2} \\ dx = \frac{dt}{2} \end{array} \right|$$

$= \int \frac{2(t+1)}{\sqrt{t}} \frac{dt}{2} = \int (t^{\frac{1}{2}} + t^{-\frac{1}{2}})\,dt =$

$= \frac{2t^{\frac{3}{2}}}{3} + 2t^{\frac{1}{2}} + C = \underline{\underline{2\frac{\sqrt{(2x-1)^3}}{3} + 2\sqrt{2x-1} + C}}$

$\underline{\underline{x > \frac{1}{2}}}$

$\int x^3\sqrt{x^2+1}\,dx$ $= \int (y-1)\sqrt{y}\,\frac{dy}{2} = \frac{1}{2}\int \left(y^{\frac{3}{2}} - y^{\frac{1}{2}}\right) dy = \frac{1}{5}y^{\frac{5}{2}} - \frac{1}{3}y^{\frac{3}{2}} + C =$

SUBSTITUCE

$$\left| \begin{aligned} y &= x^2+1 \\ dy &= 2x\,dx \\ x^2 &= y-1 \end{aligned} \right|$$

$$= \underline{\underline{\frac{1}{5}\sqrt{(x^2+1)^5} - \frac{1}{3}\sqrt{(x^2+1)^3} + C\ ,\ x\in\mathbb{R}}}$$

NEBO SUBSTITUCE

$$\left| \begin{aligned} x &= tg(t) \\ dx &= \frac{1}{\cos^2(t)}\,dt \\ t &= arctg(x) \end{aligned} \right|$$

$$= \int \cos^3(t)\sqrt{tg^2(t)+1}\,\frac{1}{\cos^2(t)}\,dt =$$

$$= \int \frac{\sin^3(t)}{\cos^3(t)}\sqrt{\frac{\sin^2(t)+\cos^2(t)}{\cos^2(t)}}\,\frac{1}{\cos^2(t)}\,dt =$$

SUBSTITUCE

$$= \int \frac{\sin^3(t)}{\cos^6(t)}\,dt = \left| \begin{aligned} u &= \cos(t) \\ du &= -\sin(t)\,dt \\ -du &= \sin(t)\,dt \end{aligned} \right| = \int \frac{u^2-1}{u^6}\,du = \ldots$$ ATD.

NEBO SUBSTITUCE

$$\left| \begin{aligned} x &= \sinh(t) \\ dx &= \cosh(t)\,dt \\ t &= argsinh(x) \end{aligned} \right|$$

$$= \int \sinh^3(t)\sqrt{\sinh^2(t)+1}\,\cosh(t)\,dt =$$

$$= \int \sinh^2(t)\sqrt{\sinh^2(t)+1}\,\cosh(t)\sinh(t)\,dt =$$

VZOREC $\cosh^2(t) - \sinh^2(t) = 1$

SUBSTITUCE

$$= \int (\cosh^2(t)-1)\cosh^2(t)\sinh(t)\,dt = \left| \begin{aligned} u &= \cosh(t) \\ du &= \sinh(t)\,dt \end{aligned} \right| =$$

$$= \int (u^2-1)u^2\,du = \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{\cosh^5(t)}{5} - \frac{\cosh^3(t)}{3} + C =$$

$$= \underline{\underline{\frac{\cosh^5(argsinh(x))}{5} - \frac{\cosh^3(argsinh(x))}{3} + C\ ,\ x\in\mathbb{R}}}$$