Matematická analýza 1, 1.paralelka - obory EEM a EK, sedmé cvičení

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1 Integrace racionálních funkcí

1.1 Napište tvar pro rozklad na parciální zlomky pro funkci $\frac{2x^2+3x+18}{(x+9)^3(x^2-4)^2(x^2+4x+8)^2}$. $=$

TYPY PARCIÁLNÍCH ZLOMKŮ

$$\frac{A}{(x-\alpha)^m} \qquad \frac{Bx+C}{(x^2+px+q)^n}$$

KOŘEN $\alpha \in \mathbb{R}$

REÁLNÉ KOŘENY NEJSOU

$$= \frac{2x^2+3x+18}{(x+9)^3(x-2)^2(x+2)^2(x^2+4x+8)^2} =$$

$$= \frac{A}{(x+9)^3} + \frac{B}{(x+9)^2} + \frac{C}{x+9} + \frac{D}{(x+2)^2} + \frac{E}{x+2} + \frac{F}{(x-2)^2} + \frac{G}{x-2} + \frac{Ix+J}{(x^2+4x+8)^2} + \frac{Kx+L}{x^2+4x+8}$$

1.2 Spočtěte.

$\int \frac{x^3-2x+5}{x^2-x-2}\,dx$

STUPEŇ POLYNOMU V ČITATELI JE VĚTŠÍ NEŽ VE JMENOVATELI, DĚLÍM

$$(x^3-2x+5):(x^2-x-2) = x+1+\frac{x+7}{x^2-x-2} = x+1+\frac{x+7}{(x-2)(x+1)}$$

$+x^2 \quad +5$

$x+7$

ROZKLAD

$$\frac{x+7}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1} \quad \Big/ \cdot (x-2)(x+1)$$

$$x+7 = A(x+1) + B(x-2)$$

POROVNÁNÍ POLYNOMŮ

$x^1: \; 1 = A+B \qquad 1 = A+B$

$x^0: \; 7 = A-2B \qquad 6 = -3B \quad \Big| \; R_2 - R_1$

$\underline{A=3}$, $\underline{B=-2}$

DOSAZENÍ KOŘENŮ

$x=-1: \; 6 = B\cdot(-3) \qquad \underline{B=-2}$

$x=2: \; 9 = A\cdot 3 \qquad \underline{A=3}$

INTEGRACE

$$\int \frac{x^3-2x+5}{x^2-x-2}\,dx = \int \left(x+1+\frac{3}{x-2}-\frac{2}{x+1}\right)dx = \underline{\frac{x^2}{2}+x+3\ln|x-2|-2\ln|x+1|+C}$$

$\underline{x \notin \{2,-1\}}$

$\int \frac{6x^2-11x-9}{(x-3)(x+1)x}\,dx = \int \left(\frac{A}{x-3} + \frac{B}{x+1} + \frac{C}{x}\right) dx = \underline{\underline{\ln|x+3| + 2\ln|x+1| + 3\ln|x| + C,\ x \notin \{3; -1, 0\}}}$

ZAKRÝVACÍ METODA – ZRYCHLENÉ DOSAZENÍ KOŘENŮ

$6x^2 - 11x - 9 = A(x+1)x + B(x-3)x + C(x-3)(x+1)$

PRO KOŘEN $x = 3$ $\quad 6\cdot 3^2 - 11\cdot 3 - 9 = A(3+1)\cdot 3 \rightarrow \underline{A} = \frac{6\cdot 3^2 - 11\cdot 3 - 9}{(3+1)\cdot 3} = \underline{1}$

TOTÉŽ ZÍSKÁM DOSAZENÍM $x=3$ DO $\frac{6x^2-11x-9}{(x-3)(x+1)x}$ PŘI ZAKRYTÍ $x-3$

PODOBNĚ PRO KOŘEN $x = -1$ $\quad \frac{6x^2-11x-9}{(x-3)(x+1)x} \rightarrow \underline{B} = \frac{6\cdot(-1)^2 - 11(-1) - 9}{(-1-3)\cdot(-1)} = \underline{2}$

A $x = 0$ $\quad \frac{6x^2-11x-9}{(x-3)(x+1)x} \rightarrow \underline{C} = \frac{6\cdot 0^2 - 11\cdot 0 - 9}{(0-3)\cdot(0+1)} = +\underline{3}$

$\int \frac{2x^2+3x+18}{x^3+9x}\,dx$

$\frac{2x^2+3x+18}{x^3+9x} = \frac{2x^2+3x+18}{x(x^2+9)} = \frac{A}{x} + \frac{Bx+C}{x^2+9}$

ZAKRÝVACÍ METODA $\quad x = 0 \quad \frac{2x^2+3x+18}{x(x^2+9)} \rightarrow \underline{A} = \frac{2\cdot 0^2 + 3\cdot 0 + 18}{(0^2+9)} = \underline{2}$

$x = 3j \quad \frac{2x^2+3x+18}{x(x^2+9)} \rightarrow B\cdot 3j + C = \frac{2(3j)^2 + 3\cdot 3j + 18}{3j} \rightarrow$

$\rightarrow -9B + 3jC = -18 + 9j + 18 \rightarrow \underline{B = 0,\ C = 3}$

POROVNÁNÍ POLYNOMŮ

$2x^2 + 3x + 18 = A(x^2+9) + (Bx+C)x$

$x^2:\ 2 = A + B \qquad \underline{B = 0}$

$x^1:\ \underline{3 = C}$

$x^0:\ 18 = 9A \qquad \underline{A = 2}$

INTEGRACE

$\int \left(\frac{2}{x} + \frac{3}{x^2+9}\right) dx = 2\ln|x| + \frac{3}{9}\int \frac{1}{\left(\frac{x}{3}\right)^2+1}\,dx = \underline{\underline{2\ln|x| + \frac{1}{3}\cdot 3\,\mathrm{arctg}\left(\frac{x}{3}\right) + C}}$

$\underline{\underline{x \neq 0}}$

2 Substituce vedoucí na racionální funkce

2.1 Spočtěte.

$$\int \frac{2\sin^3(x)+6\sin^2(x)-1}{\sin^2(x)(\sin(x)+1)^2}\cos(x)\,dx = \left| \begin{matrix} t=\sin(x) \\ dt=\cos(x)\,dx \end{matrix} \right| = \int \underbrace{\frac{2t^3+6t^2-1}{t^2(t+1)^2}}_{LS} dt =$$

ZAKRÝVACÍ METODA

$$= \int \underbrace{\left(\frac{A}{t} + \frac{-1}{t^2} + \frac{C}{t+1} + \frac{3}{(t+1)^2}\right)}_{PS} dt =$$

$$LS = PS \quad / \cdot t^2(t+1)^2$$

$$2t^3+6t^2-1 = At(t+1)^2 - 1(t+1)^2 + Ct^2(t+1) + Dt^2$$

POROVNÁNÍ POLYNOMŮ – STAČÍ 2 ROVNICE

$$t^3:\ 2 = A + C \qquad \underline{C=0}$$

$$t^1:\ 0 = A - 2 \qquad \underline{A=2}$$

$$= 2\int \frac{1}{t}\,dt - 1\int \frac{1}{t^2}\,dt + 3\int \frac{1}{(t+1)^2}\,dt = 2\ln|t| + \frac{1}{t} - 3\frac{1}{t+1} + C \ \ldots \text{ATD.}$$

$$\int \frac{\ln(x)+1}{\ln^2(x)+4\ln(x)+13}\,\frac{dx}{x} = \left| \begin{matrix} t=\ln(x) \\ dt=\frac{dx}{x} \end{matrix} \right| = \int \frac{t+1}{\underbrace{t^2+4t+13}_{\text{PARC. ZLOMEK TYP 2}}}\,dt =$$

$$= \frac{1}{2}\left(\int \underbrace{\frac{2t+4}{t^2+4t+13}\,dt}_{} - \int \underbrace{\frac{2}{(t+2)^2+9}\,dt}_{}\right) = \frac{1}{2}\ln|t^2+4t+13| - \frac{1}{9}\int \frac{1}{\left(\frac{t+2}{3}\right)^2+1}\,dt =$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C \qquad \int \frac{1}{x^2+1}\,dx = \operatorname{arctg}(x) + C$$

$$= \frac{1}{2}\ln|t^2+4t+13| - \frac{1}{3}\operatorname{arctg}\left(\frac{t+2}{3}\right) + C =$$

$$= \underline{\underline{\frac{1}{2}\ln|\ln^2(x)+4\ln(x)+13| - \frac{1}{3}\operatorname{arctg}\left(\frac{\ln(x)+2}{3}\right) + C, \quad x>0}}$$

ZAKRÝVACÍ METODA

$$\int \frac{6\,dx}{e^{2x}-e^{x}+2} = \left| \begin{array}{l} t=e^x \\ dt=e^x dx \\ dx=\frac{dt}{t} \end{array} \right| = \underbrace{\int \frac{6}{t^2-t+2}\frac{dt}{t}}_{LS} = \int \underbrace{\left(\frac{3}{t}+\frac{At+B}{t^2-t+2}\right)}_{PS} dt =$$

$$LS = PS \;/ (t^2-t+2)t$$

$$6 = 3(t^2-t+2)+(At+B)t$$

$$t^2:\ 0=3+A \qquad \underline{A=-3}$$

$$t:\ 0=-3+B \qquad \underline{B=3}$$

$$=3\int\frac{1}{t}dt+\int\frac{-3t+3}{t^2-t+2}dt = 3\ln|t| - \frac{3}{2}\left(\int\frac{2t-1}{t^2-t+2}dt+\int\frac{-1}{(t-\frac{1}{2})^2+\frac{7}{4}}dt\right)=$$

$$3\ln|t|-\frac{3}{2}\ln|t^2-t+2|+\frac{3}{2}\frac{4}{7}\int\frac{1}{\left(\frac{t-\frac{1}{2}}{\frac{\sqrt{7}}{2}}\right)^2+1}dt = 3\ln|t|-\frac{3}{2}\ln|t^2-t+2|+\frac{6}{7}\frac{\sqrt{7}}{2}\operatorname{arctg}\left(\frac{2}{\sqrt{7}}\left(t-\frac{1}{2}\right)\right)+C=$$

$$= \ldots \text{ ATD}$$

$$\int \frac{2\sqrt[3]{2x-4}}{\sqrt[2]{2x-4}(\sqrt[2]{2x-4}+\sqrt[3]{2x-4})}\,dx = \int \frac{2(2x-4)^{\frac{1}{3}}}{(2x-4)^{\frac{1}{2}}\left((2x-4)^{\frac{1}{2}}+(2x-4)^{\frac{1}{3}}\right)}dx =$$

$$= \left| \begin{array}{l} t=(2x-4)^{\frac{1}{6}} \\ t^2=(2x-4)^{\frac{1}{3}} \\ t^3=(2x-4)^{\frac{1}{2}} \\ x=\frac{t^6+4}{2} \\ dx=3t^5\,dt \end{array} \right| = \int\frac{2t^2}{t^3(t^3+t^2)}3t^5\,dt = \int\frac{6t^2}{t+1}\,dt =$$

DĚLÍM $6t^2:(t+1)=6t-6+\frac{6}{t+1}$

$$\begin{array}{l} -6t \\ \quad +6 \end{array}$$

$$=6\int t-1+\frac{1}{t+1}\,dt = 6\left(\frac{t^2}{2}-t+\ln|t+1|\right)+C=$$

$$\underline{\underline{=3\sqrt[3]{2x-4}-6\sqrt[6]{2x-4}+6\ln\left|\sqrt[6]{2x-4}+1\right|+C}}, \quad \underline{\underline{x>2}}$$

PODOBNĚ SE ŘEŠÍ PŘÍKLAD 10/7 ℓ) Z 5. DÚ.

2.2 Převeďte na integrál racionální funkce.

$\int \frac{\sin(x)\cos(x)+2}{\cos^2(x)+3}\,dx$

SUBSTITUCE

$$\left| \begin{array}{l} t = tg(x) \quad x = arctg(t) \qquad dx = \frac{1}{t^2+1}\,dt \\ \sin(x)\cos(x) = \frac{\sin(x)\cos(x)}{\sin^2(x)+\cos^2(x)} \cdot \frac{\frac{1}{\cos^2(x)}}{\frac{1}{\cos^2(x)}} = \frac{tg(x)}{tg^2(x)+1} = \frac{t}{t^2+1} \\ \cos^2(x) = \frac{\cos^2(x)}{\sin^2(x)+\cos^2(x)} \cdot \frac{\frac{1}{\cos^2(x)}}{\frac{1}{\cos^2(x)}} = \frac{1}{tg^2(x)+1} = \frac{1}{t^2+1} \end{array} \right|$$

$$\int \frac{\sin(x)\cos(x)+2}{\cos^2(x)+3}\,dx = \int \frac{\frac{t}{t^2+1}+2}{\frac{1}{t^2+1}+3} \cdot \frac{1}{t^2+1}\,dt = \underline{\underline{\int \frac{t+2t^2+2}{(1+3t^2+3)(t^2+1)}\,dt}}$$

PODOBNĚ SE ŘEŠÍ PŘÍKLAD 10/4 c) Z 5. DÚ

SPOČTĚTE

$$\int \sin^4(x)\cdot\cos^2(x)\,dx = $$

$t = \operatorname{tg} x \quad x = \operatorname{arctg} t$

$$\cos^2(x) = \frac{1}{t^2+1} \quad dx = \frac{1}{t^2+1}\,dt$$

$$\sin^2(x) = \frac{\sin^2(x)}{\sin^2(x)+\cos^2(x)} \cdot \frac{\frac{1}{\cos^2 x}}{\frac{1}{\cos^2 x}} = \frac{\operatorname{tg}^2(x)}{\operatorname{tg}^2(x)+1} = \frac{t^2}{t^2+1}$$

$$= \int \frac{t^4}{(t^2+1)^2}\,\frac{1}{t^2+1}\,dt = \int \frac{t^4}{(t^2+1)^3}\,dt$$

RACIONÁLNÍ FUNKCE, LZE, ALE VELMI PRACNÉ

LEPŠÍ DLE VZORCŮ

$$\sin^2(x) = \frac{1-\cos(2x)}{2} \qquad \cos^2(x) = \frac{1+\cos(2x)}{2}$$

$$\int \sin^4(x)\cos^2(x)\,dx = \int \frac{\left(1-\cos(2x)\right)^2\left(1+\cos(2x)\right)}{8}\,dx =$$

$$= \frac{1}{8}\int \big(\underbrace{1-\cos^2(2x)}_{\sin^2(2x)}\big)\big(1-\cos(2x)\big)\,dx =$$

$$= \frac{1}{8}\int \left(\sin^2(2x) - \sin^2(2x)\cos(2x)\right)dx =$$

$$= \frac{1}{8}\int \frac{1-\cos(4x)}{2}\,dx - \frac{1}{8}\int \sin^2(2x)\cos(2x)\,dx = \left| \begin{array}{l} t = \sin(2x) \\ dt = 2\cos(2x)\,dx \\ \frac{dt}{2} = \cos(2x)\,dx \end{array} \right| =$$

$$= \frac{1}{16}\left(x - \frac{\sin(4x)}{4}\right) - \frac{1}{16}\int t^2\,dt =$$

$$= \frac{1}{16}x - \frac{\sin(4x)}{64} - \frac{t^3}{48} + C =$$

$$= \frac{1}{16}x - \frac{\sin(4x)}{64} - \frac{\sin^3(2x)}{48} + C, \; x \in \mathbb{R}$$