Matematická analýza 1, 1.paralelka - obory EEM a EK, osmé cvičení

Karel Pospíšil

f(x) JE SPOJITÁ NA ⟨a, b⟩

$$\int_a^b f(x)\,dx = \left[F(x)\right]_a^b = F(b) - F(a)$$

1 Určitý integrál

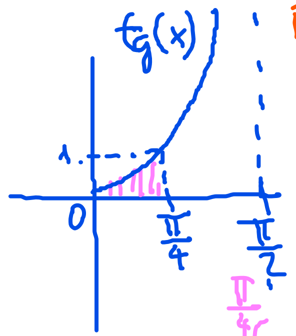
1.1 Spočtěte.

$\int\limits_0^{\pi/4} \mathrm{tg}(x)\,dx \quad [\frac{1}{2}\ln(2)]$ $=$

tg(x) SPOJITÁ NA $\langle 0, \frac{\pi}{4} \rangle$

$$= -\int_0^{\frac{\pi}{4}} \frac{-\sin(x)}{\cos(x)}\,dx = -\left[\ln|\cos(x)|\right]_0^{\frac{\pi}{4}} = -\left(\ln\left|\cos\left(\tfrac{\pi}{4}\right)\right| - \ln|\cos(0)|\right) =$$

$$= -\left(\ln\frac{\sqrt{2}}{2} - 0\right) = \ln\frac{2}{\sqrt{2}} = \underline{\underline{\frac{1}{2}\ln 2}}$$



PŘIBLIŽNĚ

$$0 < \int_0^{\frac{\pi}{4}} \mathrm{tg}(x)\,dx < \frac{\pi}{8}$$

$$\int_{-1/4}^{0} \arcsin(2x)\,dx \quad \left[-\tfrac{1}{24}\pi - \tfrac{1}{4}\sqrt{3} + \tfrac{1}{2}\right]$$

SUBSTITUCE

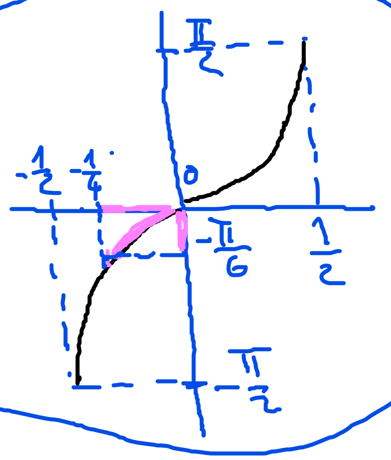
$2x = \sin(t) \qquad x = \dfrac{\sin(t)}{2}$

$t = \arcsin(2x) \qquad dx = \dfrac{\cos(t)}{2}\,dt$

$t = \arcsin(2x)$

JE SPOJ. A MÁ SPOJ. DER. NA $\langle -\frac{1}{4}, 0\rangle$

$\langle -\frac{1}{4}, 0\rangle \to \langle -\frac{\pi}{6}, 0\rangle$



$$\int_{-\frac{1}{4}}^{0} \arcsin(2x)\,dx = \int_{-\frac{\pi}{6}}^{0} t\,\frac{\cos(t)}{2}\,dt =$$

PER PARTES

$$= \begin{vmatrix} u = t & v' = \cos(t) \\ u' = 1 & v = \sin(t) \end{vmatrix} = \frac{1}{2}\left(\Big[t\sin(t)\Big]_{-\frac{\pi}{6}}^{0} - \int_{-\frac{\pi}{6}}^{0} \sin(t)\,dt\right) =$$

$$= \frac{1}{2}\left(0 - \left(-\frac{\pi}{6}\right)\sin\left(-\frac{\pi}{6}\right) + \Big[\cos(t)\Big]_{-\frac{\pi}{6}}^{0}\right) = \frac{1}{2}\left(-\frac{\pi}{6}\frac{1}{2} + 1 - \frac{\sqrt{3}}{2}\right) = \underline{\underline{-\frac{\pi}{24} + \frac{1}{2} - \frac{\sqrt{3}}{4}}}$$

NEBO PER PARTES

$$\int_{-\frac{1}{4}}^{0} \arcsin(2x)\,dx = \begin{vmatrix} u = \arcsin(2x) & v' = 1 \\ u' = \dfrac{2}{\sqrt{1-4x^2}} & v = x \end{vmatrix} =$$

$$= \Big[x\arcsin(2x)\Big]_{-\frac{1}{4}}^{0} - \int_{-\frac{1}{4}}^{0} \frac{2x}{\sqrt{1-4x^2}}\,dx =$$

SUBSTITUCE

$t = 1 - 4x^2$

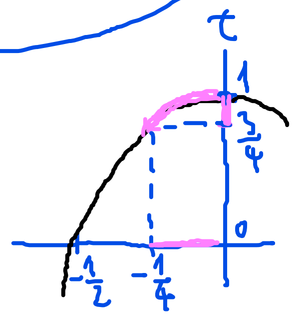
$dt = -8x\,dx$

$\dfrac{dt}{-4} = 2x\,dx$

$t = 1 - 4x^2$

JE SPOJ. A MÁ SPOJ. DER. NA $\langle -\frac{1}{4}, 0\rangle$

$\langle -\frac{1}{4}, 0\rangle \to \langle \frac{3}{4}, 1\rangle$



$$= 0 - \left(-\frac{1}{4}\arcsin\left(-\frac{1}{2}\right)\right) + \frac{1}{4}\int_{\frac{3}{4}}^{1} \frac{dt}{\sqrt{t}} = -\frac{1}{4}\frac{\pi}{6} + \frac{1}{4}\Big[2\sqrt{t}\Big]_{\frac{3}{4}}^{1} =$$

$$= -\frac{\pi}{24} + \frac{1}{4}\left(2 - 2\frac{\sqrt{3}}{2}\right) = \underline{\underline{-\frac{\pi}{24} + \frac{1}{2} - \frac{\sqrt{3}}{4}}}$$

$\int_1^5 \frac{dx}{9+\sqrt{x-1}}$ $[-9\ln(\frac{77}{81}) + 4 - 18\,\mathrm{arctgh}(\frac{2}{9})]$

$\frac{1}{9+\sqrt{x-1}}$ SPOJ. NA $\langle 1,5\rangle$

SUBSTITUCE

$t = \sqrt{x-1}$ $\quad x = t^2+1$ $\quad$ PODMÍNKY SAMOSTATNĚ

$\langle 1,5\rangle \to \langle 0,2\rangle$ $\quad dx = 2t\,dt$

$$\int_0^2 \frac{2t\,dt}{9+t} = 2\left(\int_0^2 \frac{t+9}{9+t}dt - \int_0^2 \frac{9}{9+t}dt\right) = 2\left(\int_0^2 1\,dt - 9\int_0^2 \frac{1}{9+t}dt\right) =$$

$$= 2\left(2 - 9\Big[\ln|t+9|\Big]_0^2\right) = \underline{\underline{4 - 18(\ln 11 - \ln 9)}}$$

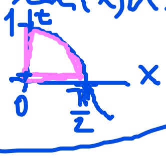
COŽ JE STEJNÉ JAKO VÝSLEDEK VÝŠE – ZKUSTE VYČÍSLIT

$$\int_0^{\pi/2} \underbrace{\sin(x)(2\cos(x)-1)e^{\cos(x)}}_{\text{SPOJ. NA } \langle 0,\frac{\pi}{2}\rangle}\,dx \quad [-e+3]$$

SUBSTITUCE

$t = \cos(x) \quad dt = -\sin(x)\,dx$

$\langle 0, \frac{\pi}{2} \rangle \to \langle 1, 0 \rangle$



$$\int_0^{\frac{\pi}{2}} \sin(x)(2\cos(x)-1)e^{\cos(x)}\,dx = -\int_1^0 (2t-1)e^t\,dt =$$

PER PARTES

$$= \begin{vmatrix} u = 2t-1 & v' = e^t \\ u' = 2 & v = e^t \end{vmatrix} = -\Big[(2t-1)e^t\Big]_1^0 + 2\Big[e^t\Big]_1^0 = -(-1-1e)+2-2e = \underline{\underline{3-e}}$$

$\int\limits_0^r \sqrt{r^2 - x^2}dx, r > 0 \quad [\left[r^2(\frac{arcsin\frac{x}{r}}{2} + \frac{sin2arcsin\frac{x}{r}}{4})\right]_0^r = r^2\frac{\pi}{4}]$

2 Nevlastní integrál

2.1 Spočtěte.

$\int_0^\infty \frac{dx}{x^2-x-2}$ [neexistuje]

$x^2-x-2=(x-2)(x+1)$

$$= \int_0^2 \frac{dx}{x^2-x-2} + \int_2^\infty \frac{dx}{x^2-x-2} = \int_0^2 \dots + \int_2^3 \dots + \int_3^\infty \dots =$$

$$= \lim_{a\to 2^-} \int_0^a \dots + \lim_{a\to 2^+} \int_a^3 \dots + \lim_{a\to\infty} \int_3^a \dots =$$

$$\int \frac{1\,dx}{(x-2)(x+1)} = \int \frac{\frac{1}{3}}{x-2} - \frac{\frac{1}{3}}{x+1}\,dx = \frac{1}{3}\left(\ln|x-2| - \ln|x+1|\right)$$

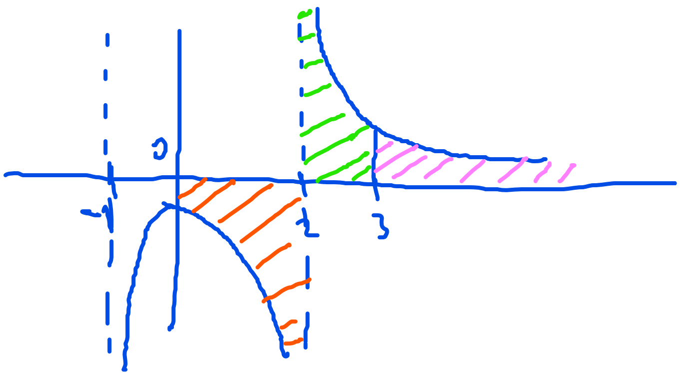
$$= \frac{1}{3}\left(\lim_{a\to 2^-}\Big[\ln|x-2| - \ln|x+1|\Big]_0^a + \lim_{a\to 2^+}\Big[\ln|x-2| - \ln|x+1|\Big]_a^3 + \lim_{a\to\infty}\Big[\ln|x-2| - \ln|x+1|\Big]_3^a\right) =$$

$$= \frac{1}{3}\left(\lim_{a\to 2^-}\left(\ln|a-2| - \ln|a+1| - \ln 2\right) + \lim_{a\to 2^+}\left(-\ln 4 - \ln|a-2| + \ln|a+1|\right) + \lim_{a\to\infty}\left(\underbrace{\ln|a-2| - \ln|a+1|}_{\ln\left|\frac{a-2}{a+1}\right| \xrightarrow{a\to\infty} \ln 1 = 0} + \ln 4\right)\right)$$

$$= \frac{1}{3}\left(-\infty - \ln 3 - \ln 2 \quad - \ln 4 + \infty + \ln 3 \quad + \infty - \infty + \ln 4\right)$$

$$= -\infty + \infty + \ln 4 =$$

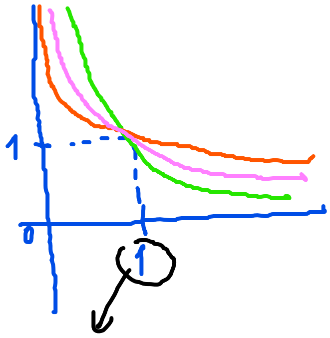
$$= \infty - \infty \to \text{NEEX.}$$



$$\int_0^9 \frac{dx}{\sqrt{x}} = \left[2\sqrt{x}\right]_0^9 = 2\cdot(3-0) = \underline{\underline{6}} \qquad \underline{\underline{K}}$$

$\int\limits_0^9 \frac{dx}{\sqrt{x^2}}$ [∞] $= \int\limits_0^9 \frac{dx}{x} = \left[\ln(x)\right]_0^9 = \ln 9 - \lim\limits_{x\to 0^+} \ln(x) = \underline{\underline{+\infty}} \quad \underline{\underline{D}}$

$$\int_9^\infty \frac{dx}{\sqrt{x^3}} = \left[-\frac{2}{\sqrt{x}}\right]_9^\infty = \lim_{x\to\infty}\left(-\frac{2}{\sqrt{x}}\right) + \frac{2}{\sqrt{9}} = \frac{2}{\sqrt{9}} = \underline{\underline{\frac{2}{3}}} \qquad \underline{\underline{K}}$$



OBECNĚ $K \in \mathbb{R}^+$

$$\int_0^1 \frac{1}{x^a}\,dx = \begin{cases} \text{KONVERGUJE} & 0<a<1 \\ \text{DIVERGUJE} & a=1 \\ \text{DIVERGUJE} & a>1 \end{cases}$$

$$\int_1^\infty \frac{1}{x^a}\,dx = \begin{cases} \text{DIVERGUJE} & 0<a<1 \\ \text{DIVERGUJE} & a=1 \\ \text{KONVERGUJE} & a>1 \end{cases}$$

$$\int_0^\infty \frac{1}{x^a}\,dx = \text{DIVERGUJE VŽDY}$$

$\int\limits_{-\infty}^{\infty} x^n dx, n \in \mathbb{N}$ [∞ pro n sudé, neexistuje pro n liché]

SUBSTITUCE

$$\int_{\frac{2}{\pi}}^{\infty} \frac{1}{x^3} sin\frac{1}{x} dx \quad [1] \quad = \left| \begin{matrix} t = \frac{1}{x} & dt = \frac{-1}{x^2} dx \\ \langle \frac{2}{\pi}, \infty) \to (\frac{\pi}{2}, 0\rangle & \end{matrix} \right| = -\int_{\frac{\pi}{2}}^{0} t \sin(t)\, dt =$$

PER PARTES

$$= \left| \begin{matrix} u = t & v' = \sin(t) \\ u' = 1 & v = -\cos(t) \end{matrix} \right| = \left[t \cos(t) \right]_{\frac{\pi}{2}}^{0} - \int_{\frac{\pi}{2}}^{0} \cos(t)\, dt =$$

$$= -\left[\sin(t) \right]_{\frac{\pi}{2}}^{0} = -(0-1) = \underline{\underline{1}}$$

NEBO ZJISTÍM JEN KONVERGENCI

SROVNÁVACÍ KRITÉRIUM, PROTOŽE $|\sin(\frac{1}{x})| \leq 1$

A $\left| \frac{1}{x^3} \sin(\frac{1}{x}) \right| \leq \frac{1}{x^3}$ NA $(\frac{2}{\pi}, \infty)$

A $\int_{\frac{2}{\pi}}^{\infty} \frac{1}{x^3} dx$ KONVERGUJE

PAK $\int_{\frac{2}{\pi}}^{\infty} \frac{1}{x^3} \sin(\frac{1}{x}) dx$ KONVERGUJE