MA2 - Jedenácté cvičení

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1 Fourierovy řady

1.1 Najděte Fourierovu řadu, Fourierovu sinovou řadu a Fourierovu kosinovou řadu funkce $f(t)$. Určete jejich součty.

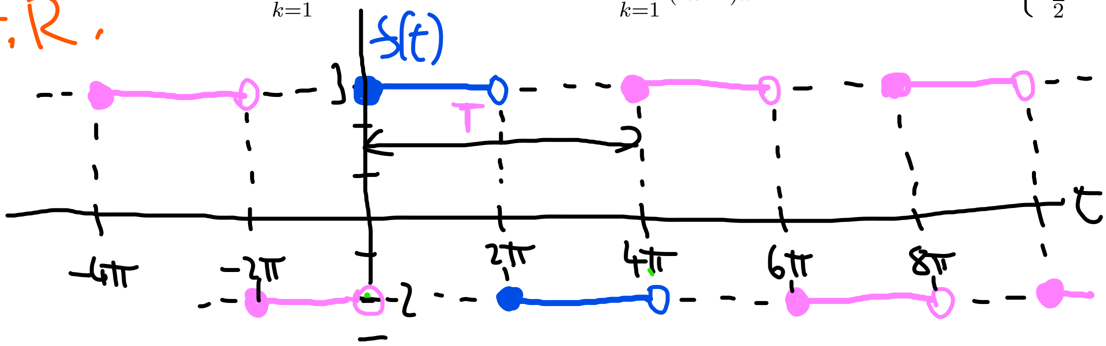
$$f(t) = \begin{cases} 3 & \text{pro } t \in \langle 0, 2\pi) \\ -2 & \text{pro } t \in \langle 2\pi, 4\pi) \end{cases}$$

$$[\tfrac{1}{2} + \sum_{k=1}^{\infty} \tfrac{10}{(2k-1)\pi} \sin \tfrac{2k-1}{2} t = \begin{cases} f(t) & \text{pro } t \in (0, 2\pi) \cup (2\pi, 4\pi) \\ \tfrac{1}{2} & \text{pro } t \in \{0, 2\pi\} \end{cases}]$$

$$[\sum_{k=1}^{\infty} \tfrac{2}{k\pi}(-5 \cos k\tfrac{\pi}{2} + 3 + 2\cos k\pi) \sin \tfrac{1}{4}kt = \begin{cases} f(t) & \text{pro } t \in (0, 2\pi) \cup (2\pi, 4\pi) \\ 0 & \text{pro } t \in \{0\} \\ \tfrac{1}{2} & \text{pro } t \in \{2\pi\} \end{cases}]$$

$$[\tfrac{1}{2} + \sum_{k=1}^{\infty} \tfrac{10}{k\pi} \sin \tfrac{1}{2}k\pi \cos \tfrac{1}{4}kt = \tfrac{1}{2} + \sum_{k=1}^{\infty} \tfrac{10}{(2k-1)\pi}(-1)^{k-1} \cos \tfrac{2k-1}{4} t = \begin{cases} f(t) & \text{pro } t \in \langle 0, 2\pi) \cup (2\pi, 4\pi) \\ \tfrac{1}{2} & \text{pro } t \in \{2\pi\} \end{cases}]$$

1) F.Ř.



VZOREC: $\omega = \frac{2\pi}{T}$

DOPLNĚNÍ $f(t)$ NA PERIODICKOU KDE: $T = 4\pi$, $\omega = \frac{2\pi}{4\pi} = \frac{1}{2}$

F. ŘADA: $f(t) \approx \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos(k\omega t) + b_k \sin(k\omega t)\right)$

KDE: $a_0 = \frac{2}{T}\int_d^{d+T} f(t)\,dt$, $a_k = \frac{2}{T}\int_d^{d+T} f(t)\cos(k\omega t)\,dt$, $b_k = \frac{2}{T}\int_d^{d+T} f(t)\sin(k\omega t)\,dt$

d VOLÍM JAK CHCI, VÝSLEDEK TÍM NEZMĚNÍM, ZKUSÍM $d = 0$

TAKŽE: $a_0 = \frac{1}{2\pi}\int_0^{4\pi} f(t)\,dt = \frac{1}{2\pi}(6\pi - 4\pi) = \underline{1}$

$\underline{a_k} = \frac{1}{2\pi}\int_0^{4\pi} f(t)\cos(\frac{k}{2}t)\,dt = \frac{3}{2\pi}\int_0^{2\pi}\cos(\frac{k}{2}t)\,dt - \frac{1}{\pi}\int_{2\pi}^{4\pi}\cos(\frac{k}{2}t)\,dt = \frac{3}{\pi k}\left[\sin(\frac{k}{2}t)\right]_0^{2\pi} - \frac{2}{k\pi}\left[\sin(\frac{k}{2}t)\right]_{2\pi}^{4\pi} =$

$= \underline{0}$

$$b_k = \frac{1}{2\pi}\int_0^{4\pi} f(t)\sin\left(\frac{k}{2}t\right)dt = \frac{3}{2\pi}\int_0^{2\pi}\sin\left(\frac{k}{2}t\right)dt - \frac{1}{\pi}\int_{2\pi}^{4\pi}\sin\left(\frac{k}{2}t\right)dt = \frac{-3}{\pi k}\left[\cos\left(\frac{k}{2}t\right)\right]_0^{2\pi} + \frac{2}{k\pi}\left[\cos\left(\frac{k}{2}t\right)\right]_{2\pi}^{4\pi} =$$

$$= \frac{-3}{k\pi}(\cos(k\pi)-1) + \frac{2}{k\pi}(1-\cos(k\pi)) = -\frac{5}{k\pi}(-1)^k + \frac{5}{k\pi} = \frac{5}{k\pi}\left(1-(-1)^k\right) =$$

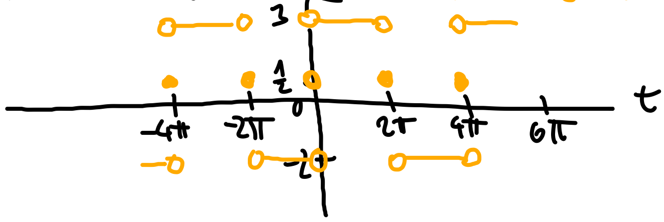
$$= \begin{cases} \frac{10}{k\pi} & \dots k \text{ LICHÉ} \\ 0 & \dots k \text{ SUDÉ} \end{cases}$$

$$f(t) \approx \frac{1}{2} + \sum_{k=1}^{\infty} \frac{5}{k\pi}\left(1-(-1)^k\right)\sin\left(\frac{k}{2}t\right) = \begin{cases} f(t) & \dots x \in (0,2\pi)\cup(2\pi,4\pi) \\ \frac{1}{2} & \dots x \in \{0, 2\pi\} \end{cases}$$

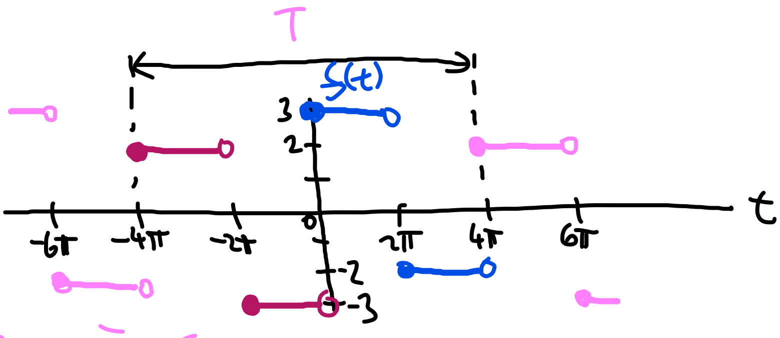
NEBO

$$f(t) \approx \frac{1}{2} + \sum_{k=1}^{\infty} \frac{10}{(2k-1)\pi}\sin\left(\frac{2k-1}{2}t\right) = \frac{1}{2} + \sum_{k=1}^{\infty}\frac{10}{(2k-1)\pi}\sin\left(\frac{2k-1}{2}t\right) = \dots \text{ VIZ VÝŠE.}$$

ŘADA KONVERGUJE K ORANŽOVÉ



② SINOVÁ FURIEROVA ŘADA



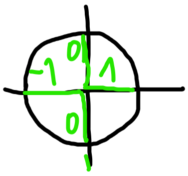
DOPLNĚNÍ NA PERIODICKOU JE LICHÉ, $T = 8\pi$, $\omega = \frac{2\pi}{8\pi} = \frac{1}{4}$

$$f(t) \approx \sum_{k=1}^{\infty} b_k \sin(k\omega t)$$

$$b_k = \frac{2}{T}\int_{d}^{d+T} f(t)\sin(k\omega t) \overset{d \text{ volím } -\frac{T}{2}}{=} \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\sin(k\omega t) \overset{f(t)\sin(k\omega t) \text{ SUDÁ}}{=} \frac{4}{T}\int_0^{\frac{T}{2}} f(t)\sin(k\omega t)$$

TAKŽE: $b_k = \frac{4}{8\pi}\int_0^{4\pi} s(t)\sin\left(\frac{k}{4}t\right)dt = \frac{3}{2\pi}\int_0^{2\pi}\sin\left(\frac{k}{4}t\right)dt - \frac{2}{2\pi}\int_{2\pi}^{4\pi}\sin\left(\frac{k}{4}t\right)dt =$

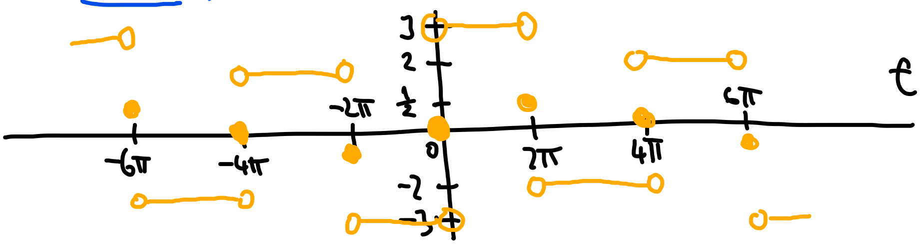
$$= -\frac{6}{k\pi}\left[\cos\left(\frac{k}{4}t\right)\right]_0^{2\pi} + \frac{4}{k\pi}\left[\cos\left(\frac{k}{4}t\right)\right]_{2\pi}^{4\pi} =$$



$$= -\frac{6}{k\pi}\left(\cos\left(k\frac{\pi}{2}\right)-1\right) + \frac{4}{k\pi}\left(\cos(k\pi)-\cos\left(k\frac{\pi}{2}\right)\right) =$$

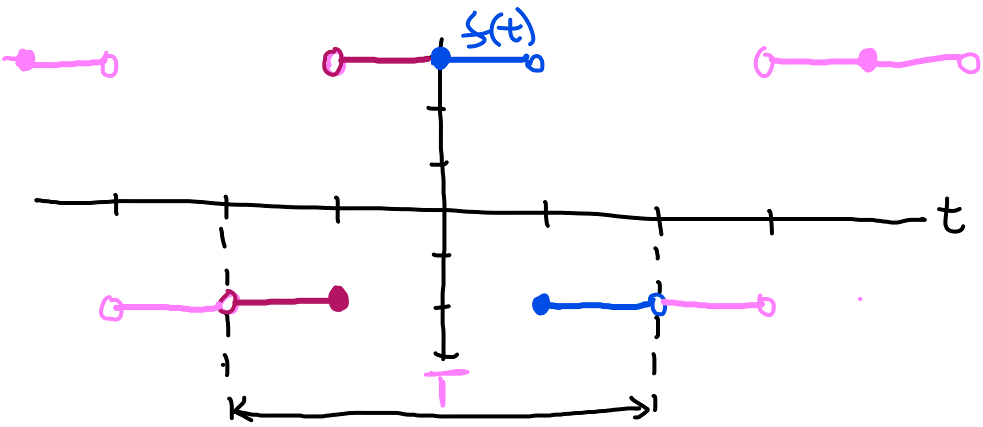
$$= \frac{6}{k\pi} + \frac{4}{k\pi}(-1)^k - \frac{10}{k\pi}\cos\left(k\frac{\pi}{2}\right)$$

TEDA: $s(t) \approx \sum_{k=1}^{\infty} \frac{2}{k\pi}\left(3+2(-1)^k - \frac{5}{k\pi}\cos\left(k\frac{\pi}{2}\right)\right)\sin\left(\frac{k}{4}t\right) = \begin{cases} s(t) & \dots t\in(0,2\pi)\cup(2\pi,4\pi) \\ 0 & \dots t=0 \\ \frac{1}{2} & \dots t=2\pi \end{cases}$



ŘADA KONVERGUJE K ORANŽOVÉ

③ KOSINOVÁ FOURIEROVA ŘADA



DOPLNĚNÍ NA PERIODICKOU JE SUDÉ, $T=8\pi$, $\omega = \frac{2\pi}{8\pi} = \frac{1}{4}$

$$s(t) \approx \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k\cos(k\omega t)$$

$$a_0 = \frac{2}{T}\int_d^{d+T} s(t)\,dt = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} s(t)\,dt = \frac{4}{T}\int_0^{T} s(t)\,dt$$

d VOLÍM $-\frac{T}{2}$ $\quad$ $s(t)$ i $s(t)\cos(k\omega t)$ SUDÉ

$$a_k = \frac{2}{T}\int_d^{d+T} s(t)\cos(k\omega t)\,dt = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} s(t)\cos(k\omega t)\,dt = \frac{4}{T}\int_0^{T} s(t)\cos(k\omega t)\,dt$$

TAKŽE: $a_k = \frac{4}{8\pi}\int_0^{4\pi} f(t)\,dt = \frac{1}{2\pi}2\pi = 1$

$$a_k = \frac{4}{8\pi}\int_0^{4\pi} f(t)\cos\left(\frac{k}{4}t\right)dt = \frac{3}{2\pi}\int_0^{2\pi}\cos\left(\frac{k}{4}t\right)dt - \frac{1}{\pi}\int_0^{4\pi}\cos\left(\frac{k}{4}t\right)dt =$$

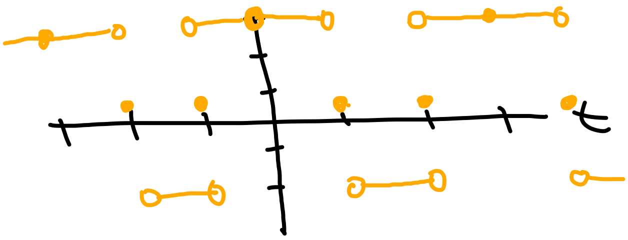
$$= \frac{6}{k\pi}\left[\sin\left(\frac{k}{4}t\right)\right]_0^{2\pi} - \frac{4}{k\pi}\left[\sin\left(\frac{k}{4}t\right)\right]_{2\pi}^{4\pi} = \frac{10}{k\pi}\sin\left(k\frac{\pi}{2}\right) = \begin{cases} 0 \dots k \text{ SUDÉ} \\ \frac{10}{k\pi}, -\frac{10}{k\pi}, \frac{10}{k\pi}, -\frac{10}{k\pi} \dots \\ \dots k \text{ LICHÉ} \end{cases}$$

TEDA:

$$f(t) \approx \frac{1}{2} + \sum_{k=1}^{\infty}\frac{10}{k\pi}\sin\left(k\frac{\pi}{2}\right)\cos\left(\frac{k}{4}t\right) = \begin{cases} f(t) & \dots t \in \langle 0, 2\pi) \cup (2\pi, 4\pi) \\ \frac{1}{2} & \dots t = 2\pi \end{cases}$$

NEBO:

$$f(t) \approx \frac{1}{2} + \sum_{k=1}^{\infty}(-1)^{k+1}\frac{10}{(2k-1)\pi}\cos\left(\frac{(2k-1)}{4}t\right) = \text{ VIZ VÝŠ}$$



ŘADA KONVERGUJE K ORANŽOVÉ

1.2 Najděte Fourierovu řadu, Fourierovu sinovou řadu a Fourierovu kosinovou řadu funkce $f(t)$. Určete jejich součty.

$$f(t) = \begin{cases} t & \text{pro } t \in \langle 0, 2) \\ t-4 & \text{pro } t \in \langle 2, 4) \end{cases}$$

[$\sum\limits_{k=1}^{\infty} 4\,\frac{(-1)^{1+k}\sin\left(\frac{1}{2}\,k\pi\,t\right)}{k\pi} = \begin{cases} f(t) & \text{pro } t \in \langle 0, 2) \cup (2, 4) \\ 0 & \text{pro } t \in \{2\} \end{cases}$]

[$\sum\limits_{k=1}^{\infty} -8\,\frac{\cos\left(\frac{1}{2}\,k\pi\right)\sin\left(\frac{1}{4}\,k\pi\,t\right)}{k\pi} = \begin{cases} f(t) & \text{pro } t \in \langle 0, 2) \cup (2, 4) \\ 0 & \text{pro } t \in \{2\} \end{cases}$]

[$\sum\limits_{k=1}^{\infty} \frac{\left(8\,\sin\left(\frac{1}{2}\,k\pi\right)k\pi+8\,(-1)^{k}-8\right)\cos\left(\frac{1}{4}\,k\pi\,t\right)}{k^2\pi^2} = \begin{cases} f(t) & \text{pro } t \in \langle 0, 2) \cup (2, 4) \\ 0 & \text{pro } t \in \{2\} \end{cases}$]

1.3 Najděte Fourierovu řadu, Fourierovu sinovou řadu a Fourierovu kosinovou řadu funkce $f(t)$. Určete jejich součty.

$$f(t) = \begin{cases} t^2 & \text{pro } t \in \langle 0, 2) \\ (t-4)^2 & \text{pro } t \in \langle 2, 4) \end{cases}$$

$[4/3 + \sum_{k=1}^{\infty} 16 \, \frac{(-1)^k \cos\left(\frac{1}{2}\, k\pi\, t\right)}{k^2\pi^2} = f(t) \text{ pro } t \in \langle 0, 4)]$

$[\sum_{k=1}^{\infty} \frac{\left(64\, \sin\left(\frac{1}{2}\, k\pi\right) k\pi + 64\, (-1)^k - 64\right) \sin\left(\frac{1}{4}\, k\pi\, t\right)}{k^3\pi^3} = f(t) \text{ pro } t \in \langle 0, 4)]$

$[4/3 + \sum_{k=1}^{\infty} 64 \, \frac{\cos\left(\frac{1}{2}\, k\pi\right) \cos\left(\frac{1}{4}\, k\pi\, t\right)}{k^2\pi^2} = f(t) \text{ pro } t \in \langle 0, 4)]$