MA2 - Sedmé cvičení

Karel Pospíšil

1 Křivkový integrál funkce.

1.1 Najděte křivkový integrál $\int_C \mathrm{d}s$, kde $C = \{(x,y); x = a(t - \sin t), y = a(1 - \cos t), 0 \le t \le \pi, a > 0\}$.

$[4a]$

$$\int_C ds = \int_0^\pi f(x(t), y(t)) \cdot \|(x'(t), y'(t))\| \, dt = \int_0^\pi \sqrt{a^2\left((1-\cos t)^2 + (\sin t)^2\right)} \, dt$$

$$= a\int_0^\pi \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t} \, dt = a\sqrt{2}\int_0^\pi \sqrt{1-\cos t} \, dt = \sqrt{2}a\int_0^\pi \sqrt{1 - \left(\cos^2\frac{t}{2} - \sin^2\frac{t}{2}\right)} \, dt$$

$$= 2a\int_0^\pi \sin\frac{t}{2} \, dt = 2a\,2\left[-\cos\frac{t}{2}\right]_0^\pi = 4a(-0+1) = \underline{\underline{4a}}$$

1.2 Najděte křivkový integrál $\int_C x^2 \mathrm{d}s$, kde $C = \{(x,y); y = ln(x), 1 \leq x \leq 2\}$.

$[\frac{1}{3}(5^{\frac{3}{2}} - 2^{\frac{3}{2}}]$

VZOREC:

$$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\cdot\|x'(t),y'(t)\|\,dt,$$

KDE $(x(t),y(t))$, $t\in\langle a,b\rangle$ JE PARAMETRIZACE C

$$\int_C x^2 ds = \int_1^2 t^2 \|t', (\ln(t))'\|\,dt = \Big|\, C = \{(t,\ln(t)) \mid t\in\langle 1,2\rangle\}$$

$$= \int_1^2 t^2\sqrt{1+\frac{1}{t^2}}\,dt = \int_1^2 t\sqrt{t^2+1}\,dt = \left|\begin{matrix} u = t^2+1 \\ du = 2t\,dt \end{matrix} \quad \langle 1,2\rangle \to \langle 2,5\rangle\right| = \frac{1}{2}\int_2^5 \sqrt{u}\,du =$$

$$= \frac{1}{2}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_2^5 = \underline{\underline{\frac{1}{3}\left(5^{\frac{3}{2}} - 2^{\frac{3}{2}}\right)}}$$

2 Křivkový integrál vektorového pole.

2.1 Najděte křivkový integrál $\int\limits_{(C)} (x^2 - 2xy)\mathrm{d}x + (y^2 - 2xy)\mathrm{d}y$, kde $C = \{(x, y); y = x^2, -1 \le x \le 1, \}$, C kladně orientovaná.

$[-\frac{14}{15}]$

$$\int\limits_{(C)} m(x,y)\,dx + n(x,y)\,dy = \int_a^b \big(m(x(t),y(t)),\, n(x(t),y(t))\big)\cdot\big(x'(t),y'(t)\big)\,dt$$

KDE $(x(t),y(t))$, $t\in\langle a,b\rangle$ JE PARAMETRIZACE C

$$\int\limits_{(C)} (x^2-2xy)\,dx + (y^2-2xy)\,dy = \int_{-1}^{1} (t^2-2t^3,\, t^4-2t^3)\cdot(1,2t)\,dt =$$

$$C = \{ (t,t^2),\ t\in\langle -1,1\rangle \}$$

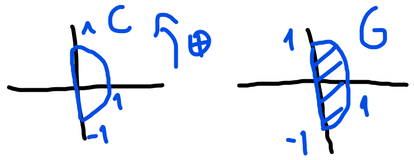
$$= \int_{-1}^{1} (t^2-2t^3+2t^5-4t^4)\,dt = \left[\frac{t^3}{3} - \frac{t^4}{2} + \frac{t^6}{3} - \frac{4}{5}t^5\right]_{-1}^{1} =$$

$$= \left(\underline{\frac{1}{3}} - \underline{\frac{1}{2}} + \underline{\frac{1}{3}} - \underline{\frac{4}{5}} + \underline{\frac{1}{3}} + \underline{\frac{1}{2}} - \underline{\frac{1}{3}} - \underline{\frac{4}{5}}\right) = \frac{2}{3} - \frac{8}{5} = \frac{10-24}{15} = -\underline{\underline{\frac{14}{15}}}$$

3 Křivkový integrál vektorového pole, Greenova věta.

3.1 Najděte křivkový integrál $\int\limits_{(C)} (x, 2y+x)\mathrm{d}\vec{s}$, kde $C = \{(x,y)|x^2+y^2=1, x \geq 0\} \cup \{(x,y)|x=0, -1 \leq y \leq 1\}$, C kladně orientovaná.

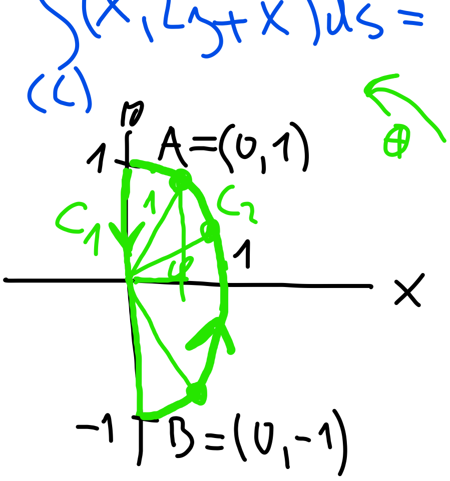
$[\frac{\pi}{2}]$



$$\int\limits_{(C)} (u(x,y), v(x,y))\,d\vec{s} = \iint\limits_{G} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$$

$$\iint\limits_{G} 1\,dx\,dy = \frac{\pi r^2}{2} = \underline{\underline{\frac{\pi}{2}}}$$

$$\int\limits_{(C)} (x, 2y+x)\,d\vec{s} = \int_0^1 (0, 2-4t)\cdot(0,-2)\,dt + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos\varphi, 2\sin\varphi+\cos\varphi)\cdot(-\sin\varphi, \cos\varphi)\,d\varphi =$$



$\vec{AB} = (0,-2)$

$C_1 = \{A + t\vec{AB} \mid t \in \langle 0,1\rangle\} = \{(0,1) + t(0,-2) \mid t \ldots$

$C_2 = \{(\cos\varphi, \sin\varphi) \mid \varphi \in \langle -\frac{\pi}{2}, \frac{\pi}{2}\rangle\}$

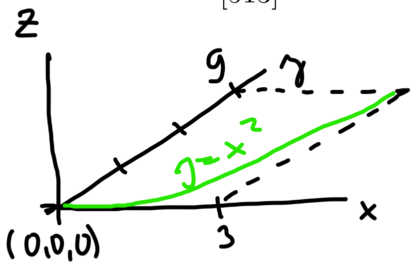
$$\int_0^1 (-4+8t)\,dt + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (-\sin\varphi\cos\varphi + 2\sin\varphi\cos\varphi + \cos^2\varphi)\,d\varphi =$$

$$= -4 + \frac{8}{2}\left[t^2\right]_0^1 + \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin 2\varphi \, d\varphi + \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1+\cos 2\varphi)\, d\varphi = -4 + 4 + \frac{1}{4}\left[-\cos 2\varphi\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} +$$

$$+ \frac{\pi}{2} + \frac{1}{4}\left[\sin 2\varphi\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{1}{4}(1-1) + \frac{\pi}{2} + \frac{1}{4}(0-0) = \underline{\underline{\frac{\pi}{2}}}$$

3.2 Jakou práci vykoná síla $\vec{F} = (3x^2, 5xy, 3)$, působící na částici pohybující se po grafu funkce v rovině $z = 0$ $y = x^2$ mezi body $(0,0,0)$ a $(3,9,0)$.

[513]



$C = \{(t, t^2, 0) \mid t \in \langle 0, 3 \rangle\}$

JEDNÁ SE VLASTNĚ O INTEGRÁL $\int_{(C)} \bar{F} \, d\bar{s}$

$$\int_{(C)} \bar{F} \, d\bar{s} = \int_0^3 (3t^2, 5t^3, 3) \cdot (1, 2t, 0) \, dt = \int_0^3 (3t^2 + 10t^4) \, dt =$$

$$= \left[t^3 + 2t^5 \right]_0^3 = 27 + 2 \cdot 27 \cdot 9 = 19 \cdot 27 = \underline{\underline{513}}$$

4 Potenciál

4.1 Pokud existuje, najděte potenciál vektorového pole .

$\vec{F} = (2x + y, 3y^2 + x)$

HLEDÁM $f(x,y)$ TAKŽE $\mathrm{GRAD}(f) = \vec{F}$

POSTUP DLE DEFINICE

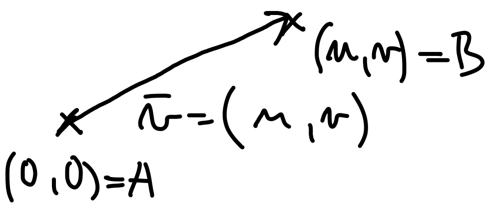
$$\left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y)\right) = (2x+y, 3y^2+x)$$

$$f(x,y) = \int (2x+y)\,dx$$

$$f(x,y) = x^2+yx+C(y) \to \frac{\partial f(x,y)}{\partial y} = x + C'(y) = 3y^2+x \to C(y) = y^3$$

$$\underline{\underline{f(x,y) = x^2+yx+y^3+C}}$$, ZKOUŠKA: $\mathrm{GRAD}(f) = (2x+y, x+3y^2)$ PLATÍ.

POSTUP S POMOCÍ KŘIVKOVÉHO INTEGRÁLU



$(x,y) = t(u,v)$ $\qquad t \in \langle 0,1\rangle$

POČÍTÁM PŘES LIBOVOLNOU KŘIVKU Z A DO B

$$\int_{(C)} (2x+y, 3y^2+x)\,d\bar{s} = \int_0^1 (2tu+tv, 3t^2v^2+tu)(u,v)\,dt =$$

$$= \int_0^1 (2tu^2+tuv+3t^2v^3+tuv)\,dt = \left[t^2u^2+t^3v^3+t^2uv\right]_0^1 = u^2+v^3+uv$$

$$\underline{\underline{f(x,y,z) = x^2+y^3+xy+C}}$$

EXISTENCE

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1-1 = 0 \to \underline{\underline{\text{ANO}}}$$

$\vec{F} = (x^2 + 2xy - y^2, x^2 - 2xy - y^2).$

(F_1, F_2, F_3)

$$\vec{F} = (y^2z^3, 2xyz^3, 3xy^2z^2)$$

EXISTENCE

$$\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \\ \bar{e}_1 & \bar{e}_2 & \bar{e}_3 \end{vmatrix} = (6xyz^2 - 6xyz^2)\bar{e}_1 - (3y^2z^2 - 3y^2z^2)\bar{e}_2 + (2yz^3 - 2yz^3)\bar{e}_3 = \bar{o}$$

$\rightarrow$ POTENCIÁL EXISTUJE

$(u,v,w) \quad (u,v,w) \quad (0,0,0)$

$$(x,y,z) = t(u,v,w) \ ; \ t \in \langle 0,1 \rangle$$

$$\int_{(c)} \bar{F}\,d\bar{s} = \int_0^1 (t^2v^2t^3w^3, 2tutvt^3w^3, 3tut^2v^2t^2w^2) \cdot (u,v,w)\,dt =$$

$$= \int_0^1 (t^5uv^2w^3 + 2t^5uv^2w^3 + 3t^5uv^2w^3)\,dt = \int_0^1 6t^5uv^2w^3\,dt =$$

$$= uv^2w^3\left[t^6\right]_0^1 = uv^2w^3$$

TAKŽE POTENCIÁL: ZKOUŠKA:

$$f(x,y,z) = xy^2z^3 + C \ , \ \mathrm{GRAD}(f) = (y^2z^3, 2xyz^3, 3xy^2z^2)$$

VZOROVÝ TEST

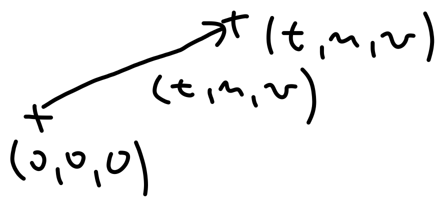
$$\bar{G} = (x,\ yz^2,\ 3)$$

$$\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\bar{e}_1 - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\bar{e}_2 + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\bar{e}_3 = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \\ \bar{e}_1 & \bar{e}_2 & \bar{e}_3 \end{vmatrix} =$$

$$(0-2yz)\bar{e}_1 - (0-0)\bar{e}_2 + (0-0)\bar{e}_3 = -2yz\cdot\bar{e}_1 \neq \bar{o} \rightarrow G \text{ NENÍ POT.}$$

$$\bar{F} = (y+z,\ x+z,\ x+y)$$

$$\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \\ \bar{e}_1 & \bar{e}_2 & \bar{e}_3 \end{vmatrix} = (1-1)\bar{e}_1 - (1-1)\bar{e}_2 + (1-1)\bar{e}_3 = \bar{o} \quad F \text{ JE POT.}$$



$$C = \{d(t,m,v),\ d \in \langle 0,1 \rangle\}$$

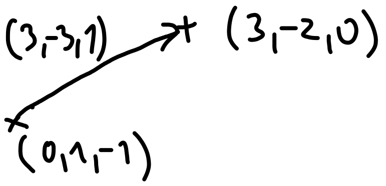
$$\int_{(C)} \bar{F}\, d\bar{s} = \int_0^1 (dm+dv,\ dt+dv,\ dt+dm)(t,m,v)\, dd$$

$$= \int_0^1 (dtm + dtv + dtm + dvm + dtv + dmv)\, dd =$$

$$= \int_0^1 (2dtm + 2dtv + 2dmv)\, dd = \left[d^2 tm + d^2 tv + d^2 mv\right]_0^1 = tm + tv + mv$$

$$f(x,y,z) = xy + xz + yz + C, \quad \text{ZK: } \mathrm{GRAD}(f) = (y+z,\ x+z,\ x+y) \quad \text{ANO}$$

C:



$$C = \{(0,1,-1) + t(3,-3,1);\ t \in \langle 0,1 \rangle\}$$

$$\int_{(C)} \bar{F}\, d\bar{s} = \int_0^1 (1-3t-1+t,\ 3t-1+t,\ 3t+1-3t)(3,-3,1)\, dt =$$

$$= \int_0^1 (-6t - 12t + 3 + 1)\, dt = \left[-\frac{18}{2}t^2 + 4t\right]_0^1 = -9 + 4 = \underline{-5}$$