MA2 - Šesté cvičení

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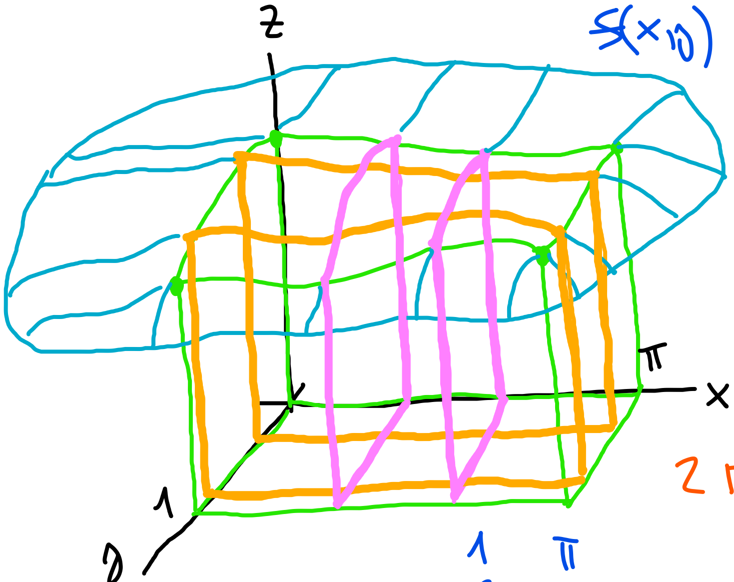
1 Dvojnásobný integrál.

1.1 Intergrujte $f(x, y) = y\cos(xy)$ přes obdélník $0 \le x \le \pi,\ 0 \le y \le 1$.



$$\int_0^\pi \left(\int_0^1 f(x,y)\, dy \right) dx$$

$$\int_0^1 \left(\int_0^\pi f(x,y)\, dx \right) dy$$

2 MOŽNÁ POŘADÍ INTEGRACE,

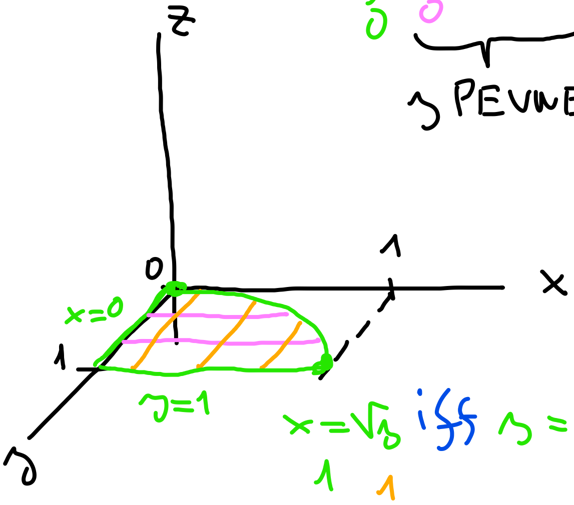
ZKUSÍM DRUHÉ:
$$\int_0^1 \left(\int_0^\pi y\cos(xy)\, dx \right) dy = \int_0^1 \left[\sin(xy)\right]_0^\pi dy =$$

$$= \int_0^1 \sin(\pi y)\, dy = -\frac{1}{\pi}\left[\cos(\pi y)\right]_0^1 = -\frac{1}{\pi}(-1-1) = \underline{\underline{\frac{2}{\pi}}}$$

PRVNÍ BY SE TĚŽKO INTEGROVALO.

1.2 Změňte pořadí integrace.

$$I = \int_0^1 \int_0^{\sqrt{y}} 1\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \Big(\underbrace{\int_0^{\sqrt{y}} 1\,\mathrm{d}x}_{\text{y PEVNÉ, URČUJE MEZE PRO X}} \Big)\,\mathrm{d}y$$



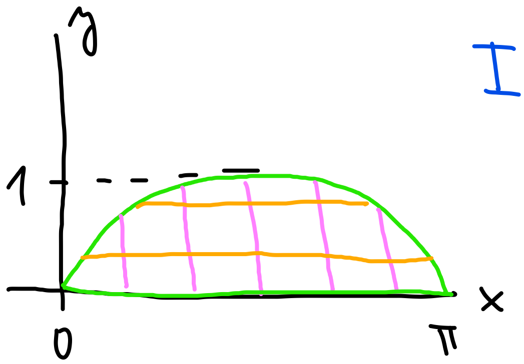
$x=\sqrt{y}$ iff $y=x^2$

$$I = \int_0^1 \Big(\underbrace{\int_{x^2}^{1} 1\,\mathrm{d}y}_{\text{X PEVNÉ, URČUJE MEZE PRO y}} \Big)\,\mathrm{d}x$$

VÝSLEDEK JE V OBOU POŘADÍCH $\frac{2}{3}$, ZKUSTE.

$$\int_0^{\pi} \int_0^{\sin x} f(x,y)\,\mathrm{d}y\,\mathrm{d}x$$

VYSTAČÍM S 2D OBRÁZKEM, OSA z JE KOLMÁ NA x A y.



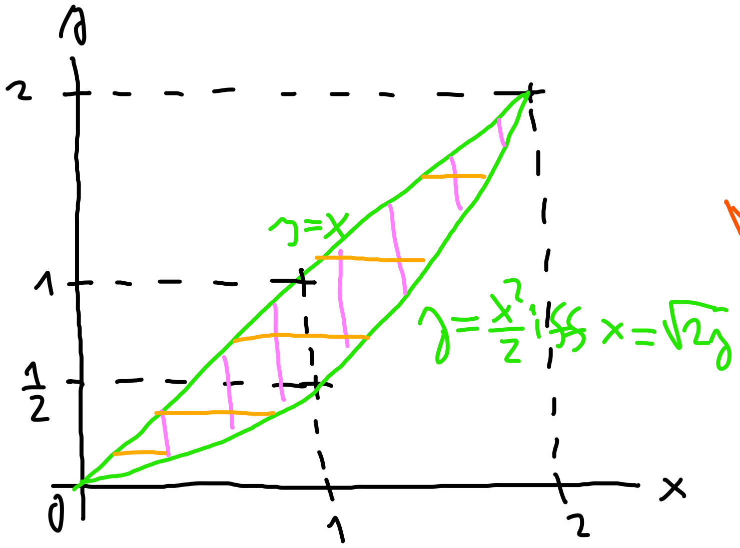
$$I = \int_0^{\pi} \left(\int_0^{\sin(x)} f(x,y)\, dy \right) dx =$$

$$= \int_0^{1} \left(\int_{\arcsin(y)}^{\pi - \arcsin(y)} f(x,y)\, dx \right) dy$$

1.3 Intergrujte.

$\iint\limits_A \frac{x}{x^2+y^2}\,\mathrm{d}x\,\mathrm{d}y\,, A = \{(x,y); \frac{x^2}{2} \le y \le x, 0 \le x \le 2\}$

$[\ln 2]$



$$\int_0^2 \left(\int_{\frac{x^2}{2}}^{x} \frac{x}{x^2+y^2}\, dy \right) dx$$

NEBO

$$\int_0^2 \left(\int_{y}^{\sqrt{2y}} \frac{x}{x^2+y^2}\, dx \right) dy$$

ZKUSÍM $\int_0^2 \left(\int_{y}^{\sqrt{2y}} \frac{x}{x^2+y^2}\, dx \right) dy = \left| \begin{array}{l} t = x^2+y^2 \\ dt = 2x\, dx \\ \langle y, \sqrt{2y} \rangle \to \langle 2y^2, 2y+y^2 \rangle \end{array} \right| =$

$$= \int_0^2 \left(\frac{1}{2} \int_{2y^2}^{2y+y^2} \frac{dt}{t} \right) dy = \int_0^2 \frac{1}{2} \left[\ln|t| \right]_{2y^2}^{2y+y^2} dy = \frac{1}{2} \int_0^2 \left(\ln|2y+y^2| - \ln|2y^2| \right) dy$$

HNUSNÉ, ZKUSÍM RADĚJI DRUHÉ POŘADÍ

$$\int_0^2 \int_{\frac{x^2}{2}}^{x} \left(\frac{x}{x^2+y^2}\, dy \right) dx = \int_0^2 \left(\frac{1}{x} \int_{\frac{x^2}{2}}^{x} \frac{1}{1+\left(\frac{y}{x}\right)^2}\, dy \right) dx =$$

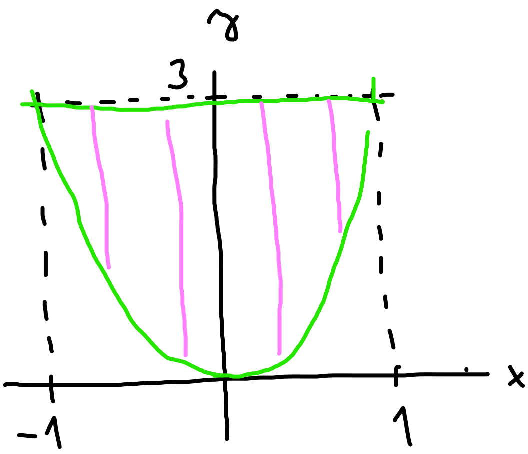
$$= \int_0^2 \left(\frac{1}{x} \left[x \,\mathrm{arctg}\left(\frac{y}{x}\right) \right]_{\frac{x^2}{2}}^{x} \right) dx = \int_0^2 \left(\mathrm{arctg}(1) - \mathrm{arctg}\left(\frac{x}{2}\right) \right) dx =$$

$$= \int_0^2 \frac{\sqrt{2}}{2}\, dx - \int_0^2 1 \cdot \mathrm{arctg}\left(\frac{x}{2}\right) dx = \left| \begin{array}{ll} u = \mathrm{arctg}\frac{x}{2} & v' = 1 \\ u' = \frac{1}{2}\frac{1}{1+\frac{x^2}{4}} & v = x \end{array} \right| = \sqrt{2} - \left(\left[x\,\mathrm{arctg}\left(\frac{x}{2}\right) \right]_0^2 - \int_0^2 \frac{2x}{4+x^2}\, dx \right) =$$

$$= \sqrt{2} - 2\frac{\sqrt{2}}{2} + \left[\ln|x^2+4| \right]_0^2 = \ln(8) - \ln(4) = \underline{\ln(2)}$$

$$\iint_A x+y\,\mathrm{d}x\,\mathrm{d}y\,,\ A=\{(x,y);3x^2\le y\le 3\}$$

$\left[\frac{36}{5}\right]$



$$\int_{-1}^{1}\int_{3x^2}^{3} x+y\,dy\,dx =$$

$$= \int_{-1}^{1}\left[xy+\frac{y^2}{2}\right]_{3x^2}^{3} dx = \int_{-1}^{1} 3x+\frac{9}{2}-3x^3-\frac{9x^4}{2}\,dx =$$

$$= \left[3\frac{x^2}{2}+\frac{9}{2}x-3\frac{x^4}{4}-\frac{9}{2}\frac{x^5}{5}\right]_{-1}^{1} =$$

$$= 6-\frac{3}{4}-\frac{9}{10}+3+\frac{3}{4}-\frac{9}{10}=9-\frac{9}{5}=\frac{36}{5}$$

$$\int f(x(u))\,x'(u)\,du = \int f(x)\,dx$$

1.4 Intergrujte s použitím substituce (polární souřadnice).

$\iint\limits_A 1\,\mathrm{d}x\,\mathrm{d}y$, kde A je kruh o poloměru jedna se středem v počátku.

SUBSTITUCE VZOREC

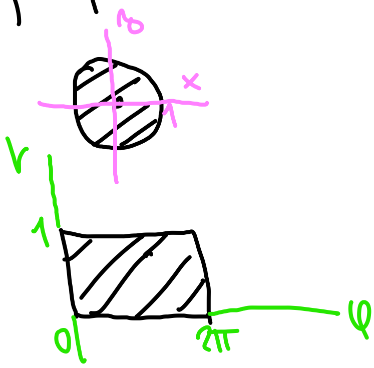
$$\iint\limits_{(x(A),y(A))} f(x(u,v),y(u,v))\,|J(x(u,v),y(u,v))|\,du\,dv = \iint\limits_A f(x,y)\,dx\,dy$$

PRO POLÁRNÍ SOUŘADNICE $x(r,\varphi)=r\cos(\varphi)$, $y(r,\varphi)=r\sin(\varphi)$

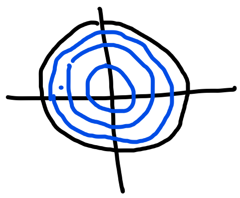
FUNKCI $f(x,y)=1$

A MNOŽINY A

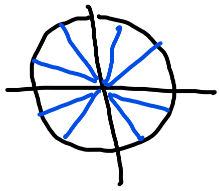
$(x(A),y(A))$



$$\iint\limits_A 1\,dx\,dy = \int_0^1\left(\int_0^{2\pi} 1\,|J(r\cos\varphi, r\sin\varphi)|\,d\varphi\right)dr$$



$$\iint\limits_A 1\,dx\,dy = \int_0^{2\pi}\int_0^1 1\,|J(r\cos\varphi, r\sin\varphi)|\,dr\,d\varphi$$



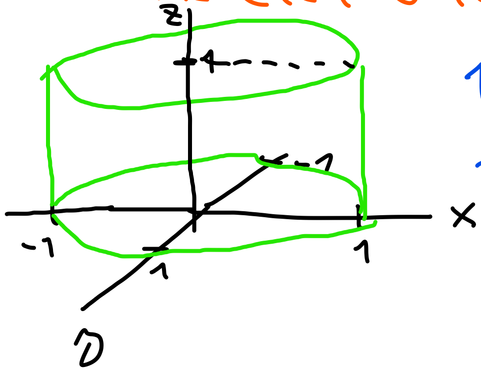
ZKUSÍM PRVNÍ POŘADÍ

$$\int_0^1\left(\int_0^{2\pi} 1\left|\begin{vmatrix}\cos\varphi & -r\sin\varphi\\ \sin\varphi & r\cos\varphi\end{vmatrix}\right|d\varphi\right)dr = \int_0^1\left(\int_0^{2\pi} r\,d\varphi\right)dr = \int_0^1 2\pi r\,dr =$$

$$= \pi\left[r^2\right]_0^1 = \underline{\underline{\pi}}$$

ZKUSTE DRUHÉ POŘADÍ.

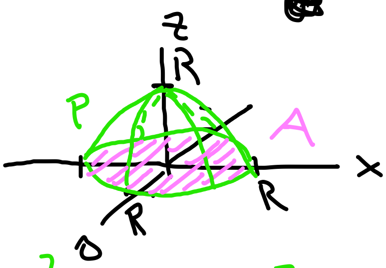
GEOMETRICKÝ VÝZNAM



1) OBJEM VÁLCE

2) OBSAH KRUHU

1.5 Odvoďte vzorec pro objem koule o poloměru R pomocí dvojného integrálu.



$$x^2+y^2+z^2=R^2 \rightarrow z=\sqrt{R^2-(x^2+y^2)}$$

ROVNICE KULOVÉ PLOCHY

FUNKCE JEJÍŽ DEF. OBOR JE A
GRAF JE PŮLKULOVÁ PLOCHA P

OBJEM KOULE JE TEDA

$$2\cdot\iint_A z(x,y)\,dx\,dy = 2\iint_A \sqrt{R^2-(x^2+y^2)}\,dx\,dy =$$

$$= \left|\begin{array}{l}\text{SUBSTITUCE POLÁRNÍ}\\ (x,y)=(r\cos\varphi, r\sin\varphi),\ J(x,y)=r\end{array}\right| = 2\int_0^{2\pi}\left(\int_0^R \sqrt{R^2-r^2}\,r\,dr\right)d\varphi =$$

$$= \left|\begin{array}{l}\text{SUBSTITUCE}\\ u=R^2-r^2 \quad du=-2r\,dr\\ \langle 0,R\rangle\to\langle R^2,0\rangle\end{array}\right| = 2\int_0^{2\pi}\left(\int_{R^2}^0 \sqrt{u}\,\frac{du}{-2}\right)d\varphi = -\frac{2}{3}\int_0^{2\pi}\left[u^{\frac{3}{2}}\right]_{R^2}^0 d\varphi =$$

$$= +\frac{2}{3}\int_0^{2\pi} R^3\,d\varphi = \underline{\underline{\frac{4}{3}\pi R^3}}$$

1.6 NAJDĚTE POLOHU TĚŽIŠTĚ PLOŠNÉHO ÚTVARU $A=\{(x,y)\mid (x^2+y^2)^2\le 2y^3\}$

ZKUSÍM POLÁRNÍ SOUŘADNICE $(r^2\cos^2\varphi + r^2\sin^2\varphi)^2 = r^4 \le 2r^3\sin^3\varphi \rightarrow r\le 2\sin^3\varphi$

$\sin\varphi\ge 0 \rightarrow \varphi\in\langle 0,\pi\rangle$

$$T=\left[\frac{\iint_A x}{\iint_A 1},\ \frac{\iint_A y}{\iint_A 1}\right]$$

$$\iint_A 1\,\underbrace{dx\,dy}_{\text{NĚKDY PÍŠU, NĚKDY NEPÍŠU}} = \int_0^{\pi}\left(\int_0^{2\sin^3\varphi} r\,dr\right)d\varphi = \frac{1}{2}\int_0^{\pi}\left[r^2\right]_0^{2\sin^3\varphi} d\varphi = 2\int_0^{\pi}\sin^6\varphi\,d\varphi = \underbrace{\dots}_{\text{ŘEŠTE SAMI}} = \underline{\frac{5}{8}\pi}$$

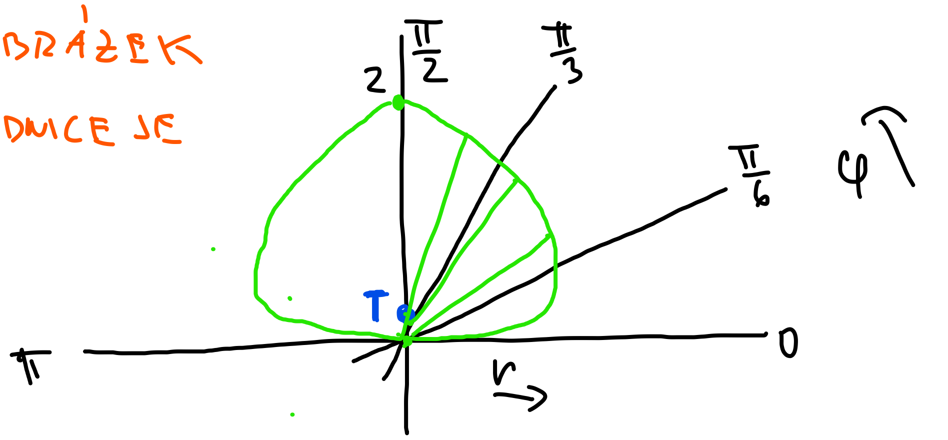
$$\iint_A x\,dx\,dy = \int_0^{\pi}\left(\int_0^{2\sin^3\varphi} r\cos\varphi\cdot r\,dr\right)d\varphi = \frac{1}{3}\int_0^{\pi}\left[r^3\cos\varphi\right]_0^{2\sin^3\varphi} d\varphi = \frac{8}{3}\int_0^{\pi}\sin^9\varphi\cos\varphi\,d\varphi = \dots = \underline{0}$$

$$\iint_A y\,dx\,dy = \int_0^{\pi}\left(\int_0^{2\sin^3\varphi} r\sin\varphi\cdot r\,dr\right)d\varphi = \frac{1}{3}\int_0^{\pi}\left[r^3\sin\varphi\right]_0^{2\sin^3\varphi} d\varphi = \frac{8}{3}\int_0^{\pi}\sin^{10}\varphi\,d\varphi = \dots = \underline{\frac{\pi}{16}}$$

TAKŽE $\underline{\underline{T}} = \left[0, \frac{\frac{1}{16}}{\frac{5}{8}}\right] = \underline{\underline{\left[0, \frac{1}{10}\right]}}$

PŘIBLIŽNÝ OBRÁZEK

(X-OVÁ SOUŘADNICE JE ZŘEJMÁ)

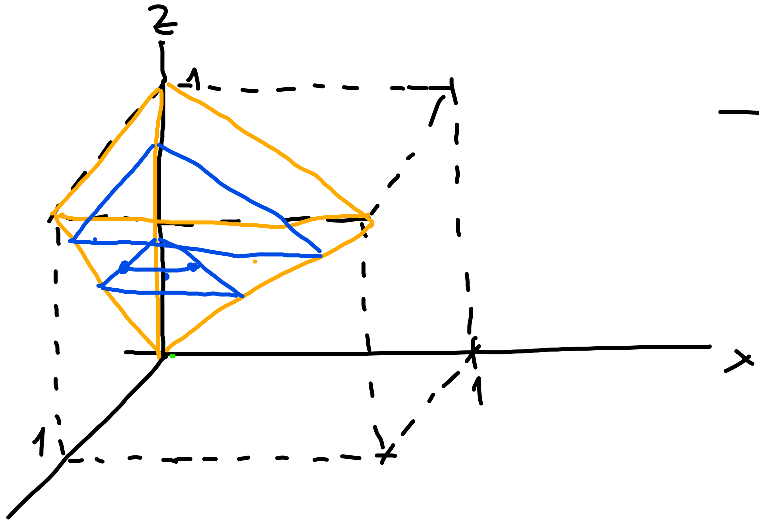


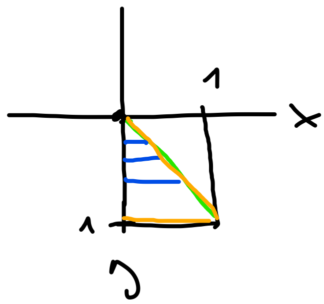
$r = 2\sin^3\varphi$

2 Trojnásobný integrál.

2.1 Načrtněte oblast integrace .

$$\int_0^1 \int_0^z \int_0^y f(x, y, z)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$$





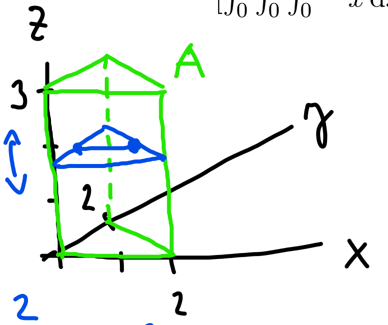
y

$$\int_0^1 \int_x^{2x} \int_0^{x+y} f(x,y,z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$$

2.2 Intergrujte .

$\iiint\limits_A x \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}z$, A je omezena plochami $x=0, y=0, z=0, z=3, x+y=2$

$[\int_0^3 \int_0^2 \int_0^{2-y} x \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}z = 4]$



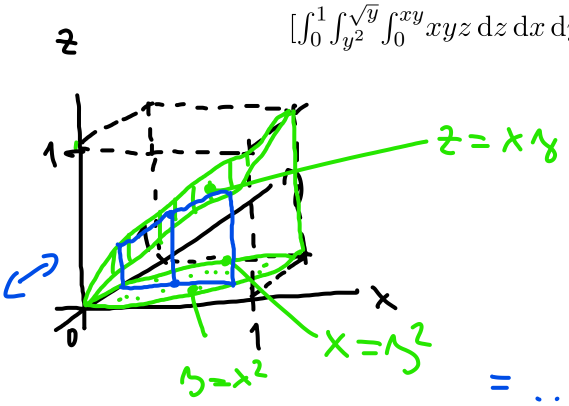
$$\int_0^3 \int_0^2 \int_0^{2-y} x \,dx \,dy \,dz =$$

$$= \frac{1}{2}\int_0^3 \int_0^2 \left[x^2\right]_0^{2-y} dy\,dz = \frac{1}{2}\int_0^3 \int_0^2 (4-4y+y^2)\,dy\,dz = \frac{1}{2}\int_0^3 \left[4y-2y^2+\frac{y^3}{3}\right]_0^2 dz =$$

$$= \frac{1}{2}\int_0^3 \left(8-8+\frac{8}{3}\right) dz = \underline{\underline{4}}$$

$\iiint\limits_A xyz\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, A je omezena plochami $x = y^2, y = x^2, z = 0, z = xy$

$[\int_0^1 \int_{y^2}^{\sqrt{y}} \int_0^{xy} xyz\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{96}]$

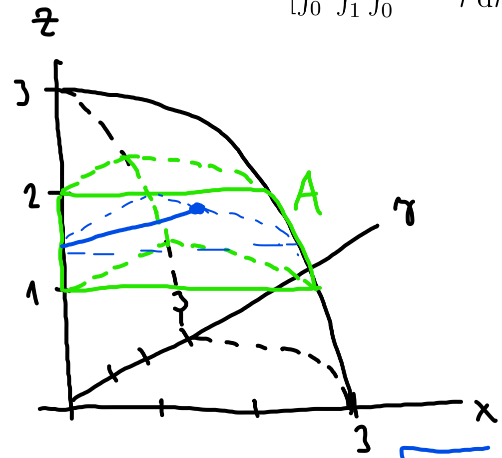


$$\int_0^1 \int_{y^2}^{\sqrt{y}} \int_0^{xy} xyz\,dz\,dx\,dy =$$

$$= \underbrace{\ldots}_{\text{SPOČTI SÁM}} = \underline{\underline{\frac{1}{96}}}$$

2.3 Najděte objem tělesa $A = \{(x,y,z)|\, x^2+y^2+z^2 \le 9,\ x \ge 0,\ y \ge 0,\ 1 \le z \le 2\}$.

$[\int_0^{\frac{\pi}{2}}\int_1^2\int_0^{\sqrt{9-z^2}} r \,\mathrm{d}r \,\mathrm{d}z \,\mathrm{d}\phi = \frac{10}{6}\pi]$



CYLINDRICKÉ SOUŘADNICE

$x = r\cos\varphi,\ y = r\sin\varphi,\ z = z$

$r^2 + z^2 = 9$

$r = \sqrt{9-z^2}$

$$J = \begin{vmatrix} \cos\varphi & \sin\varphi & 0 \\ -r\sin\varphi & r\cos\varphi & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

$$\iiint_A 1 \,dx\,dy\,dz = \int_1^2 \int_0^{\frac{\pi}{2}} \int_0^{\sqrt{9-z^2}} 1\, r \,dr\,d\varphi\,dz = \ldots = \underline{\underline{\frac{10}{6}\pi}}$$

2.4 Odvoďte vzorec pro objem koule o poloměru R pomocí trojného integrálu.

JE TO 3-ROZMĚRNÉ, SYMETRICKÉ PODLE POČÁTKU $\Rightarrow$
$\Rightarrow$ POUŽIJU SFÉRICKÉ SOUŘADNICE :

$$x = r\cos\varphi \sin\vartheta \qquad J = r^2 \sin\vartheta$$
$$y = r\sin\varphi \sin\vartheta$$
$$z = r\cos\vartheta$$

$$\int_0^{\pi}\int_0^{2\pi}\int_0^{R} 1\, r^2 \sin\vartheta \, dr\, d\varphi\, d\vartheta = \int_0^{\pi}\int_0^{2\pi}\left[\frac{r^3}{3}\sin\vartheta\right]_0^R d\varphi\, d\vartheta =$$

$$= \int_0^{\pi}\int_0^{2\pi} \frac{R^3}{3}\sin\vartheta \, d\varphi\, d\vartheta = \int_0^{\pi} 2\pi \frac{R^3}{3}\sin\vartheta\, d\vartheta = \frac{2}{3}\pi R^3 \left[-\cos\vartheta\right]_0^{\pi} =$$

$$= \frac{2}{3}\pi R^3 (1+1) = \underline{\underline{\frac{4}{3}\pi R^3}}$$

POZNÁMKA: PŮJDE-LI O HMOTNOST KOULE S HUSTOTOU $\rho(x,y,z)$, INTEGRÁL BUDE VYPADAT TAKTO:

$$\int_0^{\pi}\int_0^{2\pi}\int_0^{R} \rho(r\cos\varphi\sin\vartheta, r\sin\varphi\sin\vartheta, r\cos\vartheta)\cdot r^2\sin\vartheta\, dr\, d\varphi\, d\vartheta$$