Exam BE5B01DMG

Example	Points
1	
2	
3	
4	
5	
6	
Σ	

Name and surname:

Answer by complete sentences. Give reasons to all your assertions.

---

1. [Max: 4 points] Given a formula $\alpha = \neg Q(b,a) \Rightarrow \forall x\,(R(x) \Rightarrow \exists y\,(Q(x,y) \wedge R(y)))$ of predicate logic. Write down the formula β tautologically equivalent to $\neg\alpha$ which has negation in front of atomic formulas only.

2. [Max: 12 points] Given a relation R on the set of all pairs of real numbers $\mathbb{R} \times \mathbb{R}$ by
$$(a,b)\,R\,(c,d) \quad \text{iff} \quad a^2 - d = c^2 - b.$$
   (a) [Max: 2 points] Decide whether R is reflexive (define a reflexive relation).
   (b) [Max: 3 points] Decide whether R is symmetric (define a symmetric relation).
   (c) [Max: 3 points] Decide whether R is antisymmetric (define an antisymmetric relation).
   (d) [Max: 4 points] Decide whether R is transitive (define a transitive relation).

3. [Max: 18 points] An operation $*$ is defined on the set $S = \mathbb{Q} \times \mathbb{Q}$, i.e. the set containing all pairs of integers by:
$$(a,b) * (c,d) = (bc + a, bd).$$
   (a) [Max: 5 points] Show that the pair $(S,*)$ forms a semigroup. Write down what is a semigroup.
   (b) [Max: 3 points] Find a neutral element of the semigroup $(S,*)$. Write down what is a neutral element.
   (c) [Max: 6 points] Find all invertible elements of the monoid $(S,*)$. Write down what is an invertible element.
   (d) [Max: 4 points] Find at least one subset $A \subseteq S$ such that $(A,*)$ is a subsemigroup of $(S,*)$ that is a group and A more than two elements. Justify your answers.

4. [Max: 15 points] In $(\mathbb{Z}_{203}, +, \cdot)$
   (a) [Max: 7 points] calculate 6^{507}. A number does not suffice, you have to justify your answer.
   (b) [Max: 8 points] Use (a) to find all solutions of the equation $6^{507} \cdot x - 11 = 3(2x+1)$ (v $\mathbb{Z}_{203}$).

5. [Max: 23 points]
   (a) [Max: 4 points] Define the notion of a spanning tree of an undirected graph. Give a necessary and sufficient condition for an undirected graph to have a spanning tree.
   (b) [Max: 10 points] Given an undirected graph by its list of edges $\{u,v\}$ together with it weight $c(\{u,v\})$:

u	1	1	1	1	1	2	2	2	3	3	3	3	3	4	5	5	6	6	7
v	2	3	4	6	8	3	5	8	4	5	6	7	8	5	6	8	7	8	8
$c(\{u,v\})$	9	13	5	7	9	13	4	14	7	6	4	9	2	6	5	1	15	7	10

   Find a minimal spanning tree of G. Explain how you have got it.
   (c) [Max: 6 points] Give an example of a simple directed graph G without loops satisfying the following conditions: (1) G has 10 vertices, (2) G has three components of strong connectivity, (3) G is disconnected, and (4) G has 24 edges, 3 of which do not belong to the subgraphs induced by components of strong connectivity. Explain that your graph has all the properties. Justify all the properties you claim.
   (d) [Max: 3 points] Define a condensation of a directed graph. Draw the condensation of the graph from (c).

6. [Max: 8 points] Consider passwords of length 7 consisting of 26 small characters of Latin alphabet (6 vowels, and 20 consonants). (Always explain how you have got the result.)
   (a) [Max: 2 points] How many passwords have exactly two vowels?
   (b) [Max: 2 points] How many passwords begin with d or t, and the second character is a vowel?
   (c) [Max: 4 points] What are the answers to (a) and (b) if each character must be used at most once?

2018-01-17