Exam BE5B01DMG

Example	Points
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3	
4	
5	
6	
Σ	

Name and surname:

Answer by complete sentences. Give reasons to all your assertions.

1. [MAX: 4 POINTS] Given a $\alpha = \forall x\,[\neg Q(a,b) \vee (R(x) \Leftrightarrow \exists y\, Q(x,y))]$ of predicate logic. Write down the formula β tautologically equivalent to $\neg\alpha$ which has negation in front of atomic formulas only.

2. [MAX: 12 POINTS] Given a relation R on the set of all pairs of integers $\mathbb{Z} \times \mathbb{Z}$ by

$$(a,b)\,R\,(c,d) \quad \text{iff} \quad |a-c| = 2|b-d|.$$

   (a) [MAX: 2 POINTS] Decide whether R is reflexive (define a reflexive relation).

   (b) [MAX: 3 POINTS] Decide whether R is symmetric (define a symmetric relation).

   (c) [MAX: 3 POINTS] Decide whether R is antisymmetric (define an antisymmetric relation).

   (d) [MAX: 4 POINTS] Decide whether R is transitive (define a transitive relation).

3. [MAX: 19 POINTS] An operation $*$ is defined on the set $\mathbb{R} \times \mathbb{R}$, i.e. the set containing all pairs of real numbers by:

$$(u,v) * (x,y) = (ux - vy, uy + vx) \quad \text{multiplication and addition is in } \mathbb{R}.$$

   (a) [MAX: 5 POINTS] Show that the pair $(\mathbb{R} \times \mathbb{R}, *)$ forms a semigroup. Write down what is a semigroup.

   (b) [MAX: 3 POINTS] Find a neutral element of the semigroup $(\mathbb{R} \times \mathbb{R}, *)$. Write down what is a neutral element.

   (c) [MAX: 6 POINTS] Find all invertible elements of the monoid $(\mathbb{R} \times \mathbb{R}, *)$. Write down what is an invertible element.

   (d) [MAX: 5 POINTS] If $(\mathbb{R} \times \mathbb{R}, *)$ is not a group find a subminoind of $(\mathbb{R} \times \mathbb{R}, \star)$ which is a group and has at least 10 elements (define a submonoid). Justify your answers.

4. [MAX: 14 POINTS] Given $(\mathbb{Z}_{252}, +, \cdot)$.

   (a) [MAX: 8 POINTS] calculate 11^{652}. A number does not suffice, you have to justify your answer. Is there an element $a \in \mathbb{Z}_{252}$ for which $a^5 = 1$? Thoroughly justify your answer.

   (b) [MAX: 6 POINTS] Use (a) to find all solutions of the equation $11^{652} \cdot (x-1) = 2(5x-2)$ (in $\mathbb{Z}_{252}$).

5. [MAX: 23 POINTS]

   (a) [MAX: 6 POINTS] In a graph G with 6 vertices $\{v_1, \ldots, v_6\}$ it holds that $d(v_1) = 3$, $d(v_2) = 3$, $d(v_3) = 1$, $d(v_4) = 2$, $d(v_5) = 1$, $d(v_6) = 2$. Is it possible that G is a tree? (Define a tree.) If yes, draw an example of such a tree, if no, justify your answer.

   (b) [MAX: 5 POINTS] Give the smallest possible number m such that any simple graph G without loops with 30 vertices and m edges contains a circuit. The number itself does not suffice, you have to justify your answer.

   (c) [MAX: 9 POINTS] Given a directed graph with vertices $\{1, \ldots, 10\}$ and the set of edges ($PV(e)$ denotes the initial vertex of e, $KV(e)$ the terminal vertex of e)

PV	1	1	2	2	3	3	4	5	5	5	6	6	6	7	8	8	9	9	10	10
KV	3	10	3	5	1	5	6	1	2	7	4	7	9	6	4	9	7	8	1	8

   Find strongly connected components of G. Justify your answer. (Define a strongly connected component.)

   (d) [MAX: 3 POINTS] Draw the condensation of the graph from (c). Justify your answer.

6. [MAX: 8 POINTS] A drawer contains 5 pairs of socks of beige color, 4 pairs of socks of green color, 4 pairs of socks of dark gray color, and 1 pair of socks of black color. How many single socks do we have to take out of the drawer to make sure that

   (a) [MAX: 2 POINTS] we have two socks of the same color?

   (b) [MAX: 2 POINTS] that we have two socks of different colors?

   (c) [MAX: 4 POINTS] that we have a pair of beige colour? colors?

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