Exam BE5B01DMG

21 January 2019

Name and surname:

Give reasons for your assertions, the results themselves do not suffice.

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1. [4 points] Given the formula α of predicate logic

$$\alpha = \big(\forall x\, \forall y\, (P(y) \land P(x))\big) \land \big[Q(a,b) \Rightarrow \big((\forall x\, P(x)) \land (\forall y\, P(y))\big)\big],$$

write the formula β tautologically equivalent to $\neg\alpha$ which has negation in front of atomic formulas only.

2. [20 points] A relation R on the set of all pairs of real numbers $\mathbb{R} \times \mathbb{R}$ is given by

$$(a,b)\,R\,(c,d) \qquad \text{if and only if} \qquad a^2 + ad = c^2 + bc.$$

   (a) [4 points] Find all pairs (a,b) for which $(a,b)\,R\,(3,2)$.
   (b) [4 points] Define a reflexive relation and decide if R is reflexive.
   (c) [4 points] Define a symmetric relation and decide if R is symmetric.
   (d) [4 points] Define an antisymmetric relation and decide if R is antisymmetric.
   (e) [4 points] Define a transitive relation and decide if R is transitive.

3. [22 points] Consider the monoid $(\mathbb{Z}_{25}, \odot)$ of equivalence classes of integers modulo 25.

   (a) [4 points] How many elements does $(\mathbb{Z}_{25}^\star, \odot)$ have?
   (b) [7 points] List all invertible elements of $(\mathbb{Z}_{25}, \odot)$ and find the inverse of six of them.
   (c) [7 points] Write the definition of the order of an element in a group and compute the order of $[3]_{25}$, $[7]_{25}$, $[9]_{25}$ and $[11]_{25}$ in $(\mathbb{Z}_{25}^\star, \odot)$.
   (d) [4 points] Is there an element of $(\mathbb{Z}_{25}^\star, \odot)$ with order 6?

4. [16 points] In $\mathbb{Z}_{412}$ consider the equation $4(x+2) + 9^{616}x = 3$.

   (a) [8 points] Calculate 9^{616} in $\mathbb{Z}_{412}$.
   (b) [8 points] Find all solutions of the equation.

5. [15 points]

   (a) [5 points] Is there a graph G with vertices $V = \{v_1, \ldots, v_7\}$ whose degrees are $d(v_1) = 3$, $d(v_2) = 2$, $d(v_3) = 4$, $d(v_4) = 1$, $d(v_5) = 2$, $d(v_6) = 0$, $d(v_7) = 3$?
   (b) [10 points] Given a graph G by its list of edges with corresponding weights:

u	1	1	1	2	2	2	2	3	5	5	5	6	8	8	9	9
v	2	6	8	4	6	7	8	4	1	2	6	8	3	7	5	8
$c(\{u,v\})$	4	3	6	2	14	1	6	15	1	12	4	8	13	15	2	8

   Find a minimal spanning tree for G and its weight.

6. [13 points] Consider the set $A := \{1, \ldots, 10\}$.

   (a) [2 points] How many subsets does A have?
   (b) [3 points] How many subsets of A have exactly 4 elements?
   (c) [4 points] How many subsets of A have at most 8 elements?
   (d) [4 points] How many subsets of A contain every even number?