Exam BE5B01DMG

28 January 2019

Name and surname:

Give reasons for your assertions, the results themselves do not suffice.

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1. [4 points] Given the formula α of predicate logic

$$\alpha = \big(\forall y\,(R(x,y) \wedge P(y))\big) \vee \big[R(a,b) \Rightarrow \big((\forall x\,P(x)) \Leftrightarrow (\forall y\,P(y))\big)\big],$$

   write the formula β tautologically equivalent to $\neg\alpha$ which has negation in front of atomic formulas only.

2. [20 points] A relation R on the set of all natural numbers $\mathbb{N}$ is given by

$$k\,R\,n \qquad \text{if and only if} \qquad \text{the greatest common divisor of } k \text{ and } n \text{ is } 1.$$

   (a) [4 points] Find all k for which $2\,R\,k$; find all k that are not in relation with 10.
   (b) [4 points] Define a reflexive relation and decide if R is reflexive.
   (c) [4 points] Define a symmetric relation and decide if R is symmetric.
   (d) [4 points] Define an antisymmetric relation and decide if R is antisymmetric.
   (e) [4 points] Define a transitive relation and decide if R is transitive.

3. [22 points] An operation $*$ is defined on the set $G := \{(a,b) \in \mathbb{Q} \times \mathbb{Q} \colon a \neq 0\}$ by the rule

$$(a,b) * (c,d) = \left(ac, ad + \frac{b}{c}\right).$$

   (a) [3 points] Prove that $*$ is a binary operation on G.
   (b) [4 points] Write the definition of semigroup and prove that $(G,*)$ is a semigroup.
   (c) [4 points] Write the definition of neutral element and find a neutral element of $(G,*)$.
   (d) [6 points] Write the definitions of invertible element and group; is $(G,*)$ a group?
   (e) [5 points] Is $\{(a,0) \colon a \in \mathbb{Q},\ a \neq 0\}$ a subgroup of $(G,*)$?

4. [16 points] In $\mathbb{Z}_{148}$ consider the equation $5^{509}(2x+1) = 2x + 25$.

   (a) [8 points] Calculate 5^{509} in $\mathbb{Z}_{148}$.
   (b) [8 points] Find all solutions of the equation.

5. [15 points]

   (a) [4 points] Consider a graph with vertices $V = \{v_1, \dots, v_7\}$ whose degrees are $d(v_1) = 3$, $d(v_2) = 2$, $d(v_3) = 4$, $d(v_4) = 1$, $d(v_5) = 2$, $d(v_6) = 1$, $d(v_7) = 3$. Can G be a tree?
   (b) [7 points] Given a directed graph G with vertices $\{1, \dots, 12\}$ and edges given by

IV(e)	1	2	2	2	3	3	4	5	5	5	6	6	7	7	8	8	9	9	10	10
TV(e)	3	5	8	9	4	9	10	4	6	8	5	8	4	10	2	3	1	7	4	7

   Find strongly connected components of G.
   (c) [4 points] Define the condensation of a graph and draw the condensation of the graph from (b).

6. [13 points] Admissible passwords are strings of 6 characters chosen from 26 letters of Latin alphabet (20 consonants and 6 vowels) and each letter can be used at most once.

   (a) [3 points] How many passwords begin with c or h and the second letter is a vowel?
   (b) [5 points] How many passwords can be formed if every password must contain at least one consonant and one vowel?
   (c) [5 points] How many passwords can be formed if every password must contain the letter a?