1 Tutorial 1 – October 3rd, 2017

1.1 Write down the truth tables of the following formulas. Decide whether they are tautologies:

a) $(x \Rightarrow y) \Rightarrow x$;

b) $\neg\neg\neg x \Leftrightarrow \neg x$;

c) $(x \vee y) \Leftrightarrow (y \Rightarrow x)$;

d) $((x \vee y) \vee z) \Leftrightarrow (x \vee (y \vee z))$;

e) $(x \Rightarrow (y \vee z)) \vee ((y \wedge z) \Rightarrow x)$.

Solution of e).

Denote $\alpha = (x \Rightarrow (y \vee z)) \vee ((y \wedge z) \Rightarrow x)$. Then the truth table is

x	y	z	$x \Rightarrow (y \vee z)$	$(y \wedge z) \Rightarrow x$	α
0	0	0	1	1	1
0	0	1	1	1	1
0	1	0	1	1	1
0	1	1	1	0	1
1	0	0	0	1	1
1	0	1	1	1	1
1	1	0	1	1	1
1	1	1	1	1	1

Since the column corresponding to α contains all 1's, α is a tautology.

1.2 For which truth valuations u is the formula

a) $(x \Rightarrow \neg x) \wedge (\neg x \Rightarrow x)$ true;

b) $x \Rightarrow (x \Rightarrow y)$ false;

c) $x \wedge (y \Rightarrow (z \vee x))$ true;

d) $(x \Rightarrow \neg x) \Leftrightarrow \neg x$ false;

e) $(x \vee \neg y) \Rightarrow (\neg x \wedge y)$ true;

f) $(((x \vee y) \wedge z) \wedge y) \vee x$ false?

Solution of f). We give two different solutions:

1. We can use the truth table for $\alpha = (((x \vee y) \wedge z) \wedge y) \vee x$

x	y	z	$x \vee y$	$(x \vee y) \wedge z$	$((x \vee y) \wedge z) \wedge y$	α
0	0	0	0	0	0	0
0	0	1	0	0	0	0
0	1	0	1	0	0	0
0	1	1	1	1	1	1
1	0	0	1	0	0	1
1	0	1	1	1	0	1
1	1	0	1	0	0	1
1	1	1	1	1	1	1

We can see that α is false for all valuations in which x is false and at least one of y and z is also false.

2. We can also proceed without the truth table. Since $\alpha = \beta \vee x$ (for $\beta = ((x \vee y) \wedge z) \wedge y)$, α is true whenever x is true. So α could be false only for x false. Moreover, for α to be false, also β must be false. On the other hand, $\beta \models\!\!\dashv y \wedge z$, therefore α is false for those truth valuations for which x and $z \wedge y$ are both false, which means that x is false and at least one of y and z is false as well.

1.3 Decide whether the following formulas are tautologies, contradictions, or satisfiable formulas that are not tautologies:

a) $(x \Rightarrow y) \Rightarrow (x \vee y)$;

b) $((x \Rightarrow y) \Rightarrow (\neg x \wedge y)) \vee \neg y$;

c) $((x \wedge y) \vee (\neg x \wedge \neg y)) \Leftrightarrow ((\neg x \vee \neg y) \wedge (x \vee y))$;

d) $(x \Rightarrow (x \Rightarrow y)) \Rightarrow y$;

e) $((x \Rightarrow z) \wedge (y \Rightarrow z)) \Rightarrow ((x \wedge y) \Rightarrow z)$;

f) $((x \Rightarrow z) \vee (y \Rightarrow z)) \Rightarrow (x \Rightarrow y)$.

Solution of f).

Denote $\alpha = ((x \Rightarrow z) \vee (y \Rightarrow z)) \Rightarrow (x \Rightarrow y)$. Again we show two different solutions.

1. We can use the truth table:

x	y	z	$x \Rightarrow z$	$y \Rightarrow z$	$(x \Rightarrow z) \vee (y \Rightarrow z)$	$x \Rightarrow y$	α
0	0	0	1	1	1	1	1
0	0	1	1	1	1	1	1
0	1	0	1	0	1	1	1
0	1	1	1	1	1	1	1
1	0	0	0	1	1	0	0
1	0	1	1	1	1	0	0
1	1	0	0	0	0	1	1
1	1	1	1	1	1	1	1

Hence α is a satisfiable formula but not a tautology.

2. We can also proceed as follows: The formula α is satisfiable, since for example if $u(x) = u(y) = 0$ (and $u(z)$ arbitrary), the formula α is true. Indeed, in this case $u(x \Rightarrow y) = 1$, therefore any formula of the form $\beta \Rightarrow (x \Rightarrow y)$ is true as well.

On the other hand, α is not a tautology, since for $u(x) = 1$, $u(y) = 0$ we have $u(x \Rightarrow y) = 0$ and at the same time $u(y \Rightarrow z) = 1$ regardless the value $u(z)$. Therefore, $u((x \Rightarrow z) \vee (y \Rightarrow)) = 1$ and α is false.

1.4 Show that the following semantical consequences are valid:

a) $\{\alpha \Rightarrow \beta, \beta \Rightarrow \gamma\} \models \alpha \Rightarrow \gamma$;

b) $\{\alpha \Rightarrow \beta, \neg\beta\} \models \neg\alpha$;

c) $\{\alpha \vee \beta, \alpha \Rightarrow \gamma, \beta \Rightarrow \gamma\} \models \gamma$;

d) $\{\alpha \Rightarrow \beta, \alpha \Rightarrow \neg\beta\} \models \neg\alpha$;

e) $\{(\alpha \wedge \beta) \Rightarrow \gamma, (\alpha \wedge \neg\beta) \Rightarrow \gamma\} \models \alpha \Rightarrow \gamma$.X

Solution of c).

Denote $S = \{\alpha \vee \beta, \alpha \Rightarrow \gamma, \beta \Rightarrow \gamma\}$. We show two different solutions.

1. We will use the truth table.

α	β	γ	$\alpha \vee \beta$	$\alpha \Rightarrow \gamma$	$\beta \Rightarrow \gamma$	S	γ
0	0	0	0	1	1	0	0
0	0	1	0	1	1	0	1
0	1	0	1	1	0	0	0
0	1	1	1	1	1	1	1
1	0	0	1	0	1	0	0
1	0	1	1	1	1	1	1
1	1	0	1	0	0	0	0
1	1	1	1	1	1	1	1

We can see from the truth table that whenever S is true in u formula γ is true as well. Therefore, γ is a consequence of S.

2. We show that if in a truth valuation u the formula γ is false, then at least one of the formulas from S is false as well.

Assume that $u(\gamma) = 0$. Then either $u(\alpha) = 1$ but then $u(\alpha \Rightarrow \gamma) = 0$ (and $u(S) = 0$); or $u(\alpha) = 0$. Similarly, $u(\beta) = 1$ yields $u(\beta \Rightarrow \gamma) = 0$ (and $u(S) = 0$); or $u(\beta) = 0$. But if $u(\alpha = 0 = u(\beta)$ then $u(\alpha \vee \beta) = 0$ (and again $u(S) = 0$. Therefore, there is no u for which $u(S) = 1$ and $u(\gamma) = 0$, which proves that $S \models \gamma$.

1.5 Write down values of the boolean function g given bellow and try to simplify it:

a) $g(x, y, z) = (x + \overline{y} + z)\,(\overline{x} + \overline{y} + \overline{z})\,(\overline{x} + y + \overline{z})$;

b) $g(x, y, z) = (x + y + z)\,(\overline{x} + \overline{y} + \overline{z})\,(\overline{x} + \overline{y} + z)\,(x + \overline{y} + \overline{z})$.

Solution of b).

We will display the values of $g(x, y, z)$ in the following table. The eight rows express the eight combinations of 0 and 1 we can substitute into g. For example,

$$g(0, 0, 0) = (0 + 0 + 0)\,(1 + 1 + 1)\,(1 + 1 + 0)\,(0 + 1 + 1) = 0 \cdot 1 \cdot 1 \cdot 1 = 0.$$

Similarly, $g(0, 0, 1)$, etc. We get:

x	y	z	$g(x, y, z)$
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	0

To simplify the boolean function $g(x, y, z)$ we will use the the distributivity law:

$$(\overline{x} + \overline{y} + \overline{z})\,(\overline{x} + \overline{y} + z) = (\overline{x} + \overline{y})\,(z + \overline{z}) = (\overline{x} + \overline{y})\,1 = (\overline{x} + \overline{y}).$$

and

$$(\overline{x} + \overline{y} + \overline{z})\,(x + \overline{y} + \overline{z}) = (\overline{x} + x)\,(\overline{y} + \overline{z}) = 1\,(\overline{y} + \overline{z}) = (\overline{y} + \overline{y}).$$

Hence

$$g(x, y, z) = (x + y + z)\,(\overline{x} + \overline{y})\,(\overline{y} + \overline{y}).$$

Answers

1.1 a) not a tautology, b) a tautology, c) not a tautology, d) a tautology, e) a tautology.

1.2 a) In no truth valuation. b) For the truth value for which $u(x) = 1$, $u(y) = 0$. c) For all truth valuations for which we have $u(x) = 1$. d) In no truth valuation. e) For truth valuation with $u(x) = 0$ and $u(y) = 1$. f) For all truth valuations for which the logical variable x is false and at least one from logical variables y, z is also false. It means $u(x) = 0$ and at the same time $u(y) = 0$ or $u(z) = 0$.

1.3 a) A satisfiable formula which is not a tautology. b) A satisfiable formula which is not a tautology. c) A contradiction. d) A satisfiable formula which is not a tautology. e) A tautology. f) A satisfiable formula which is not a tautology.

1.5 a) $g(x, y, z) = x\overline{z} + \overline{x}y + \overline{y}\overline{z} = (\overline{x} + \overline{z})\,(x + \overline{y} + z)$.