10 Tutorial 10 – December 5th, 2017

10.1 Given a Boolean algebra B with operations $\wedge$, $\vee$, complement $\overline{a}$ of a, the smallest element $\mathbf{0}$, and the greatest element $\mathbf{1}$. Show that for every $a, b \in B$ it holds that

$$\text{if } a \sqsubseteq b \text{ then } a \wedge \overline{b} = \mathbf{0} \text{ and } \overline{a} \vee b = \mathbf{1}.$$

Solution. We show the first part, the second part can be proved analogously.

We have $a \sqsubseteq b$ if and only if $a \wedge b = a$. Therefore,

$$a \wedge \overline{b} = (a \wedge b) \wedge \overline{b} = a \wedge (b \wedge \overline{b}) = a \wedge \mathbf{0} = \mathbf{0}.$$

10.2 Given a Boolean algebra B with operations $\wedge$, $\vee$, complement $\overline{a}$ of a, the smallest element $\mathbf{0}$, and the greatest element $\mathbf{1}$. Show that for every $a, b \in B$ we have

$$a \sqsubseteq b \quad \text{if and only if} \quad \overline{b} \sqsubseteq \overline{a}.$$

10.3 Given a Boolean algebra B with operations $\wedge$, $\vee$, complement $\overline{a}$ of a, the smallest element $\mathbf{0}$, and the greatest element $\mathbf{1}$. Show that for every $a, b, c \in B$ we have

$$\text{if } a \sqsubseteq b \text{ then } a \wedge c \sqsubseteq b \wedge c, \quad a \vee c \sqsubseteq b \vee c.$$

Solution. We show that $a \wedge c \sqsubseteq b \wedge c$; the other equality can be proved similarly.

We know that $a \sqsubseteq b$ if and only if $a \wedge b = a$. Let us calculate

$$(a \wedge c) \wedge (b \wedge c) = (a \wedge b) \wedge c = a \wedge c.$$

Therefore, $a \wedge c \sqsubseteq b \wedge c$ holds.

10.4 Given a Boolean algebra B with operations $\wedge$, $\vee$, complement $\overline{a}$ of a, the smallest element $\mathbf{0}$, and the greatest element $\mathbf{1}$. Define a new operation $|$ on B by

$$a \,|\, b = \overline{a} \vee \overline{b}.$$

Show that

$$a \vee b = (a \,|\, a) \,|\, (b \,|\, b) \quad \text{and} \quad a \wedge b = (a \,|\, b) \,|\, (a \,|\, b).$$

Solution. We show the first part.

Let us calculate

$$(a \,|\, a) \,|\, (b \,|\, b) = (\overline{a} \vee \overline{a}) \,|\, (\overline{b} \vee \overline{b}) = \overline{a} \,|\, \overline{b} = \overline{\overline{a}} \vee \overline{\overline{b}} = a \vee b.$$

10.5 Draw a tree T on the set of vertices $V = \{1, \ldots, 8\}$.

10.6 Draw all different trees with 5 vertices. (Two trees are the same if they differ only in names of vertices.)

Solution. Since the sum of degrees must be 8 and since there are at least two vertices of degree one, the sum of degrees of 3 vertices must be 6. One possibility is $2+2+2$ which gives a path; the second possibility is $1+2+3$, and the last possibility is $1+1+4$. Since for every case there is only one tree with the prescribed degrees, there are only these three different trees.

10.7 Draw a simple undirected graph G which has 8 vertices, 10 edges, and 2 components of connectivity.

10.8 Given a tree T on the set of vertices $V = \{1, \ldots, 6\}$. If G is a graph obtained from T by adding 2 edges (between vertices from V) how many circuits G can have? Give the smallest and the biggest number of circuits.

Answers

10.8 A tree with 2 added edges can have either 2 or 3 circuits. Indeed, the first edge closes just one circuit, say C_1. The second edge can close either a circuit C_2 which is vertex disjoint with C_1, or C_2 can have a common edge with C_1. In the first case, the graph contains two circuits, in the second case three.