11 Tutorial 11 – December 12th, 2017

11.1 Find a minimal spanning tree in the undirected graph given by the following matrix of weights

$$\begin{pmatrix} - & 9 & 13 & 5 & - & 7 & - & 9 \\ 9 & - & 13 & - & 4 & - & - & 14 \\ 13 & 13 & - & 6 & 5 & 4 & 9 & 2 \\ 5 & - & 6 & - & 4 & - & - & - \\ - & 4 & 5 & 4 & - & 5 & - & 1 \\ 7 & - & 4 & - & 5 & - & 15 & 8 \\ - & - & 9 & - & - & 15 & - & 7 \\ 9 & 14 & 2 & - & 1 & 8 & 7 & - \end{pmatrix}$$

Solution. First we sort the edges (we state the weight of an edge in the brackets)
$e_1 = \{5,8\}\,(1)$, $e_2 = \{3,8\}\,(2)$, $e_3 = \{2,5\}\,(4)$, $e_4 = \{3,6\}\,(4)$, $e_5 = \{4,5\}\,(4)$, $e_6 = \{1,4\}\,(5)$, $e_7 = \{3,5\}\,(5)$, $e_8 = \{5,6\}\,(5)$, $e_9 = \{3,4\}\,(6)$, $e_{10} = \{1,6\}\,(7)$, $e_{11} = \{7,8\}\,(7)$, $e_{12} = \{6,8\}\,(8)$, $e_{13} = \{1,2\}\,(9)$, $e_{14} = \{1,8\}\,(9)$, $e_{15} = \{3,7\}\,(9)$, $e_{16} = \{1,3\}\,(13)$, $e_{17} = \{2,3\}\,(13)$, $e_{18} = \{2,8\}\,(14)$, $e_{19} = \{6,7\}\,(15)$.
Put $L = \emptyset$.

Now we will go through edges in the given order and include e_i into L if and only if it does not close a circuit.

1. e_1 does not close a circuit, hence $L := \{\{5,8\}\}$.
2. e_2 does not close a circuit, hence $L := \{\{5,8\},\{3,8\}\}$.
3. e_3 does not close a circuit, hence $L := \{\{5,8\},\{3,8\},\{2,5\}\}$.
4. e_4 does not close a circuit, hence $L := \{\{5,8\},\{3,8\},\{2,5\},\{3,6\}\}$.
5. e_5 does not close a circuit, hence $L := \{\{5,8\},\{3,8\},\{2,5\},\{3,6\},\{4,5\}\}$.
6. e_6 does not close a circuit, hence $L := \{\{5,8\},\{3,8\},\{2,5\},\{3,6\},\{4,5\},\{1,4\}\}$.
7. e_7 closes a circuit formed by e_1, e_2 and e_7, hence L is the same as in 6.
8. e_8 closes a circuit formed by e_1, e_2, e_4 and e_8, hence L is the same as in 6.
9. e_9 closes a circuit formed by e_5, e_1, e_2 and e_9, hence L is the same as in 6.
10. e_{10} closes a circuit formed by e_6, e_5, e_1, e_2, e_4 and e_{10}, hence L is the same as in 6.
11. e_{11} does not close a circuit, hence $L := \{\{5,8\},\{3,8\},\{2,5\},\{3,6\},\{4,5\},\{1,4\},\{7,8\}\}$.

Since L contains $8 - 1 = 7$ edges, we have

$$L = \{\{5,8\},\{3,8\},\{2,5\},\{3,6\},\{4,5\},\{1,4\},\{7,8\}\}$$

is the set of edges of a minimal spanning tree of G. The weight (price) of L is

$$c(L) = 1 + 2 + 4 + 4 + 4 + 5 + 7 = 27.$$

11.2 Find a minimal spanning tree in the undirected graph given by the following matrix of weights

$$\begin{pmatrix} - & 5 & 9 & 3 & 2 & 5 & 1 \\ 5 & - & 18 & 7 & 19 & 1 & 7 \\ 9 & 18 & - & 6 & 19 & 10 & 3 \\ 3 & 7 & 6 & - & 14 & 8 & 9 \\ 2 & 19 & 19 & 14 & - & 7 & 8 \\ 5 & 1 & 10 & 8 & 7 & - & 4 \\ 1 & 7 & 3 & 9 & 8 & 4 & - \end{pmatrix}$$

11.3 Find an example of an undirected weighted graph which has a unique minimal spanning tree, or show that such a graph does not exist.

11.4 Find an example of an undirected weighted graph which has precisely two minimal spanning trees, or show that such a graph does not exist.

11.5 Given a directed graph $G = (V, E)$, $V = \{1, \ldots, 12\}$, E is given by the following table (u is the initial vertex of e, v is the terminal vertex of e).

u	1	1	2	4	4	4	4	4	4	5	5	6	6	7	7	8	8	10	10	11	11	11	12
v	2	9	9	2	5	7	9	10	12	3	10	5	10	1	2	6	9	2	9	1	2	5	2

Decide whether G has a topological sort or not. If the answer is yes, find one topological sort of vertices.

Solution. The graph G is acyclic if and only if it has a topological sort of vertices. We will use the algorithm for finding a topological sort of vertices; if we succeed in sorting all vertices, the graph is acyclic; if not, the graph is not acyclic.

First, we calculate the in-degrees of vertices of G. We get

v	1	2	3	4	5	6	7	8	9	10	11	12
$d^-(v)$	2	6	1	0	3	1	1	0	5	3	0	1

The set M contains all vertices with in-degree 0; therefore $M := \{4, 8, 11\}$. We set $i := 1$.

Since M is non-empty, we choose an arbitrary element from M, say 4, and put $v_1 = 4$.
Now, for every edges with initial vertex 4 we decrease the in-degree of the terminal vertex w by 1. If we get $d^-(w) = 0$ we insert w in M.

Hence,

$$d^-(2) = 5, d^-(5) = 2, d^-(7) = 0, d^-(9) = 4, d^-(10) = 2, d^-(12) = 0,$$

and

$$M := \{8, 11, 7, 12\}, i := 2.$$

Since M is non-empty, we choose an arbitrary element from M, say 8, and put $v_2 = 8$.
Now, for every edges with initial vertex 8 we decrease the in-degree of the terminal vertex w by 1. If we get $d^-(w) = 0$ we insert w in M.

Hence,

$$d^-(6) = 0, d^-(9) = 3, \quad \text{and} \quad M := \{11, 7, 12, 6\}, \ \ i := 3.$$

Similarly, we put $v_3 = 11$. Hence,

$$d^-(1) = 1, d^-(2) = 4, d^-(5) = 1, \quad \text{and} \quad M := \{7, 12, 6\}, \ \ i := 4.$$

We put $v_4 = 7$. Hence,

$$d^-(1) = 0, d^-(2) = 3, \quad \text{and} \quad M := \{12, 6, 1\}, \ \ i := 5.$$

We put $v_5 = 12$. Hence,

$$d^-(2) = 2, \quad \text{and} \quad M := \{6, 1\}, \ \ i := 6.$$

We put $v_6 = 6$. Hence,

$$d^-(5) = 0, d^-(10) = 1, \quad \text{and} \quad M := \{1, 5\}, \ \ i := 7.$$

We put $v_7 = 1$. Hence,

$$d^-(2) = 1, d^-(10) = 1, \quad \text{and} \quad M := \{5\}, \ \ i := 8.$$

We put $v_8 = 5$. Hence,

$$d^-(3) = 0, d^-(10) = 0, \quad \text{and} \quad M := \{3, 10\}, \ \ i := 9.$$

We put $v_9 = 3$. Since there is not edge with initial vertex 3, we get

$$M := \{10\} \quad \text{and} \quad i := 10.$$

We put $v_{10} = 10$. Hence,

$$d^-(2) = 0, d^-(9) = 1, \quad \text{and} \quad M := \{2\}, \ \ i := 11.$$

We put $v_{11} = 2$. Hence,

$$d^-(9) = 0, \quad \text{and} \quad M := \{9\}, \ \ i := 12.$$

We put $v_{12} = 9$. Hence,

$$M := \emptyset \quad \text{and} \quad i := 13.$$

Since $M = \emptyset$, the algorithm terminates. The sequence which was found is

$$v_1 = 4, v_2 = 8, v_3 = 11, v_4 = 7, v_5 = 12, v_6 = 6, v_7 = 1, v_8 = 5, v_9 = 3, v_{10} = 10, v_{11} = 2, v_{12} = 9$$

is a topological sort of vertices. Therefore, the graph G is acyclic.

11.6 Given a directed graph $G = (V, E)$, $V = \{1, \dots, 12\}$, E is given by the following table (u is the initial vertex of e, v is the terminal vertex of e).

u	1	3	3	3	3	4	5	5	5	6	6	6	6	6	6	6	8	8	8	9	12	12	12	12
v	4	4	7	9	10	7	2	4	11	1	3	4	5	7	9	12	9	10	11	11	1	2	3	8

Decide whether G has a topological sort or not. If the answer is yes, find one topological sort of vertices.

11.7 Given a directed graph $G = (V, E)$, $V = \{1, \dots, 12\}$, E is given by the following table (u is the initial vertex of e, v is the terminal vertex of e).

u	1	2	2	2	2	2	3	4	4	5	5	6	6	6	6	6	7	7	7	8	11	11	12
v	3	1	6	7	9	12	4	8	11	3	8	4	7	9	10	12	1	5	10	3	3	10	8

Decide whether G has a topological sort or not. If the answer is yes, find one topological sort of vertices.

11.8 Draw a simple directed acyclic graph which has 9 vertices, and 15 edges.

Answers

11.2 The set of edges of a minimal spanning tree is

$$L = \{\{1, 7\}, \{2, 6\}, \{1, 5\}, \{1, 4\}, \{3, 7\}, \{6, 7\}\}.$$

The weight (price) of L is

$$c(L) = 1 + 1 + 2 + 3 + 3 + 4 = 14.$$

11.3 For example, the graph from 11.2 has a unique minimal spanning tree.

11.4 The following weighted graph (given by its matrix of weights) has precisely two minimal spanning trees.

$$\begin{pmatrix} - & 1 & - & - \\ 1 & - & 1 & 2 \\ - & 1 & - & 2 \\ - & 2 & 2 & - \end{pmatrix}$$

Indeed, $L_1 = \{\{1,2\},\{2,3\},\{3,4\}\}$ and $L_2 = \{\{1,2\},\{2,3\},\{2,4\}\}$ are the only minimal spanning trees (with $c(L_1) = c(L_2) = 4$).

11.6 The graph G is acyclic, since it has a topological sort of vertices, one of them is

$$v_1 = 6, v_2 = 5, v_3 = 12, v_4 = 1, v_5 = 2, v_6 = 3, v_7 = 8, v_8 = 4, v_9 = 9, v_{10} = 10, v_{11} = 7, v_{12} = 11.$$

11.7 The graph G is not acyclic; for example, edges $(3,4)$, $(4,8)$, and $(8,3)$ form a cycle.

Notice, that when constructing a topological sort, the algorithm halts when $v_1 = 2$, $v_2 = 6$, $v_3 = 7$, $v_4 = 9$, $v_5 = 12$, $v_6 = 5$, and $v_7 = 1$. After that, the set M is empty and not all vertices were sorted.